STATIONARY SOLUTIONS TO KOMPANEETS EQUATION FOR RELATIVISTIC PROCESSES IN ASTROPHYSICAL OBJECTS

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The purpose of the present work is to obtain new analytical solutions to the relativistically corrected Kompaneets equation describing the photon spectrum in magnetar's magnetosphere. In the assumption of stationarity, the solutions, expressed in terms of Heun Triconfluent functions, can be employed to compute the photons density as a function of the dimensionless frequency.

Key words: Comptonization processes, diffusion kinetic equation, Heun Triconfluent functions.

1. INTRODUCTION

At an elementary level, a photon – electron interaction *via* scattering is unanimously known as the Compton process. If we are to penetrate the problem with accuracy, then it must be said that the continuous change in the distribution frequency of the photons during the multiple scatterings they suffer within a medium containing thermal electrons is called the *comptonization* process.

Recently, this phenomenon was observed in the interaction of the microwave background radiation with the matter belonging to the clusters of galaxies. In few words, the photons, which are found in the cosmic microwave background (CMB) radiation, when passing through the regions containing hot, ionized gas (clusters), are suffering multiple scatterings, due to collisions with high temperature electrons. As a result, in the vicinity of the clusters of galaxies, the CMB spectrum is distorted [1].

Many studies have been dedicated to the problem of the isotropy of the CMB radiation and several reliable explanations were formulated [2, 3].

Recently, the CMB anisotropy has provided valuable information on the cluster mass, properties and gaseous content, as well as on its cosmological evolution [4]. In the same time, a connection between CMB observations and the Galactic motion through diverse physical processes has been established [5].

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Once the photon's energy increases due to Compton scatterings, the CMB radiation, in the direction of a cluster, is altered in the sense that there are fewer low energy photons and more higher energy photons than one would expect. The radio interferometer in the direction of a galaxy cluster sees a deficit of CBR radiation, since the "missing" photons are shifted to higher energy and this distortion is called the *Sunyaev–Zel'dovich effect* (SZE) [1].

The Compton type scattering interactions mentioned above are modeled by the Kompaneets kinetic equation (also known as photon diffusion equation) [6] which expresses the time change rate of the photon occupation number, due to the scattering of an isotropic radiation by an isotropic electron gas. In its traditional form [7], the Kompaneets equation is a reliable tool for different nonrelativistic astrophysics phenomena, in particular generated by the inverse Compton scattering (ICS) mechanisms. These occur when the low energy photons are scattered by high energy electrons, the opposite situation (when the energy is transferred from the photons to the electrons) being known as the direct Compton scattering. In astrophysics, the ICS has a more significant place, compared to the direct one, since it is the basic process responsible for the γ - rays generation [8].

It has been proved that ICS can be the dominant high-energy emission mechanism in majority of pulsars [9]. Besides pulsars, there is a wide range of astrophysical objects, as for example Active Galactic Nuclei (AGN), supernova remnants or clusters of galaxies, in which high energy radiation is generated through this mechanism.

Recently, Rephaeli concluded that the celebrated diffusion kinetic equation elaborated by Kompaneets is not so accurate for high frequencies, since the electrons have huge velocities resulting in large changes in the photon's energy, fact which imposes relativistic corrections [4].

The main goal of the present work is to derive new stationary closed form solutions to the Kompaneets equation, in the relativistic case. For an extensive study of the non-relativistic one, we recommend the numerical scheme developed in [10] and the analytical approach performed in [11] and [12].

2. THE KOMPANEETS EQUATION

In the non-relativistic regime characterized by the following conditions:

$$\begin{cases} h\nu << k_B T \\ k_B T << m_e c^2 \end{cases} \tag{1}$$

the Kompaneets kinetic equation describing the diffusion of soft photons (in the frequency space) in magnetar's magnetospheres is [13]

$$\frac{\partial n}{\partial t}=\frac{k_B T}{m_e c^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[Nc\sigma_T x^4\left(\frac{\partial n}{\partial x}+n+n^2\right)\right]. \tag{2}$$

The physical quantities involved in (2) are: x is the dimensionless frequency, $x=\frac{h\nu}{k_B T}$, with $h\nu$ representing the photon energy and T being the electron temperature, n represents the density of photons in the spectral interval $\mathrm{d}x$, N is the electron number density (assuming homogenous), σ_T is the Thomson cross section and c is the speed of light.

In the elastic limit and integrating in the phase space, one can switch to the comptonization parameter, η, (to which the SZE is only sensitive) by

$$\eta=\int^t \mathrm{d}t' \; Nc\sigma_T \frac{k_B T}{m_e c^2}$$

so that the Kompaneets equation (2) turns into the simpler form

$$\frac{\partial n}{\partial \eta}=\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x}+n+n^2\right)\right]. \tag{3}$$

The three terms in the parentheses are responsible for the following physical processes: the first term corresponds to the change in frequency due to the Doppler effect, the second one stands for the Compton effect (it also describes the recoil effect) and the third term accounts for induced Compton scattering.

Intensive studies were dedicated to solve the equation (3) and the corresponding solutions have been obtained via numerical methods of investigation [8, 10] or by performing different approximations. For example, when the induced scattering is dominant, the Kompaneets equation has an additional symmetry and one can employ the Lie group analysis to construct exact group invariant solutions [14].

In the stationary regime, one gets from (3) the Riccati-type differential equation

$$\frac{\partial n}{\partial x}+n+n^2=\frac{Q}{x^4}, \tag{4}$$

where the constant Q has been defined in [12] as the photon flux in the considered frequency domain, related to the current density by $j(x)=Qx^2$.

Obviously, once Q is increasing, the number of scattered photons, solutions of (4), get farer from the initially Bose-Einstein equilibrium distribution (corresponding to $Q=0$).

In our paper [11], devoted to the resonant Compton scatterings in pulsar's magnetosphere, we have solved the stationary non-relativistic equation (4). Firstly, by writing n as

$$n = \frac{\Psi'}{\Psi}, \tag{5}$$

with

$$\Psi = \exp\left[-\frac{x}{2}\right]\cdot\sqrt{x}\cdot f(x),$$

we have come to the following equation for the unknown function *f(x)*

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}f}{\mathrm{d}x} - \left(\frac{Q}{x^4} + \frac{1}{4x^2} + \frac{1}{4}\right) f = 0 .$$

With the new variable

$$y = \frac{x^2 - 1}{x^2 + 1},$$

this finally transforms into the following second order differential equation:

$$\left(y^2 - 1\right)^3 \frac{\mathrm{d}^2 f}{\mathrm{d}y^2} + 2y\left(y^2 - 1\right)^2 \frac{\mathrm{d}f}{\mathrm{d}y} + \left[Qy^2 + \left(-2Q + \frac{1}{2}\right)y + Q + \frac{1}{2}\right] f = 0, \tag{6}$$

whose solutions has been written in terms of Heun Double Confluent functions. Recently, considerable attention has been given to different types of Heun's functions because they are the most general ones, including the well-known hypergeometric or Mathieu's functions, as particular cases [15].

Since in [11] we have discussed the resulting photon distribution spectrum, in agreement with the analysis developed in [12], let us work out now the necessary condition for a polynomial form of *f(y)*, following the theory developed in [16]. Generally, for a differential equation which can be cast into the form

$$\left[F(D) + P\left(y, \frac{\mathrm{d}}{\mathrm{d}y}\right)\right] f(y) = 0 ,$$

where $F(D) = \sum_n a_n D^n$, with $D = y\,\frac{\mathrm{d}}{\mathrm{d}y}$, is a diagonal operator in the space of monomials and $P\left(y, \frac{\mathrm{d}}{\mathrm{d}y}\right)$ is an arbitrary polynomial function of y and $\frac{\mathrm{d}}{\mathrm{d}y}$, (except D), one has to impose the condition

$$F(D)y^n = 0\,. \tag{7}$$

By inspecting the equation (6), it turns out that the essential operator $F(D)$ is

$$F(D) = 3D^2 - D + \left(Q + \frac{1}{2}\right)$$

and the condition (7) is leading, for negative values of Q, to the following non-trivial quantization law

$$|Q| = \left[n(3n-1) + \frac{1}{2}\right].$$

In [12], it has been assumed that the constant Q may be either positive or negative. The positive values are corresponding to a constant photon supply at $x = 0$ and a sink at $x = \infty$, while the negative ones are for a supply at $x = \infty$ and a sink at $x = 0$.

3. THE RELATIVISTIC VERSION

As stated in [17], when it comes to analyze the interaction between energetic photons and hot electrons ($h\nu, k_B T_e > 0.01 m_e c^2$), relativistic corrections to the Kompaneets equation are required.

For low temperatures, so that the terms proportional to $k_B T_e / m_e c^2$ can be neglected, Sazonov and Sunyaev have derived the following (more general) form of the photon frequency distribution as a consequence of the Compton interaction of the monochromatic radiation with thermal electrons [17],

$$\frac{\partial n(\nu)}{\partial t} = \frac{h}{m_e c^2} \frac{1}{\nu^2} \frac{\partial}{\partial \nu}\left[\nu^4 \left(n + a\frac{h\nu^2}{m_e c^2}\frac{\partial n}{\partial \nu}\right)\right]. \tag{8}$$

The term describing the frequency diffusion of photons has been introduced by Ross *et al.* [18] and Illarionov *et al.* [19] and it becomes significant when dealing with the scattering of hard radiation $(h\nu \sim 0.1 m_e c^2)$ on cold electrons $(kT_e << h\nu)$.

One may notice in (8) that this term is multiplied by the positive constant *a,* whose numerical value $a = 0.7$ is coming from the Fokker-Planck expansion of the integro-differential kinetic equation describing the comptonization of photons.

In the stationary regime and with the notation

$$x = \frac{h\nu}{m_e c^2}\,, \tag{9}$$

the equation (8) becomes

$$ax^2 \frac{\partial n}{\partial x} + n = \frac{Q}{x^4},$$

where the positive constant Q can be associated to the photon flux, as in [12].

In our analysis, we are going to generalize the above form which can be easily integrated, by adding the n^2 term, describing the induced Compton scattering contribution.

Thus, we come to the Riccati-type differential equation

$$ax^2 \frac{\partial n}{\partial x} + n + n^2 = \frac{Q}{x^4}, \tag{10}$$

which, in terms of the new variable

$$y = -\frac{1}{ax}, \tag{11}$$

reads

$$\frac{\partial n}{\partial y} + n + n^2 = Qa^4 y^4. \tag{12}$$

Once we are writing down $n(y)$ as in (5), where the derivative is with respect to y, the equation (12) turns into the following second order differential equation

$$\Psi'' + \Psi' - Qa^4 y^4 \Psi = 0. \tag{13}$$

Under the change of function

$$\Psi = \exp\left[-\frac{y}{2} - \frac{y^3}{3} a^2 \sqrt{Q}\right] \cdot u(y), \tag{14}$$

the equation (13) leads to the following equation for the unknown function u

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - 2a^2 \sqrt{Q} y^2 \frac{\mathrm{d}u}{\mathrm{d}y} - \left(2ya^2\sqrt{Q} + \frac{1}{4}\right) u = 0, \tag{15}$$

which, with $z = \sigma y$, becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} - 2a^2 \sqrt{Q} \frac{z^2}{\sigma^3} \frac{\mathrm{d}u}{\mathrm{d}z} - \left(2a^2 \sqrt{Q} \frac{z}{\sigma^3} + \frac{1}{4\sigma^2}\right) u = 0. \tag{16}$$

One may notice that, for $\sigma = \left(\frac{2a^2 \sqrt{Q}}{3} \right)^{\frac{1}{3}}$, the Eq. (16) has exactly the canonical form of the so-called Heun Triconfluent equation [15]:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - (\gamma + 3z^2) \frac{\mathrm{d}w}{\mathrm{d}z} + \left(\alpha + \beta z - 3z \right) w = 0 , \tag{17}$$

whose solutions are the Heun Triconfluent functions:

$$w = HeunT(\alpha, \beta, \gamma, z) . \tag{18}$$

By comparing (16) and (17), one can identify the parameters:

$$\begin{cases} \alpha = -\frac{1}{4\sigma^2} = -\frac{1}{4} \left(\frac{3}{2a^2 \sqrt{Q}} \right)^{\frac{2}{3}} \\ \beta = \gamma = 0 \end{cases} . \tag{19}$$

and the explicit form of the variable, $z = \left(\frac{2a^2 \sqrt{Q}}{3} \right)^{\frac{1}{3}} y$.

Finally, with

$$u = HeunT \left(-\frac{1}{4} \left(\frac{3}{2a^2 \sqrt{Q}} \right)^{\frac{2}{3}}, 0, 0, \left(\frac{2a^2 \sqrt{Q}}{3} \right)^{\frac{1}{3}} y \right), \tag{20}$$

the expression of the function Ψ defined in (14) is

$$\Psi = \exp \left[-\frac{y}{2} - \frac{y^3}{3} a^2 \sqrt{Q} \right] \cdot HeunT \left(-\frac{1}{4} \left(\frac{3}{2a^2 \sqrt{Q}} \right)^{\frac{2}{3}}, 0, 0, \left(\frac{2a^2 \sqrt{Q}}{3} \right)^{\frac{1}{3}} y \right). \tag{21}$$

Envisaging (5), we get the concrete expression of the photon number density:

$$n = -\frac{1}{2} - a^2 \sqrt{Q} y^2 + \frac{HeunT'}{HeunT} = -\frac{1}{2} - \frac{\sqrt{Q}}{x^2} + \left(\frac{2a^2 \sqrt{Q}}{3} \right)^{\frac{1}{3}} \cdot \frac{HeunTprime}{HeunT} , \tag{22}$$

Where $HeunT\,' = \dfrac{\mathrm{d}HeunT}{\mathrm{d}y}$, while what is termed in the most recent version of Maple symbolic algebra system by HeunTprime is defined as

$$HeunTprime = \frac{\mathrm{d}}{\mathrm{d}z} HeunT\left[\alpha, \beta, \gamma, z\right].$$

In the expression (22), one may check (using Maple) that the terms in the r.h.s. are competing against each other once the last term turns positive. By imposing the natural (physical) condition $n > 0$, we get a frequency range, for any given value of the model parameter $b = a^2 \sqrt{Q}$.

4. CONCLUSIONS

In the last years, both the non-relativistic and the relativistic Kompaneets equations have been seen as valuable tools for probing the cluster environment and the global properties of the universe.

For low frequencies, one may use the Kompaneets equation in its traditional form (2), while for clusters observed by modern telescopes at much higher frequencies, it has been stated that the relativistic treatment is more suitable.

The aim of the present paper is to find closed form solutions to the relativistic Kompaneets equation, in the stationary case and for low temperatures regime.

We have started our analysis by adding to the equation (8), derived in [15], the induced Compton scattering contribution, n^2. The obtained photon distribution spectrum, written in terms of the Heun Triconfluent functions, leads to ranges for the dimensionless frequency, once we are fixing the numerical value of the model parameter $b = a^2 \sqrt{Q}$.

A direct generalization of the theory presented in our paper is to study the interaction between energetic photons and hot electrons, with $h\nu, k_B T_e > 0.01 m_e c^2$. In this case, the main corrections to the relativistic Kompaneets equation would be of order $h\nu / (m_e c^2)$ and $k_B T_e / (m_e c^2)$.

We conclude by saying that the relation between different forms of the Kompaneets equation and Heun's second order linear differential equations is quite new [12]. Even though the way from the general theory, which is well understood [15], to the physical interpretation of the solutions is quite complicated, in the last years, a whole range of physical processes, from quantum optics [20] to black holes [21], have been discussed in terms of Heun-type solutions.

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