ADDENDUM TO THE SAMPLING THEOREM: IMPROVED ACCURACY INTERPOLATION FOR SAMPLED FUNCTIONS

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The form in which the interpolated function given by the sampling theorem is generally used is a truncated version, due to the impossibility of taking in the infinite number of terms. However, in this paper we challenge this habit by introducing new functions, with a finite number of terms, some better approximating the original function and all having a circular character, necessary in Fourier computation for which the sampling theorem is mostly used. We use both integer and semi-integer sampling for generality. We also study both the cases of even and odd number of samples. We had in mind at all times the 2D sampling theorem for optics applications. However, for simplicity we worked in 1D. The generalization to 2D is straightforward.

Key words: sampling theorem, interpolation, circularity.

1. INTRODUCTION OR THE TRUNCATED INTERPOLATION FUNCTION

The sampling theorem, as in the classical outline owed to Goodman [1], is generally wrongly applied for functions sampled over a limited domain and with only an approximate limited bandwidth. It is unavoidable in practice to work with functions of limited definition domain and also it is unavoidable to assume that these functions have a limited bandwidth so that the main results of the sampling theorem may be applied to it, although functions with limited definition domain have necessarily an unlimited bandwidth (and viceversa). What is generally done to obtain the interpolation function for a sampled limited definition domain function is to use a truncated version of the sampling theorem, as outlined by Goodman [1]. For instance, we know that the interpolated version for a 1D, bandwidth limited function g is of the form [1, 2]

\[ g(t) = \sum_{n=-\infty}^{\infty} g \left( \frac{n}{2f_s} \right) \text{sinc}(n - 2f_s t), \]

with the interpolation function sinc being defined as

\[ \text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}, \quad (2) \]

where \( f_s \) is a sampling frequency respecting the condition

\[ f_s \geq 2 f_N, \quad (3.a) \]

with

\[ |f| \leq \frac{f_N}{2}. \quad (3.b) \]

where \( f_N \) is the Nyquist frequency, the function \( g \) bandwidth is \( \Delta f \), and \( f \) is an arbitrary frequency with a corresponding non-null value in the Fourier spectrum of \( g \). The index \( n \) of the sum from (1) runs from \(-\infty\) to \(+\infty\) because a function of limited bandwidth has an infinite domain over which has non-null values. In practice, of course, it is impossible to calculate such a sum and one has to truncate it. In this case the expression (1) of the interpolation function becomes imprecise. The problem is that we are concerned with discrete calculation, the only type of numeric calculation possible on computers, and generally available to humans, when the Fourier transform of a function is not a closed form function, which is the general and most frequent case.

Suppose we have a function \( g \) defined over the domain \((-\Delta t/2, \Delta t/2)\) centered in origin. Suppose also that the sampling interval \( \delta t = \Delta t/N \), where \( N \) is an even integer corresponding to a frequency \( f_S \) large enough so that the Fourier spectrum amplitude values for \(|f| \geq f_S\) are very small and, therefore, negligible. Then Eq. (1) can be rewritten as

\[ g_T(t) = \sum_{n=-N/2}^{N/2-1} g_s \text{sinc}(n-2f_S t), \quad g_s \equiv g\left(\frac{n}{2f_s}\right) \equiv g\left(\frac{n}{\Delta f}\right) \equiv g(n\delta t) \equiv g(t_s), \quad (4) \]

where we chose \( f_s = f_N \) for simplicity. The interpolated function (4) is only an approximation of the true function \( g \). For lack of a better term we shall call the function from Eq. (4) the “T” function because it is a truncated version of the complete (infinite series) function and we subscripted it accordingly.

### 2. WHAT DISCRETE, FINITE ELEMENTS CALCULATIONS BRING US: THE “V” INTERPOLATION FUNCTION

The problem we are asking now is whether we can improve on Eq. (4), whether there is an interpolation function that is more accurate than the one inspired from [1, 2]. First let us note that, according to the discrete inverse Fourier transform definition we have
where \( \delta f \) is, of course, \( \Delta f/N \). We also know that according to the discrete Fourier transform definition we have

\[
G_m = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} g_n \exp\left(-i2\pi m \frac{t}{N\delta t}\right) = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} g_n \exp\left(-i2\pi \frac{mn}{N}\right). \tag{6}
\]

Introducing definition (6) of \( G_m \) in Eq. (5) we obtain the \( g \) as a function of its samples \( g_n \), that is an interpolation function of \( g \).

\[
g_v(t) = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} \sum_{n=-N/2}^{N/2-1} g_n \exp\left(i2\pi \frac{m}{N} \frac{t}{\delta t}\right) \exp\left(-i2\pi \frac{mn}{N}\right) = \sum_{n=-N/2}^{N/2-1} h_n(t) g_n, \tag{7.a}
\]

\[
= \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \sum_{m=-N/2}^{N/2-1} \frac{1}{N} \exp\left[i2\pi \frac{m}{N} \left(\frac{t}{\delta t} - n\right)\right] = \sum_{n=-N/2}^{N/2-1} h_n(t) g_n, \tag{7.b}
\]

where we noted with \( h_n(t) \) the interpolation coefficient. This coefficient can be calculated and the result is

\[
h_n(t) = \exp\left[-i \frac{\pi}{N} \left(\frac{t}{\delta t} - n\right)\right] \frac{\sin\left[\pi \left(\frac{t}{\delta t} - n\right)\right]}{N \sin\left[\pi / N \left(\frac{t}{\delta t} - n\right)\right]}. \tag{8}
\]

One may notice that the interpolation coefficient from Eq. (8) does not resemble at all the sinc function from Eqs. (1, 4), which is the interpolation coefficient in Goodman. We obtained a different function; one that we shall call the “V” function. However, if we make \( N \to \infty \), then \( h_n \) tends to the function \( \text{sinc}(n-t/\delta t) \), which is to be expected, since \( h_n \) was calculated having in mind a limited definition domain for \( g \), and, therefore, a limited \( N \), while Eq. (1) was calculated having in mind an infinite definition domain. However, the opposite process, to recover expression (8) for the interpolation coefficient by making the period or the number \( N \) finite in Eq. (1) is not easy and, in the final instance, it is not reciprocal. The rigorous, non-truncated Goodman expression does not tend to expression (8)!

**3. THE FALSE RECIPROCAL OR THE “C” INTERPOLATION FUNCTION**

Let us see what we meant above in detail. Let us rewrite Eq. (1) taking into account the limited period of the function \( g \). We know that a discrete function to which we can apply the direct and inverse Fourier transforms is circular, that is it repeats periodically to infinity [3-5]. Then Eq. (1) can be written as

\[
g(t) = \frac{1}{\sqrt{N}} \sum_{m=-N/2}^{N/2-1} G\left(f_m\right) \exp\left(i2\pi f_m t\right) = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} G_m \exp\left(i2\pi \frac{m}{N} \frac{t}{\delta t}\right). \tag{5}
\]
\[ g(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{N/2-1} g_{n+mN} \text{sinc}\left(\frac{t}{\delta t} - n + mN\right) = \] 
\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{N/2-1} g_n \frac{\sin\left[\pi(t/\delta t - n)\right]}{\pi(t/\delta t - n + mN)} = \] 
\[ = \sum_{n=-N/2}^{N/2-1} g_n \sin\left[\frac{\pi}{\delta t - n}\right] \frac{1}{N} \sum_{m=-\infty}^{\infty} \frac{1}{(t/\delta t - n)/N + m}. \] 

We used in Eq. (9) the circularity property of \( g_n \), the periodicity of the sine function and the fact that we chose \( N \) even. The inner sum in (9.c) converges to
\[ \sum_{m=-\infty}^{\infty} \frac{1}{(t/\delta t - n)/N + m} = \pi \cot\left[\frac{\pi(t/\delta t - n)}{N}\right], \] 
although it is not easy to prove. In literature we found only one instance when the result of the sum is listed and proven [6]. For the interested reader a different proof of Eq. (10), using residues calculus, is given in the Appendix. Also, one can prove Eq. (10) by using a computer program for symbolic calculation. At any rate, introducing (10) in (9.c) we obtain
\[ g_c(t) = \sum_{n=-N/2}^{N/2-1} g_n \cos\left[\frac{\pi}{N}\left(\frac{t}{\delta t} - n\right)\right] \frac{\sin\left[\frac{\pi(t/\delta t - n)}{N}\right]}{N} \frac{\sin\left[\frac{\pi}{N}(t/\delta t - n)\right]}{\pi}. \] 

We arranged the terms in Eq. (11) in this manner on purpose to ease the comparison to the interpolation coefficient (8). Indeed, the interpolation coefficient from (11) is similar to the one from (8), except for the fact that the imaginary part of the exponential from (8) is gone in (11). We obtained yet another function, one that we shall call the “C” function. Off the top we may say that the interpolation formula (11) is better, because if \( g \) is a real function there is no danger to have an imaginary component in formula (11) while on formula (8) it is possible that the imaginary parts of each term of the sum do not cancel out. On the other hand formula (8) has a strong argument in its favor in the fact that it issues from a discrete formulation of the Fourier transform, and the discrete formulations are of immense practical importance, which makes it hard to give up on them. We already used function “V” successfully, and the success was due to the fact that it issued from the problem at hand [7]. Probably the best way to assess the quality of the two formulas is to see how well they approximate some particular functions. Also it remains the mystery of why the formulas (8) and (11) are not identical to be solved.
4. COMPARISON BETWEEN THE “T”, “V” AND “C” FUNCTIONS

To recapitulate, we have three candidates for the interpolation function, the “T”, “V” and “C” functions. They differ only slightly, yet enough to be well differentiated:

\[ g_T(t) = \sum_{n=-N/2}^{N/2-1} g_s \sin\left(\frac{t}{\delta} - n\right), \]  \hspace{1cm} (12.a)

\[ g_V(t) = \sum_{n=-N/2}^{N/2-1} g_s \exp\left[-i \frac{\pi}{N} \left(\frac{t}{\delta} - n\right)\right] \frac{\sin\left[\pi \left(t/\delta - n\right)\right]}{N \sin\left[\pi / N \left(t/\delta - n\right)\right]}, \]  \hspace{1cm} (12.b)

\[ g_C(t) = \sum_{n=-N/2}^{N/2-1} g_s \cos\left[\frac{\pi}{N} \left(\frac{t}{\delta} - n\right)\right] \frac{\sin\left[\pi \left(t/\delta - n\right)\right]}{N \sin\left[\pi / N \left(t/\delta - n\right)\right]}. \]  \hspace{1cm} (12.c)

One fact that differentiates these functions we can state off the top: \( g_T \) is not a periodic (circular) function while the other two are. This is an impediment for \( g_T \) because the discrete nature is equivalent to circularity [3-5]. The denominators of \( g_T \) are linear with respect to the argument; this is an indication that the function is not periodic. The other functions, \( g_V \) and \( g_C \) are constructed out of periodic functions, so there is hope for them. And the graphical representation for Fig. 1 shows exactly that, the “T” function is not and the others are periodic. One may say that because the “T” function is not circular, we endeavor to make it circular using two approaches: one was using the Fourier discrete formalism which involved the circularity, the second was to include in the truncated version of the interpolation function all the (infinite) periods left out.

The non-circular character of “T” versus the circular character of “V” and “C” may be seen in Fig. 1.a-c. The interpolated function is the rect function.
One simple way to compare the functions is to see how well they reproduce the rectangle or the Heaviside π function. This function is expressed as

\[
g(t) = \begin{cases} 
1, & |t| \leq 1/2 \\
1/2, & |t| = 1/2 \\
0, & \text{in rest}
\end{cases}.
\] (13)

To add some variety and credibility to our numeric and graphical analysis, we added another function to our battery of tests, a function that for lack of a better name we just called it “vic”:

\[
\text{vic}(t) = \frac{3}{4} + \frac{1}{4} \cos(24\pi t) \exp \left[ -\left( \frac{t}{0.2} \right)^{16} \right].
\] (14)

This is a function especially designed for Fourier analysis. It has a relatively quick sinusoidal variation that will yield some salient maxima in the Fourier spectrum and an overall Gaussian envelope which produces a fast vanishing Fourier spectrum toward the high frequencies, although the spectrum is not rigorously bandwidth-limited. It is real and looks like
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Fig. 2 – The representation of the “vic” function.

Fig. 3 – The representation of the differences between the interpolation functions “T”, “C” and “V” and the original “vic” function. The difference is expressed in decibels, for better illustration of the small quantities involved.

One way to estimate the quality of an interpolation function is like in Fig. 3, where the difference between the original “vic” function and the interpolation of type “T”, “V” and “C” are shown together on a logarithmic scale, due to the smallness of the differences. The parameters of both the original functions $g$ relevant from the point of view of the sampling theorem are $\Delta t = 4 \, \mu m$, the period of the function; $\delta f = 1/\Delta t = 0.25 \, \mu m^{-1}$, the unit step in the frequency space; $\Delta f = 80 \, \mu m^{-1}$, double the Nyquist frequency; $\delta t = 1/\Delta f = 0.05 \, \mu m$, the unit sampling step, and $N = \Delta t/\delta t = \Delta f/\delta f = 80$. The comparison of the three interpolation functions for the “vic” function is shown graphically in Fig. 3. One may see that the “C” function is somewhat superior to the others. The worse seem to be the “V” function. It was necessary a representation of the differences rather than a comparison of the interpolation functions themselves, because then they are virtually indistinguishable. These observations cannot be generalized to rules. Pending on the nature of the function and of the chosen sampling we encountered a variety of
situations that prevents us from ascribing superiority to one type of function with respect to the quality of the interpolation. In general they are all quite good interpolation functions and, pending on the situation, one may be preferable to another.

5. SEMI-INTEGER VALUES FOR SAMPLES

Partly for completeness and partly because the sampling \( g_n \) from Eq. (4) is a little unbalanced (more samples in the left side of zero) we decided to include among the candidate functions those with samples made at semi-integer values of the step interval \( \delta x \). In this way the samples are perfectly centered in the \( \Delta x \) domain. The samples will be

\[
g_{\frac{N}{2}+\frac{1}{2}}, \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1,
\]

or

\[
g_n, \quad n = -\frac{N}{2} + \frac{1}{2}, \ldots, \frac{N}{2} - \frac{1}{2}.
\]

Now we have three new contestants for the best interpolation function. The truncated version is quite obviously what form will have

\[
g_{TSI} (t) = \sum_{n=-(N-1)/2}^{(N-1)/2} g_n \sin \left( \frac{t}{\delta t} - n \right).
\]

The “SI” subscript refers to semi-integer, obviously. \( g_V \) may be calculated easily according to steps of the proof ranging from Eq. (5) to Eq. (8). But the result is quite different. The derivations yield a \( g_V \) that is missing the exponential coefficient:

\[
g_{VSI} (t) = \sum_{n=-(N-1)/2}^{(N-1)/2} g_n \sin \left( \frac{\pi}{N} \left( \frac{t}{\delta t} - n \right) \right)
\]

Another unexpected result we obtain by calculating the “C” function for the semi-integer sampling. Namely nothing changes. The “C” function for the semi-integer samples has the same form as for integer samples. They are not the same, mind you, they just have the same form; they differ by their samples:

\[
g_{CSI} (t) = \sum_{n=-(N-1)/2}^{(N-1)/2} g_n \cos \left[ \frac{\pi}{N} \left( \frac{t}{\delta t} - n \right) \right] \sin \left( \frac{\pi}{N} \left( \frac{t}{\delta t} - n \right) \right) \sin \left( \frac{\pi}{N} \left( \frac{t}{\delta t} - n \right) \right)
\]
For the residue calculus shown in the Appendix, the function $g_{CSI}$ presents the same case, the same situation, the results are based again on the residues of

$$\pi \cot \left[ \frac{\pi}{N} \left( \frac{t}{\delta t} - n \right) \right].$$

Possibly a better way to estimate the quality of the interpolation function is to compute the $\chi^2$ functions for the differences with respect to the original functions,

$$\chi^2 = \sum_{m=-N/2}^{N/2-1} \left| g_{m}^{\text{interpolation}} - g_{m}^{\text{original}} \right|^2.$$  \hspace{1cm} (19)

The calculated results were put into Table 1. They don’t show any clear preference for a certain type of interpolation with one remarkable exception, which is not for a certain interpolating function but for an original function. The “vic” function shows a capacity for being well interpolated. We attribute again this property to the lack of corners which are difficult to reproduce with the help of elementary continuous functions.

<table>
<thead>
<tr>
<th>original function</th>
<th>interpolation function</th>
<th>$g_T$</th>
<th>$g_V$</th>
<th>$g_C$</th>
<th>$g_{TSI}$</th>
<th>$g_{VSI}$</th>
<th>$g_{CSI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>rect</td>
<td></td>
<td>4.76$10^{-3}$</td>
<td>4.76$10^{-3}$</td>
<td>4.75$10^{-3}$</td>
<td>1.37$10^{-3}$</td>
<td>1.37$10^{-3}$</td>
<td>1.37$10^{-3}$</td>
</tr>
<tr>
<td>vic</td>
<td></td>
<td>9.85$10^{-9}$</td>
<td>9.96$10^{-9}$</td>
<td>9.86$10^{-9}$</td>
<td>9.89$10^{-9}$</td>
<td>9.86$10^{-9}$</td>
<td>9.96$10^{-9}$</td>
</tr>
</tbody>
</table>

6. REMARKS

For the case of integer-type unbalanced sampling there is a remark that can be made. What if in the computation of $g$, we use the circularity of the $g$ function, in the case $g$ is continuous at the endings of its period? Then we have $g_{N/2} = g_{N/2}$

Taking advantage of this property, and continuing our not so rewarding policy of balancing the sampling, we rewrite the formula for $g_V$ in Eq. (7.a) with the following equation

$$g_V(t) = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} \exp \left( i2\pi \frac{m}{N} \frac{t}{\delta t} \right) \times \left[ \frac{1}{2} g_{-N/2} \exp(i\pi) + \sum_{m=-N/2}^{N/2-1} g_{m} \exp \left( -i2\pi \frac{mn}{N} \right) \right] + \frac{1}{2} g_{N/2} \exp(-i\pi) \right].$$ \hspace{1cm} (20)

The $\frac{1}{2}$ factor is a consequence of the rigorous definition of the rect function which is $\frac{1}{2}$ at the boundaries. The definition (20) seems like an average of two $g_V$ functions, one containing the first boundary but not the last and the second the
opposite way around, and the intermediary elements are mediated with each other. The function (20) yields “C” as interpolation function, as in Eq. (12.c). We found a way to reconcile the “V” and the “C” function for integer sampling. However, for semi-integer sampling we were unable to find such reconciliation. This reconciliation is, only relative. In the end the “V” function depends primarily on the chosen sampling. The sampling chosen in Eq. (20) is artificial and impractical.

Another thing that should be mentioned is that, although the “C” function and even the “T” function can sometimes interpolate better the original function, especially the “C” function, in discrete calculation such as those from [7], we have to use the “V” function, because it issued naturally in the course of those calculations.

7. EXTENSION TO ODD NUMBER OF SAMPLES

Out of desire to give our work as much completeness and perspective, we extended our research to the case odd \( N \). Off the top it seems trivial, and the interpolation function should not change drastically over plus or minus one sample. Yet it does, at least in form. In order to keep to a minimum the wild richness of trivial possibilities, we made the convention that the odd number of samples corresponds to an odd number of steps with one more in the right end of the domain. We made the symbolic calculation for the functions corresponding to the odd counterpart and the results were arrayed in Table 2. No striking regularities are noticed.

### Table 2

The interpolation functions for various possibilities, \( N \) even or odd, the samples integer or semi-integer and the types of function: “T”, “V” and “C”. We used the notations \( \omega_n = (t/\delta t – n) \) for making space in the table. For odd \( N \) we chose the positive arm of the samples longer with a step \( \delta t \). The fact that some functions have the same form may be deceiving; they still have different samples \( g_n \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>type</th>
<th>integer ( t_n )</th>
<th>semi-integer ( t_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>T</td>
<td>( \sum_{n=-N/2}^{N/2} g_n \text{ sinc} \omega_n )</td>
<td>( \sum_{n=-(N-1)/2}^{(N-1)/2} g_n \text{ sinc} \omega_n )</td>
</tr>
<tr>
<td></td>
<td>V</td>
<td>( \sum_{n=-N/2}^{N/2} g_n \exp\left(-i \frac{\pi}{N} \omega_n \right) \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} )</td>
<td>( \sum_{n=-(N-1)/2}^{(N-1)/2} g_n \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} )</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>( \sum_{n=-N/2}^{N/2} g_n \cos\left(\frac{\pi}{N} \omega_n \right) \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} )</td>
<td>( \sum_{n=-(N-1)/2}^{(N-1)/2} g_n \cos\left(\frac{\pi}{N} \omega_n \right) \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} )</td>
</tr>
<tr>
<td>odd</td>
<td>T</td>
<td>( \sum_{n=(N-1)/2}^{N/2} g_n \text{ sinc} \omega_n )</td>
<td>( \sum_{n=N/2+1}^{N/2} g_n \text{ sinc} \omega_n )</td>
</tr>
</tbody>
</table>
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Table 2 (continued)

| V | \( \sum_{n=-N/2}^{N/2} g_n \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} \) | \( \sum_{n=-N/2}^{N/2} g_n \exp\left(-i \frac{\pi n \omega_s}{N} \right) \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} \) |
| C | \( \sum_{n=-N/2}^{N/2} g_n \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} \) | \( \sum_{n=-N/2}^{N/2} g_n \frac{\sin(\pi \omega_n)}{N \sin(\pi/N \omega_n)} \) |

One detail that is different in the case of odd \( N \) is that for the calculus of the “C” function the equivalent of the inner sum from (9c) tends to a different function, which is shown by a modified version of the residue calculation from the Appendix. Namely the series are

\[
\sum_{m=0}^{\infty} \frac{1}{\sin \left( \frac{\pi}{N} \left( \frac{t}{\delta t} - n \right) \right)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m + 1}. \tag{21}
\]

Using (21) we obtain the functions from the odd C row of Table 2.

8. CONCLUSION

A very old topic of physics and engineering, the sampling theorem, is challenged from a new point of view, and new interpolating functions are generated that may replace the one used out of habit. These new functions are at least as good interpolations as the old function and they either come from the practice of discrete calculations (the “V” function) or from calculating the limit to infinity of the interpolating function (the “C” function).

APPENDIX

First we rewrite the sum from Eq. (10) as follows

\[
\sum_{m=-\infty}^{\infty} \frac{1}{x + m} = \frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{1}{x^2 - m^2}, \tag{A1}
\]

where we noted \((t/\delta t - n)/N = x\). This is a problem of residues calculus and this reformulation helps us by allowing the choosing of a meromorphic function which is easier to prove that converges to zero when the argument tends to infinity. This meromorphic function is
The function (A2) is proportional to \(1/z^2\) when \(z\) tends to infinity while the perimeter of a circle of radius \(|z|\) is \(2\pi|z|\). Therefore the integration of the function over a circle of radius \(z \to \infty\) (the big circle in Figs. A1 and A2) tends to zero like \(1/z\). The integration over the small circles from Fig. A1, or, more clearly in Fig. A2, that surround the singularities of function (A2) gives the residue values. For the singularities that are integer numbers \(m\) the integration variable has the form

\[
f(z) = \frac{\cot(\pi z)}{z^2 - x^2}. \tag{A2}
\]
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Fig. A2 – Continuation from Figure A1. The contour of the curvilinear integral after the cuts eliminated each other by bringing them close. Only the circles with clockwise sense remain surrounding the singularities of the meromorphic function.

\[ z = m + re^{i\theta}, \quad (A3) \]

where \( r \) is the radius of the shrinking circle \( C_m \) and \( \theta \) goes from 0 to \( 2\pi \). Then the residue is

\[
\text{Res}\left[f(z)\right]_{z=m} = -\lim_{r \to 0} \frac{1}{2\pi i} \oint_{C_m} \frac{\cot\left(\pi z\right)}{z^2 - x^2} \, dz =
\]

\[ = -\lim_{r \to 0} \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{\cot\left(\pi \left(m + re^{i\theta}\right)\right)}{\left[m + re^{i\theta}\right]^2 - x^2} ire^{i\theta} \, d\theta = \quad (A4.a) \]

\[
= \lim_{r \to 0} \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{(-1)^{m} re^{i\theta}}{x^2 - m^2} \frac{-1}{\pi} \sin\left(\pi re^{i\theta}\right) \, d\theta = \frac{1}{\pi} \frac{1}{x^2 - m^2}. \quad (A4.b)
\]
The minus sign in Eq. (A4) is due to the fact that the circulation sense in the circles $C_m$ is clockwise. The same thing is valid for $x$ and $-x$. For the singularity $x$ the integration variable is

$$z = x + re^{i\theta}.$$  

The residue is then

$$\text{Res}[f(z)]_{z=x} = -\lim_{r \to 0} \frac{1}{2\pi i} \int_0^{2\pi} \cot\left(\frac{x + re^{i\theta}}{x^2 - m^2}\right) ire^{i\theta} d\theta = -\frac{\cot(\pi x)}{2x}. \quad (A6.a)$$

Similar calculations for the singularity $-x$ leads to

$$\text{Res}[f(z)]_{z=-x} = -\frac{\cot(\pi x)}{2x}. \quad (A6.b)$$

We know the sum of the residues (A4) and (A6) is zero. Combining them we obtain

$$\pi \cot(\pi x) = \frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{1}{x^2 - m^2}, \quad (A7)$$

which, if we compare it to (A1), is exactly what we wanted to obtain.

There is another case treated in the article that we do not intend to treat fully in this appendix, but only to indicate the way it can be done, more so since it is very similar to the case just presented above. It is the sum

$$\frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{(-1)^m}{x^2 - m^2} = \frac{\pi}{\sin(\pi x)}. \quad (A8)$$

The function $\pi/\sin(\pi x)/(x^2-x^2)$ is meromorphic with all the singular points of $\pi \cot(\pi x)/(x^2-x^2)$. Relation (A8) can be proved also, similarly to (A7), and help to obtain the form of the “C” function for odd $N$, namely without the cosine coefficient.

REFERENCES


