PERTURBATION OF DISPERSIVE SHALLOW WATER WAVES WITH ROSENAU-KdV-RLW EQUATION AND POWER LAW NONLINEARITY

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This paper studied dynamical features of dispersive shallow water waves that are modeled by Rosenau-KdV-RLW equation. This model is generalized to power law nonlinearity. Soliton perturbation theory is applied to obtain the adiabatic dynamics of soliton parameters. Ansatz method also obtains exact 1-soliton solution to perturbed Rosenau-KdV-RLW equation. Finally, semi-inverse variational principle gives an analytical 1-soliton solution to this model.

Key words: solitons; singular solitons; dispersion; integrability.

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1. INTRODUCTION

Dynamics of shallow water waves that is observed along lake shores and beaches has been a research area for the past few decades in the area of oceanography [1-25]. There are several models proposed in this context. They are Korteweg-de Vries (KdV) equation, Boussinesq equation, Peregrine equation, regularized long wave (RLW) equation, Kawahara equation, Benjamin-Bona-Mahoney equation, Bona-Chen equation and several others. These models are derived from first principles under various different hypothesis and approximations. They are all well studied and very well understood.

The dynamics of dispersive shallow water waves, on the other hand, is captured with slightly different models. A few of these models are Rosenau-Kawahara equation, Rosenau-KdV equation, and Rosenau-RLW equation [3, 9, 15, 18-21]. There are quite interesting results, from these models, that have been reported in several journals in the past few years. This paper will study the conjunction of Rosenau-KdV and Rosenau-RLW equation in order to consider the
equal-width effect. Thus, the focus of this research paper will be on Rosenau-KdV-RLW (R-KdV-RLW) equation with perturbation terms so that the effect of several perturbation terms is visible. The results of R-KdV-RLW equation without the effects of perturbation have been recently reported [21].

Soliton perturbation theory will be applied to obtain the adiabatic parameter dynamics of these solitary waves. The 1-soliton solution of the perturbed R-KdV-RLW equation will be derived. In this context, ansatz approach will come into play. Later, semi-inverse variational principle (SVP) will be employed to obtain an analytical solitary wave solution to the model. There are several constraint conditions that will be displayed in order for the solutions to exist.

2. MATHEMATICAL MODEL

The dimensionless form of the R-KdV-RLW equation without perturbation is

\[ q_t + a q_x + b_1q_{xxx} + b_2q_{xxt} + c q_{xxxx} + k (q^n)_x = 0 \] (1)

where \( q(x,t) \) represents the wave profile, with \( x \) and \( t \) being spatial and temporal variables that are independent of each other. The coefficient of \( a \) is responsible for drifting of the waves, while the coefficients of \( b_j \) for \( j = 1, 2 \) are the third order dispersions. Here \( b_1 \) is the coefficient of third order spatial dispersion, while \( b_2 \) accounts for spatio-temporal dispersion in order to consider equal-width effect. Inclusion of this spatio-temporal dispersion surely makes this model well-posed [16]. The coefficient of \( c \) is the fifth order dispersion. Finally, \( k \) represents the coefficient of nonlinearity while \( n \) is the power law nonlinearity parameter and \( n > 1 \). This section is a re-visitation of results that are already reported earlier during 2014 [21]. The results that are listed here are simply for quick and easy reference.

Revisiting the solitary wave solution, the 1-soliton solution for (1) is [21]

\[ q(x,t) = A \text{sech}^{n-1}[B(x - vt)] \] (2)

where the amplitude \( A \) of the solitary wave is

\[ A = \left[ \frac{\{D_1 - (n^2 + 2n + 5)ac\}^2}{8(n+1)^2b_1b_2 + (n^2 + 2n + 5)\{D_1 - (n^2 + 2n + 5)ac\}} \right]^{\frac{1}{n-1}} \times \left[ \frac{(n+3)(3n+1)}{16(n+1)ck} \right]^{\frac{1}{n-1}} \] (3)

while the inverse width is given by
\[ B = \frac{n-1}{n+1} \left[ D_1 - \frac{(n^2 + 2n + 5)ac}{32hc} \right]^{\frac{1}{2}} \]  

(4)

where

\[ D_1 = \sqrt{a^2c^2(n^2 + 2n + 5)^2 + 16(n + 1)^2 b_1c(b_1 - ab_2)} \]  

(5)

The amplitude-width relation is

\[ A = \left[ \frac{8(n + 1)(3n + 1)b_1cB^4}{k(n - 1)^2 \{(n - 1)^2 b_2 + 4(n^2 + 2n + 5)cB^2\}} \right]^{\frac{1}{n-1}} \]  

(6)

and speed of soliton is either

\[ v = \frac{a(n - 1)^4 + 16(n - 1)^4 b_1 B^2}{(n - 1)^4 + 16(n - 1)^2 b_2 B^2 + 256cB^2} \]  

(7)

or

\[ v = \frac{b_1(n - 1)^2}{(n - 1)^2 b_2 + 4cB^2(n^2 + 2n + 5)} \]  

(8)

2.1. CONSERVATION LAW

The two conserved quantities for (1) are momentum \((M)\) and energy \((E)\) and are given by [21]

\[ M = \int_{-\infty}^{\infty} qdx = \frac{A}{B} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{-2}{n-1}\right)}{\Gamma\left(\frac{1}{2} + \frac{2}{n-1}\right)} \]  

(9)

and

\[ E = \int_{-\infty}^{\infty} \{q^2 - b_2(q_{xx})^2 + c(q_{xx})^2\} dx = \frac{A^2 \left( (n - 1)^2(n + 7)(3n + 5) - 16b_2(n - 1)(3n + 5)B^2 + 256(n + 2)cB^4 \right)}{B(n - 1)^2(n + 7)(3n + 5)} \times \]  

(10)

\[ \times \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{4}{n-1}\right)}{\Gamma\left(\frac{1}{2} + \frac{4}{n-1}\right)} \]
These results are all taken from an earlier work where unperturbed R-KdV-RLW was studied [21].

3. SOLITON PERTURBATION THEORY

In presence of perturbation, the perturbed R-KdV-RLW equation is given by [1, 11, 20]

\[ q_t + aq_x + b_1q_{xxx} + b_2q_{xxt} + c_4q_{xxxx} + k(q^n)_x = \varepsilon R \]  

(11)

where \( R \) represents perturbation terms while \( \varepsilon \) is the perturbation parameter. Here,

\[ R = \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q'' q_x + \lambda q q_{xxx} + \nu q q_{xxxx} + \]  

\[ + \sigma q_x q_{xxx} + \xi q q_{xxxx} + \eta q_x q_{xxxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q_{xxxxxxx} \]  

(12)

From the perturbation terms, the shoaling effect is captured by \( \alpha \), while \( \beta \) introduces dissipation. Higher order nonlinear dispersion is indicated in the coefficient of \( \delta \). Fifth order spatial dispersion is the coefficient of \( \psi \) while higher order stabilization is introduced by \( \rho \). The rest of the terms are accounted for Whitham hierarchy [1, 11].

In presence of these perturbation terms, the adiabatic change of momentum and energy are given by [20]

\[ \frac{dM}{dt} = \varepsilon \int_{-\infty}^{+\infty} Rdx \]  

(13)

and

\[ \frac{dE}{dt} = 2\varepsilon \int_{-\infty}^{+\infty} qRdx \]  

(14)

Substituting \( R \) from (12) into (13) and (14) and using the 1-soliton solution (2) in order to carry out the integration gives

\[ \frac{dM}{dt} = \frac{x\alpha A}{B} \left[ \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{2}{n-1}\right)} \right] = \varepsilon \alpha M \]  

(15)

and
\[
\frac{dE}{dt} = 2\varepsilon A^2 \left\{ \alpha \frac{16B^3}{B} \frac{16\beta B}{(n-1)(n+7)} + \frac{256\rho B^3(n+2)}{(n-1)^2(n+7)(3n+5)} \right\} \times \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{4}{n-1} \right)}{\Gamma \left( \frac{1}{2} + \frac{4}{n-1} \right)} \tag{16}
\]

Eq. (15) shows that momentum dissipates adiabatically with time. This law of dissipation is

\[
M(t) = M_0 e^{\alpha t} \tag{17}
\]

where \(M_0\) is the initial momentum of the soliton. For large time,

\[
\lim_{t \to \infty} M(t) = 0 \tag{18}
\]

provided \(\alpha < 0\).

From the law of adiabatic variation of energy given by Eq. (16), the fixed point is [20]

\[
\frac{1}{\beta} = \frac{\beta(n-1)(3n+5) \pm (n-1)\sqrt{(3n+5)D}}{32\rho(n+2)} \tag{19}
\]

where the discriminant is:

\[
D = \beta^2(3n+5) - 4\alpha\rho(n+2)(n+7) \tag{20}
\]

This fixed point given by Eq. (19) is valid, whenever

\[
\beta^2(3n+5) > 4\alpha\rho(n+2)(n+7) \tag{21}
\]

However, if

\[
\beta^2(3n+5) < 4\alpha\rho(n+2)(n+7) \tag{22}
\]

then

\[
D = 4\alpha\rho(n+2)(n+7) - \beta^2(3n+5) \tag{23}
\]

Finally, if

\[
4\alpha\rho(n+2)(n+7) = \beta^2(3n+5) \tag{24}
\]

\(D = 0\), in which case, the fixed value of the soliton width is
\[
\bar{B} = \left[ \frac{\beta(n-1)(3n+5)}{32\rho(n+2)} \right]^{\frac{1}{2}}
\]  
(25)

that carries with it the constraint condition
\[
\beta \rho > 0
\]  
(26)

The fixed point value of the amplitude, as obtained from (6) is
\[
\bar{A} = \left[ \frac{\beta(3n+5) \pm \sqrt{(3n+5)D}}{8b_2\rho(n-1)(n+2) + c(n+2n+5)\beta(3n+5) \pm \sqrt{(3n+5)D}} \right]^{\frac{1}{n-1}} \times \\
\left[ \frac{(n+1)(n+3)(3n+1)cB}{16\kappa\rho(n-1)(n+2)} \right]^{\frac{1}{n-1}}
\]  
(27)

The slow change in the velocity of the soliton is
\[
v = \frac{a(n-1)^4 + 16(n-1)^4b_1B^2}{(n-1)^4 + 16(n-1)^2b_2B^2 + 256cB^4} + \frac{\varepsilon}{M} \int_{-\infty}^{\infty} xRdx
\]  
(28)

that leads to
\[
v = \frac{a(n-1)^4 + 16(n-1)^4b_1B^2}{(n-1)^4 + 16(n-1)^2b_2B^2 + 256cB^4} - \\
-\varepsilon B \left[ \delta A^m \frac{\Gamma\left(\frac{2m+2}{n-1}\right)}{m+1} + \frac{16A^2B^2\nu(n-1)^2 + \sigma(n+17)}{3(n-1)(n+5)(n+11)} \frac{\Gamma\left(\frac{6}{n-1}\right)}{\Gamma\left(\frac{1}{2} + \frac{6}{n-1}\right)} \right] + \\
+ \frac{8AB^2((n-1)(3n+5)(\gamma - 3\lambda) - 16(n+2)(3\xi - \eta - 5\kappa)B^2)}{(n-1)^2(n+7)(3n+5)} \frac{\Gamma\left(\frac{4}{n-1}\right)}{\Gamma\left(\frac{1}{2} + \frac{4}{n-1}\right)} \times \\
\frac{\Gamma\left(\frac{1}{2} + \frac{2}{n-1}\right)}{\Gamma\left(\frac{2}{n-1}\right)}
\]  
(29)

or
\[ v = \frac{b(n-1)^2}{b_2(n-1)^2 + 4(n^2 + 2n + 5)cB^2} + \frac{\varepsilon}{M} \int_{-\infty}^{+\infty} x R dx \] 

(30)

which simplifies to

\[ v = \frac{(n-1)^2 b_1}{(n-1)^2 b_2 + 4cB^4(n^2 + 2n + 5)} + \mathcal{O}\left(\frac{\delta A^m}{m+1}, \frac{\Gamma\left(\frac{1}{2} + \frac{2m+2}{n-1}\right)}{\Gamma\left(\frac{6}{n-1}\right)} \right) \]

\[ + \frac{8AB^2}{(n-1)(3n+5)(3\gamma - 3\lambda) - 16(n+2)(3\xi - \eta - 5\kappa)} \frac{\Gamma\left(\frac{4}{n-1}\right)}{\Gamma\left(\frac{4}{n-1}\right)} \times \frac{\Gamma\left(\frac{1}{2} + \frac{2}{n-1}\right)}{\Gamma\left(\frac{2}{n-1}\right)} \]

(31)

by virtue of (2) and (12). This shows that in presence of perturbation terms, dispersive solitary wave will move along the shore with a fixed amplitude and width given by (27) and (19) [or (25)] respectively. Also, the speed of the perturbed soliton will be either (29) or (31).

4. ANSATZ APPROACH

The perturbed R-KdV-RLW equation will now be integrated to obtain the exact 1-soliton solution. Only a few perturbation terms, that are compatible to integrability, will be considered. This study will be split into three sections where solitary waves, shock waves and singular solitary waves will be studied. The ansatz approach will be our tool of integration for this section. It is only the Hamiltonian perturbation terms that will be considered as strong perturbation. The remaining non-Hamiltonian or dissipative perturbation terms are all discarded since such terms do not permit integrability [20].
4.1. SOLITARY WAVE SOLUTION

The perturbed R-KdV-RLW equation that is going to be considered in this paper is [1, 11]

\[
q_t + a q_x + b q_{xxx} + c q_{xxxx} + k(q^n)_x = \\
= \gamma q_x q_{xx} + \lambda q q_{xxx} + \nu q q_x q_{xx} + \sigma q^2 + \\
+ \xi q q_{xxx} + \eta q_x q_{xx} + \psi q_{xxxx} + \kappa q q_{xxxx} 
\]  
(32)

The starting hypothesis is given by [9, 20, 21]

\[
q(x,t) = A\text{sech}^p[B(x-\nu t)]
\]  
(33)

for an unknown exponent \( p \) whose value will be available by the aid of balancing principle. It must be noted that

\[
p > 0
\]  
(34)

Next, substituting hypothesis (33) into (32) and simplifying leads to

\[
AB^p(v-a-b_1p^2B^2 + b_2p^4B^4 + cvp^4B^4 + \psi p^4B^4)\text{sech}^p\tau + \\
+AB^3p(p+1)(p+2)(b_1-b_2v-2(p^2+2p+2)(cv+\psi)B^2)\text{sech}^{p+2}\tau + \\
+AB^5p(p+1)(p+2)(p+3)(p+4)(cv+\psi)\text{sech}^{p+4}\tau - \\
-k\nu A^n B\text{sech}^{np}\tau + A^2B^3p^3(\gamma + \lambda + \xi p^2B^2 + \eta p^2B^2 + \kappa p^2B^2)\text{sech}^{2p}\tau - \\
-A^2B^3p(p+1)(\gamma p + (p+2)\lambda + 2p^2(p+1)\eta B^2 + \\
+2(p^2+2p+2)(\xi p + (p+2)\kappa)B^2)\text{sech}^{2p+2}\tau + \\
+A^3B^3p(p+1)(p+2)(p+3)\xi p + (p+1)\eta + (p+3)(p+4)\kappa \text{sech}^{2p+4}\tau + \\
+A^3B^3p^3(\sigma + \nu)\text{sech}^{3p}\tau - A^3B^3p^3(p\sigma + (p+1)\nu)\text{sech}^{3p+2}\tau = 0
\]  
(35)

where

\[
\tau = B(x-\nu t)
\]  
(36)

By balancing principle, equating exponents \( np \) and \( p + 4 \), gives

\[
p = \frac{4}{n-1}
\]  
(37)

Setting the coefficients of the linearly independent functions \( \text{sech}^{p+\frac{j}{2}}\tau \) for \( j = 0; 2; 4; \ \text{sech}^{2p+\frac{j}{2}}\tau \) for \( j = 0; 2; 4 \) and \( \text{sech}^{3p+\frac{j}{2}}\tau \) for \( j = 0; 2 \) to zero leads to

\[
v = \frac{a(n-1)^4+16(n-1)^2b_1B^2-256\psi B^4}{(n-1)^4+16(n-1)^2b_2B^2+256cB^4}
\]  
(38)
or

\[
v = \frac{b_1(n-1)^2 - 4\psi B^2(n^2 + 2n + 5)}{(n-1)^2 b_2 + 4cB^2(n^2 + 2n + 5)} \tag{39}
\]

\[
\begin{bmatrix}
\zeta \\
\eta \\
\kappa \\
\gamma \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix} \tag{40}
\]

and

\[
\begin{bmatrix}
\xi \\
\eta \\
\kappa \\
\gamma \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
-\frac{(n-1)^2 (n^2 - 4n - 9)}{8B^2 (3n^3 + 17n^2 + 29n - 1)} \\
\frac{(n-1)(3n+1)(n^3 + 4n^2 + 9n - 6)}{16B^2 (3n^3 + 17n^2 + 29n - 1)} \\
\frac{(n-1)(n^3 + 9n^2 + 13n + 11)}{16B^2 (3n^3 + 17n^2 + 29n - 1)} \\
\frac{1}{0} \\
1
\end{bmatrix} \tag{41}
\]

Equating these two expressions for velocity \(v\) from (38) and (39) and then solving the bi-quadratic equation for the width \(B\) leads to

\[
B = \frac{n-1}{n+1} \left[ -\frac{(ac + \psi)(n^2 + 2n + 5) + D_2}{32(b_1c + b_2\psi)} \right]^{\frac{1}{2}} \tag{42}
\]

where

\[
D_2 = \sqrt{(ac + \psi)^2(n^2 + 2n + 5)^2 + 16(n + 1)^2(b_1c + b_2\psi)(b_1 - ab_2)} \tag{43}
\]

The relation between the soliton amplitude \((A)\) and the inverse width \((B)\) is given by

\[
A = \left[ \frac{8(n+1)(n+3)(3n+1)(b_1c + b_2\psi)B^4}{k(n-1)^2 (n-1)^2 b_2 + 4(n^2 + 2n + 5)cB^2} \right]^{\frac{1}{n-1}} \tag{44}
\]

Substituting (42) into (44) leads to

\[
A = \left[ \frac{\{- (ac + \psi)(n^2 + 2n + 5) + D_2\}^2}{8b_2 (b_1c + b_2\psi)(n+1)^2 + c\{- (ac + \psi)(n^2 + 2n + 5) + D_2\} (n^2 + 2n + 5)} \right]^{\frac{1}{n-1}} \times \left[ \frac{(n+3)(3n+1)}{16k(n+1)} \right]^{\frac{1}{n-1}} \tag{45}
\]
This shows that perturbed R-KdV-RLW equation given by (32) can be integrated provided $\sigma$ and $\nu$ are both zero. Additionally, other perturbation terms due to $\xi$, $\eta$, and $\kappa$ are linearly dependent on $\gamma$ and $\lambda$ as given by (41). It is worthy to note that soliton amplitude and width given by (42) and (45) collapse to their respective expressions given by (6) and (4) upon setting $\psi = 0$, as expected.

4.2. SHOCK WAVE SOLUTION

For shock wave solution to (32), the starting hypothesis will be [6, 21]

$$q(x,t) = A \tanh^p \tau$$  \hspace{1cm} (46)

where the definition of $\tau$ will stay the same as given by Eq. (36) and again $\rho > 0$. In this case, $A$ and $B$ are free parameters. Typically, these shock wave solutions are also referred to as topological solitons or kink solutions. In this context, the free parameter $A$ refers to as the distance between the two stable states, while $B$ is related to the angle of inclination of the curve that connects the two states to the $x$-axis. Substituting (46) into (32) and simplifying leads to

$$[a - v + (b_2\nu - b_1)B^2(3p^2 - 3p + 2) - 2(cv + \psi)B^4(5p^4 - 10p^3 + 25p^2 - 20p + 8)]\tanh^{p-1}\tau -$$

$$= \nu - [a - v + (b_2\nu - b_1)B^2(3p^2 - 3p + 2) - 2(cv + \psi)B^4(5p^4 + 10p^3 + 25p^2 + 20p + 8)]\tanh^{p+1}\tau +$$

$$+ B^2(p - 1)(p - 2)[b_1 - b_2\nu + 5(cv + \psi)B^2] \tanh^{p-3}\tau -$$

$$- B^2(p + 1)(p + 2)[b_1 - b_2\nu + 5(cv + \psi)B^2(3p^2 + 3p + 4)] \tanh^{p+1}\tau -$$

$$- (cv + \psi)B^4(p - 1)(p - 2)(p - 3)(p - 4) \tanh^{p+5}\tau -$$

$$- knA^{-1}\tanh^{p-1}\tau - knA^{n=1}\tanh^{p+1}\tau -$$

$$- AB^2(p - 1)[\nu + (p - 2)\lambda] - \xi p(5p^2 - 13p + 14)B^2 -$$

$$- \eta p(p - 1)(5p - 2)B^2 - 5k(p - 2)(p^2 - 3p + 4)B^2 \tanh^{2p-3}\tau +$$

$$+ AB^2(p + 1)[\nu + (p + 2)\lambda] - \xi p(5p^2 + 13p + 14)B^2 -$$

$$- \eta p(p + 1)(5p + 2)B^2 - 5k(p + 2)(p^2 + 3p + 4)B^2 \tanh^{2p+1}\tau +$$

$$+ AB^2[p(p3 - 1) + \lambda(3p^2 - 3p + 2) - 2\xi p(5p^3 - 6p^2 + 13p - 4)B^2 -$$

$$- 2\xi p(5p^2 - 4p + 1)B^2 - 2\kappa(5p^4 - 10p^3 + 25p^2 + 20p + 8)B^2]\tanh^{2p+1}\tau -$$

$$- AB^4(p - 1)(p - 2)[p(p - 3)\xi + p(p - 1)\eta + (p - 3)(p - 4)\kappa] \tanh^{2p-5}\tau -$$

$$- AB^4(p + 1)(p + 2)[p(p + 3)\xi + p(p + 1)\eta + (p + 3)(p + 4)\kappa] \tanh^{2p+5}\tau -$$

$$- A^2B^2p^2[\sigma p + \nu(p - 1)] \tanh^{p-3}\tau + A^2B^2p^2[\sigma p + \nu(p + 1)] \tanh^{p+3}\tau +$$

$$+ A^2B^2p[3\sigma p + \nu(p - 1)] \tanh^{p-1}\tau - A^2B^2p[3\sigma p + \nu(p + 1)] \tanh^{p+1}\tau = 0$$
Again by balancing principle, the same value of $p$ as in Eq. (37) is recovered. This leads to the shock wave solution to (32) as

$$q(x,t) = A \tanh^{\frac{4}{p-3}}[B(x - vt)]$$  \hspace{1cm} (48)

From (47), setting the coefficients of two standalone linearly independent functions $\tanh^{p-3} \tau$ and $\tanh^{p-5} \tau$ to zero leads to two different values of the parameter $p$. These two cases will be studied individually in the following two subsections.

4.2.1. CASE-I: $p = 1; n = 5$

When $p = 1$, Eq. (37) implies $n = 5$. This means that the perturbed R-KdV-RLW equation given by Eq. (32) is re-casted to

$$q_x + aq_t + b_1 q_{xxx} + b_2 q_{xxt} + c q_{xxxx} + k(q^5)_x =$$

$$= 3q_x q_{xx} + \lambda q_x q_{xxt} + \nu q_x q_{xxx} + \sigma q_x^3 +$$

$$+ \xi q_x q_{xxxx} + \eta q_x q_{xxx} + \Psi q_{xxxx} + \kappa q_{xxxx}$$  \hspace{1cm} (49)

whose solution therefore is

$$q(x,t) = A \tanh[B(x - vt)]$$  \hspace{1cm} (50)

From the remaining linearly independent functions, the velocity of the shock wave is therefore given by

$$v = \frac{a - 2b_1 B^2 - 16\psi B^4}{16cB^4 - 2b_2 B^2 + 1}$$  \hspace{1cm} (51)

or

$$v = \frac{a - 8b_1 B^2 - 136\psi B^4}{136cB^4 - 8b_2 B^2 + 1}$$  \hspace{1cm} (52)

Equating the two expressions of the velocity leads to biquadratic equation for the parameter $B$ as

$$24(h_1 + h_2 \psi)B^4 - 20(ac + \psi)B^2 + (ab_2 - b_1) = 0$$  \hspace{1cm} (53)

Solving for parameter $B$ gives:

$$B = \left[ \frac{5(ac + \psi) \pm \sqrt{25(ac + \psi)^2 - 6(h_1 + h_2 \psi)(ab_2 - b_1)}}{12(h_1 + h_2 \psi)} \right]^{\frac{1}{2}}$$  \hspace{1cm} (54)
From other linearly independent functions in (47), relation between free parameters is:

\[
A = \left( \frac{24(cv + \psi)B^4}{k} \right)^{\frac{1}{2}}
\]

or

\[
A = \left( \frac{6B^2(b_1 - b_2v + 40(cv + \psi)B^2)}{5k} \right)^{\frac{1}{2}}
\]

Additional linearly independent functions also give the following system of four equations:

\[
\begin{align*}
\sigma &= 0 \\
\sigma + 2v &= 0 \\
3\sigma + 2v &= 0 \\
3\sigma + 4v &= 0
\end{align*}
\]

and this system has a unique solution:

\[
\begin{bmatrix}
\sigma \\
v
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

The last system of equations, from remaining linearly independent functions are

\[
\begin{align*}
\gamma + 3\lambda - 32\xi B^2 - 14\eta B^2 - 120\kappa B^2 &= 0 \\
\gamma + \lambda - 8\xi B^2 - 2\eta B^2 - 8\kappa B^2 &= 0 \\
\gamma + 2\lambda - 14\xi B^2 - 5\eta B^2 - 34\kappa B^2 &= 0 \\
2\xi + \eta + 10\kappa &= 0
\end{align*}
\]

that leads to the solution

\[
\begin{bmatrix}
\xi \\
\eta \\
\kappa \\
\gamma \\
\lambda
\end{bmatrix} = 
\begin{bmatrix}
\frac{1}{4B^2} \\
\frac{1}{2B^2} \\
\frac{1}{2B^2} \\
0 \\
\frac{1}{4B^2}
\end{bmatrix} + 
\begin{bmatrix}
\frac{1}{2B^2} \\
\frac{2B^2}{7} \\
\frac{1}{2B^2} \\
0 \\
\frac{1}{2B^2}
\end{bmatrix} \lambda.
\]
4.2.2. CASE-II: \( p = 2; \ n = 3 \)

For \( p = 2 \), Eq. (37) leads to \( n = 3 \). This means that perturbed R-KdV-RLW equation is given by

\[
q_t + aq_x + bq_{xxx} + cq_{xxx} + dq_{xxxx} + k(q^3)_x = \\
= \gamma q_xq_{xx} + \lambda q_{xxx} + \nu q_x(q_x + \sigma q_x^3) + \\
+ \xi q_xq_{xxx} + \eta q_xq_{xxx} + \psi q_{xxxx} + \kappa q_{xxxx}
\]

whose solution therefore is

\[
q(x,t) = A \tanh^2[B(x - vt)]
\]

The speed of the soliton is

\[
v = \frac{a - 8b_1B^2 - 126\psi B^4}{136cB^4 - 8b_2B^2 + 1}
\]

or

\[
v = \frac{a - 20b_1B^2 - 616\psi B^4}{616cB^4 - 20b_2B^2 + 1}
\]

From these two expressions of the velocity the biquadratic equation for the parameter \( B \) is

\[
184(b_1c + b_2\psi)B^4 - 40(ac + \psi)B^2 + (ab_2 - b_1) = 0
\]

which solves to

\[
B = \left[ \frac{10(ac + \psi) \pm \sqrt{100(ac + \psi)^2 - 46(b_1c + b_2\psi)(ab_2 - b_1)}}{92(b_1c + b_2\psi)} \right]^{\frac{1}{2}}
\]

Similarly, proceeding as in the last sub-section, the relation between the parameters is

\[
A = \left[ \frac{120(cv + \psi)B^4}{k} \right]^{\frac{1}{7}}
\]

or

\[
A = \left[ \frac{4B^2(b_1 - b_2v + 70(cv + \psi)B^2)}{k} \right]^{\frac{1}{7}}
\]
Again, the linearly independent functions from (47) lead to the system of equations
\[\begin{align*}
2\sigma - v &= 0 \\
2\sigma + 3v &= 0 \\
6\sigma + 5v &= 0 \\
6\sigma + 7v &= 0 \\
\end{align*}\] (72)

that has a unique solution:
\[
\begin{bmatrix}
\sigma \\
v
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \quad \text{(73)}
\]

The final set of linear equations for \(\xi, \eta, \kappa, \gamma,\) and \(\lambda\) are
\[\begin{align*}
\gamma - 8\xi B^2 - 8\eta B^2 &= 0 \quad \text{(74)} \\
\gamma + 2\zeta - 60\xi B^2 - 36\eta B^2 - 140\kappa B^2 &= 0 \quad \text{(75)} \\
5\gamma + 4\lambda - 76\xi B^2 - 52\eta B^2 - 68\kappa B^2 &= 0 \quad \text{(76)} \\
7\gamma + 10\lambda - 188\xi B^2 - 116\eta B^2 - 308\kappa B^2 &= 0 \quad \text{(77)} \\
5\xi + 3\eta + 6\kappa &= 0 \quad \text{(78)}
\end{align*}\]

that gives the unique solution:
\[
\begin{bmatrix}
\xi \\
\eta \\
\kappa \\
\gamma \\
\lambda
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \quad \text{(79)}
\]

Hence for this case, Eq. (64), further reduces to
\[
q_t + aq_x + bq_{xxx} + c q_{xxxx} + k(q^3)_x = \psi q_{xxxxx} \quad \text{(80)}
\]

that integrates to the topological soliton solution given by (65) for parameters and constraints that are listed above.

### 4.3. SINGULAR SOLITON SOLUTION

In order to obtain singular soliton solution to (32), the starting hypothesis is [20, 21]
\[ q(x,t) = A \operatorname{csch}^p[B(x - vt)] \]  

where \( A \) and \( B \) are free parameters. Substituting Eq. (81) into Eq. (32) gives

\[
\begin{align*}
\psi & = A^2 \psi^4 - A^2 \psi^3 + A^2 \psi^2 - A^2 \psi + A^2 \\
\lambda & = A^2 \psi^4 - A^2 \psi^3 + A^2 \psi^2 - A^2 \psi + 2A^2 \psi + \psi \\
\xi & = A^2 \psi^4 - A^2 \psi^3 + A^2 \psi^2 - A^2 \psi + 2A^2 \psi + \psi \\
\eta & = A^2 \psi^4 - A^2 \psi^3 + A^2 \psi^2 - A^2 \psi + 2A^2 \psi + \psi \\
\kappa & = A^2 \psi^4 - A^2 \psi^3 + A^2 \psi^2 - A^2 \psi + 2A^2 \psi + \psi \\
\sigma & = A^2 \psi^4 - A^2 \psi^3 + A^2 \psi^2 - A^2 \psi + 2A^2 \psi + \psi \\
\nu & = A^2 \psi^4 - A^2 \psi^3 + A^2 \psi^2 - A^2 \psi + 2A^2 \psi + \psi
\end{align*}
\]

Following the same course as in the last couple of sections, singular soliton solution of (32) is

\[ q(x,t) = A \operatorname{csch}^{p-1}[B(x - vt)] \]  

where the free parameters, velocity and constraints are still governed by (36)-(43).

**5. SEMI-INVERSE VARIATIONAL PRINCIPLE**

This section applies another integration tool to integrate the perturbed R-KdV-RLW equation given by (32). This is called SVP. This technique is an inverse problem approach and can only retrieve solitary wave solution. From the perturbed R-KdV-RLW equation, only nonlinear dispersion and fifth order dispersion terms, given by the coefficients of \( \delta \) and \( \psi \) respectively, are retained [20]. Only these two perturbation terms permit integrability of (32) by SVP. Therefore, the perturbed R-KdV-RLW equation that will be studied here is

\[ q_{tt} + aq_x + b_1 q_{xx} + b_2 q_{xxx} + c q_{xxxx} + k(q^n)_x = \delta q^{m} q_x + \psi q_{xxxx} \]  

The starting hypothesis is a wave of permanent form that is given by

\[ q(x,t) = g(s) \]  

where \( g(s) \) represents the wave profile of the perturbed solitary wave and

\[ s = x - vt \]  

with \( v \) being the speed of the wave. Substituting (85) into (84) leads to
\[(v - a)g^2 + (b_2y - b_1)(g')^2 + (cv + \psi)\{2g'g'' - (g^*)^2\} - \]
\[\frac{-2k g^{n+1}}{n + 1} + 2\delta \frac{g^{m+2}}{(m+1)(m+2)} = K \]  \hspace{1cm} (87)

where \(K\) is an integration constant. Next, the stationary integral \(J\) is defined as

\[J = \int_{-\infty}^{\infty} K ds = \]
\[= \int_{-\infty}^{\infty} \left[ (v-a)g^2 - (b_1 - b_2y)(g')^2 + (cv + \psi)\{2g'g'' - (g^*)^2\} - \right. \]
\[\left. \frac{-2k g^{n+1}}{n + 1} + 2\delta \frac{g^{m+2}}{(m+1)(m+2)} \right] ds \]  \hspace{1cm} (88)

Using the 1-soliton solution hypothesis as

\[g(x - vt) = A \text{sech}^{\frac{4}{n-1}}[B(x - vt)] \]  \hspace{1cm} (89)

simplifies the stationary integral, given by (88), to

\[J = \frac{2\delta A^{2m+2}}{(m+1)(m+2)B} \Gamma \left( \frac{2m + 4}{n - 1} \right) \Gamma \left( \frac{1}{2} \right) \]
\[+ \left[ \frac{(v-a)A^2}{B} + \frac{16(b_2y - b_1)A^2B}{(n-1)(n+7)} \frac{768(n+2)(cv + \psi)A^2B^3}{(n-1)^2(n+7)(3n+5)} \right. \]
\[- \frac{32(n+3)kA^{n+1}}{(n+1)(n+7)(3n+5)B} \]  \hspace{1cm} (90)

SVP states that the soliton amplitude \(A\) and inverse width \(B\) are solutions of the coupled system [8, 12, 13, 18, 20, 21]

\[\frac{\partial J}{\partial A} = 0 \]  \hspace{1cm} (91)

\[\frac{\partial J}{\partial B} = 0 \]  \hspace{1cm} (92)
From (90), Eqs. (91) and (92) respectively simplify to

\[ v - a + \frac{16(b_2v - b_1)B^2}{(n-1)(n+7)} - \frac{768(n+2)(cv + \psi)B^4}{(n-1)^2(n+7)(3n+5)} + \]
\[ + \frac{\delta A^m}{(m+1) \Gamma\left(\frac{m+4}{n-1}\right) \Gamma\left(\frac{1}{2} + \frac{4}{n-1}\right)} \Gamma\left(\frac{4}{n-1}\right) - \frac{16(n+3)kA^{n-1}}{(n+7)(3n+5)} = 0 \]  

(93)

and

\[ v - a - \frac{16(b_2v - b_1)B^2}{(n-1)(n+7)} + \frac{2304(n+2)(cv + \psi)B^4}{(n-1)^2(n+7)(3n+5)} + \]
\[ + \frac{2\delta A^m}{(m+1)(m+2) \Gamma\left(\frac{2m+4}{n-1}\right) \Gamma\left(\frac{1}{2} + \frac{4}{n-1}\right)} \Gamma\left(\frac{4}{n-1}\right) - \frac{32(n+3)kA^{n-1}}{(n+1)(n+7)(3n+5)} = 0 \]

(94)

Subtracting (93) from (94) leads to the bi-quadratic equation for the inverse width of the soliton

\[ 256P_2B^4 - 32(b_2v - b_1)(n-1)(n+7)B^2 - P_1(n-1)^2(n+7)^2 = 0 \]  

(95)

where

\[ P_1 = \frac{m\delta A^m}{(m+1)(m+2) \Gamma\left(\frac{2m+4}{n-1}\right) \Gamma\left(\frac{1}{2} + \frac{4}{n-1}\right)} \Gamma\left(\frac{4}{n-1}\right) - \frac{16(n-1)(n+3)kA^{n-1}}{(n+1)(n+7)(3n+5)} \]  

(96)

and

\[ P_2 = \frac{12(n+2)(n+7)(cv + \psi)}{(3n+5)} \]  

(97)

The solution of (95) is

\[ B = \frac{1}{4} \left[ \frac{(n-1)(n+7)}{P_2} \left\{ (b_2v - b_1) \pm \sqrt{(b_2v - b_1)^2 + 4P_2} \right\} \right] \]  

(98)

whenever
\[(b_2 v - b_1)^2 + P_1 P_2 > 0 \]  
\[(99)\]

and

\[P_2 \left( b_2 v - b_1 \right) \pm \sqrt{\left( b_2 v - b_1 \right)^2 + P_1 P_2} > 0 \]  
\[(100)\]

Hence, 1-soliton solution of (84), from SVP, is given by (2) where the soliton width is expressed in Eq. (98) and the amplitude can be obtained after substituting (98) into (93) or (94). The speed of the soliton is located in (93) or (94). The constraint conditions (99) and (100) must hold for the soliton solution to exist.

6. CONCLUSIONS

This paper is an extended and generalized version of various previous results that are reported earlier [9]. In this area of research, the R-KdV equation along with its perturbation terms were reported earlier in 2011 and 2013 [13, 20]. Concurrently, analytical and numerical results on Rosenau-RLW equation were reported during 2009 and 2010. Later, interesting results from the combined equation R-KdV-RLW equation were published in 2014 without any perturbation terms [21]. This current paper extends these earlier results further along to address the perturbed R-KdV-RLW equation.

This paper studied the dynamics of perturbed soliton solutions to the R-KdV-RLW equation with power law nonlinearity. The soliton perturbation theory concluded that soliton momentum will dissipate in presence of shoaling. If in addition to shoaling, fourth order dispersion and dissipation terms are included, the solitons will move with a fixed amplitude and width for the appropriate sign of the discriminant. The exact soliton solutions were obtained for the perturbed R-KdV-RLW equation by the ansatz approach. Additionally, SVP reveals an analytical 1-soliton solution to the model, which is not necessarily an exact solution.

The results obtained in this paper shows that the future of this problem holds on a very strong footing. These results can be studied further along. There are several other integration tools that will be adopted to display additional results. Some of these mechanisms are $G'/G$-expansion approach, exp-function method, Lie symmetry method, simplest equation approach, tanh-coth method, first integral method and several others. Additionally, the stochastic perturbation terms will be considered. The corresponding Langevin equation will be formulated and the mean free velocity of the soliton will be computed. Later, this model will be further extended to vector-coupled R-KdV-RLW equation in order to study two-layered dispersive water waves that typically appears in oil spill along sea shores. These will serve as a generalized version to Bona-Chen or Gear-Grimshaw models.
REFERENCES

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