APPLICATION OF HYBRID FUNCTIONS FOR SOLVING
OSCILLATOR EQUATIONS

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This work is devoted to studying the oscillator fractional differential equations in the vibration research field. To this end, the Bagley-Torvik, Rayleigh, and Van der Pol equations with fractional damping are considered as certain types of oscillator fractional equations. Operational matrix of fractional integration, based upon block-pulse functions and the second kind Chebyshev polynomials, is applied to reduce the fractional equations into an algebraic system that can be solved by an appropriate method. It is shown that by increasing the number of iterations, the accuracy of the approximate solutions can be improved. The numerical results are included to demonstrate the validity and applicability of the operational matrix for solving oscillator fractional differential equations.

\textbf{Key words}: Fractional differential equations, Bagley-Torvik equation, Van der Pol equation, Rayleigh equation, Operational matrix, Hybrid functions.

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1. INTRODUCTION

The fractional calculus is a field of science that involves both integrals and derivatives of any arbitrary order. Dynamical systems can be efficiently characterized by fractional differential equations [1 – 7]. In recent years, study on application of the fractional order equations in science has attracted increasing attention. For instance, Caputo found that the obtained results with fractional order equation are in good agreement with experimental results when he applied fractional derivative to model viscoelasticity [8, 9]. Next, Bagley and Torvik [10, 11] formulated the motion of a rigid plate immersing in a Newtonian fluid. The so-called Bagley-Torvik equation is a typical system of a linear oscillator with one degree of freedom with fractional derivative of order $\frac{3}{2}$. In general, there exists no exact solution for Bagley-Torvik equation hence some approaches for numerical and analytical solutions of it are investigated by some authors [12, 13]. Also, the modeling of the concept to fractional derivatives can be used for the Van

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der Pol equation and the Rayleigh equation with non-linear damping. Due to Van der Pol equation is a staple model for oscillatory processes, much research has been focused on the numerical solution of this equation [14, 15].

In recent decades some numerical methods have been expanded to achieve an accurate solution of the obtained differential equations of physical modeling [16 − 22]. In this work, it has been suggested a method based on operational matrix for solving fractional differential equations, especially with an emphasis on solving dynamical systems that arise in physics. Note that investigating a method for solving a fractional differential equation may not be useful if that solution never appears in the actual model. The properties of operational matrix are applied to reduce the fractional differential equation to a system of algebraic equations. We notice that the coefficients matrix of algebraic equations is sparse, then the number of computations is reduced and this method provides a fast algorithm.

The outline of this paper is as follows. First, some concepts of fractional calculus are given in Section 2. In Section 3, function approximation by hybrid basis functions is given. Also, the hybrid operational matrix of the fractional integration is discussed. In Section 4, we explain how that operational matrices can be applied for solving dynamical systems such as Bagley-Torvik equation, fractional Van der Pol, and Rayleigh equation. Also, we report some numerical examples and demonstrate the accuracy of the proposed method. Finally, Section 5 contains a brief conclusion.

2. PRELIMINARY AND BASIC DEFINITIONS

This section, reviews some basic definitions and properties of fractional integral and derivative which are applied further in this work [23].

Definition 1 The Riemann-Liouville fractional integral operator of order \( \alpha \), is defined by

\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad \alpha > 0, \quad t > 0.
\]

where \( \Gamma(\cdot) \) denotes the Gamma function and for \( \alpha = 0 \), we set \( I^0 f(t) = f(t) \).

Definition 2 Let \( n = \lceil \alpha \rceil \) (\( \lceil . \rceil \) denotes ceiling function, \( \lceil x \rceil = \min \{ z \in \mathbb{Z} : z \geq x \} \)), the operator \( D^{\alpha} \), defined by

\[
D^\alpha f(t) = D^n I^{n-\alpha} f(t),
\]

is called the Riemann-Liouville fractional differential operator of order \( \alpha \). For \( \alpha = 0 \), we set \( D^0 = I \), the identity operator.

The one type of fractional derivative is Caputo fractional derivative, which is frequently used in applications.
Definition 3 The Caputo fractional derivative of \( f \in L^1[0,b] \), is defined as

\[
D^\alpha f(t) = \begin{cases} 
I^{n-\alpha} D^n f(t), & n-1 < \alpha < n, n \in \mathbb{N}, \\
\frac{d^n}{dt^n} f(t), & \alpha = n.
\end{cases}
\]

Lemma 1 Let \( \alpha, \beta \geq 0, c_1, c_2 \in \mathbb{R} \) and \( f(t), g(t) \in L^1[0,b] \). Then

1) \( I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) \),
2) \( I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t) \),
3) \( D^\alpha (c_1 f(t) + c_2 g(t)) = c_1 D^\alpha f(t) + c_2 D^\alpha g(t) \),

hold almost everywhere on \([0,b] \).

Note that for \( n-1 < \alpha < n, n \in \mathbb{N} \),

\[
I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0+) \frac{t^k}{k!}.
\]

Definition 4 The second kind Chebyshev polynomials are defined on interval \([-1,1]\) by

\[
U_m(t) = \frac{\sin(m+1)\theta}{\sin\theta}, \quad t = \cos\theta, \quad m = 0, 1, 2, \ldots
\]

These polynomial functions are orthogonal with respect to the weight function \( w(t) = \sqrt{1-t^2} \), on the interval \([-1,1]\) and satisfy the following recursive formulas

\[
U_0(t) = 1, \quad U_1(t) = 2t,
\]

\[
U_{m+1}(t) = 2U_m(t) - U_{m-1}(t), \quad m = 1, 2, \ldots
\]

Definition 5 The hybrid functions \( h_{nm}, n = 1, 2, \ldots, N, \ m = 0, 1, \ldots, M-1, \) on the interval \([0,t_f]\) are defined as,

\[
h_{nm}(t) = \begin{cases} 
U_m \left( \frac{2N}{t_f} t - 2n + 1 \right), & t \in \left[ \frac{n-1}{N} t_f, \frac{n}{N} t_f \right], \\
0, & \text{o.w.}
\end{cases}
\]

where \( n \) and \( m \) are the order of the block-pulse functions and the second kind Chebyshev polynomials, respectively.

Since \( h_{nm}(t) \) consists of the block-pulse functions and the Chebyshev polynomials, which are both complete and orthogonal, so a set of the hybrid functions based on them is a complete orthogonal set.

3. FUNCTION APPROXIMATION AND OPERATIONAL MATRIX

In this section, we apply the second kind Chebyshev polynomials to approximate an arbitrary function. Then, the operational matrix of fractional integration by hybrid basis functions will be discussed.
3.1. FUNCTION APPROXIMATION

A function \( f(t) \in L^2_w[0, t_f] \), may be expanded as \([24, 25]\)

\[
f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} h_{nm}(t) \approx \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} h_{nm}(t) = C^T H_\mu(t). \tag{2}
\]

Where \( \mu = NM \),

\[
C = [c_{10}, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, c_{N0}, \ldots, c_{N(M-1)}]^T,
\]

and

\[
H_\mu(t) = [h_{10}, \ldots, h_{1(M-1)}, h_{20}, \ldots, h_{2(M-1)}, \ldots, h_{N0}, \ldots, h_{N(M-1)}]^T.
\]

In the following theorem we present an upper bound to estimate the error that shows approximation convergence of the hybrid functions and indicates that the obtained results by these basis functions are reliable.

**Theorem 1** Assume that \( f(t) \in C^M[0, 1] \) is a real-valued function with bounded \( M^{th} \) derivative, i.e., \( f^{(M)}(t) \leq L \). If \( Y = \text{Span}\{h_{10}, \ldots, h_{1(M-1)}, h_{20}, \ldots, h_{2(M-1)}, \ldots, h_{N0}, \ldots, h_{N(M-1)}\} \), and \( C^T H_\mu(t) \) defined by relation (2), be the best approximation \( f(t) \) out of \( Y \), then by using hybrid basis functions, the mean error bound is presented as follow:

\[
\|f(t) - C^T H_\mu(t)\|_w \leq \frac{L}{NM M!}.
\]

We divide the interval \([0, 1]\) into subintervals \([\frac{n-1}{N}, \frac{n}{N}]\), \( n = 1, 2, \ldots, N \), with the restriction that \( f_{n-1} \) is the best approximation on the subinterval \([\frac{n-1}{N}, \frac{n}{N}]\), \( n = 1, 2, \ldots, N \), that approximates \( f \). Consider the Taylor polynomial \( y_{n-1}(t) = f(t_{n-1}) + f'(t_{n-1})(t - t_{n-1}) + \ldots + f^{(M-1)}(t_{n-1}) \frac{(t-t_{n-1})^{M-1}}{(M-1)!} \), we know that

\[
|f(t) - y_{n-1}(t)| \leq f^{(M)}(\zeta) \frac{(t-t_{n-1})^M}{M!}, \quad \zeta \in \left[\frac{n-1}{N}, \frac{n}{N}\right]. \tag{3}
\]
Since $C^T H_\mu(t)$ is the best approximation of $f(t)$ and $y_{n-1} \in Y$, by using Eq. (3) one has

$$\|f(t) - C^T H_\mu(t)\|_w^2 = \int_0^1 [f(t) - C^T H_\mu(t)]^2 w^2(t) dt$$

$$\approx \sum_{n=1}^N \int_{2^{n-1}}^{2^n} [f(t) - f_{n-1}(t)]^2 (1 - t^2) dt$$

$$\leq \sum_{n=1}^N \int_{2^{n-1}}^{2^n} [f(t) - y_{n-1}(t)]^2 dt$$

$$\leq \sum_{n=1}^N \int_{2^{n-1}}^{2^n} \left[ \left( \frac{1}{N} \right)^M \frac{1}{M!} \sup_{\zeta \in \left[ \frac{2^{n-1}}{N}, \frac{2^n}{N} \right]} f(M)(\zeta) \right]^2 dt$$

$$\leq \frac{L^2}{N^{2M} M!^2}.$$ 

By taking the square roots, we get the error estimate of approximate $f(t)$ with $C^T H_\mu(t)$. Obviously, by considering assumptions of this theorem, we infer that $C^T H_\mu(t) \to f(t)$ as $M, N$ are sufficiently large.

### 3.2. OPERATIONAL MATRIX OF THE FRACTIONAL INTEGRATION

Let $\mu = N \times M$; the integration of hybrid functions should be expandable into hybrid functions with the coefficients matrix $P^\alpha_{\mu \times \mu}$. The fractional integration of the vector $H_\mu(t)$ defined in Eq. (2), is given by

$$I^\alpha H_\mu(t) \approx P^\alpha_{\mu \times \mu} H_\mu(t),$$

where $P^\alpha_{\mu \times \mu}$ is the $\mu \times \mu$ operational matrix for fractional integration, and can be obtained as follows

$$P^\alpha_{\mu \times \mu} \approx \Phi_{\mu \times \mu} F^\alpha \Phi_{\mu \times \mu}^{-1}.$$ (4)

In above relation, square matrix $\Phi_{\mu \times \mu}$ by using collocation points $t_i = \frac{2i-1}{2\mu}$, $i = 1, 2, \ldots, \mu$, is defined with

$$\Phi_{\mu \times \mu} = \begin{bmatrix} H \left( \frac{1}{2\mu} \right) & H \left( \frac{3}{2\mu} \right) & \ldots & H \left( \frac{2\mu-1}{2\mu} \right) \end{bmatrix}. $$

Also

$$F^\alpha = \frac{1}{\mu^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \ldots & \varepsilon_{\mu-1} \\ 0 & 1 & \varepsilon_1 & \ldots & \varepsilon_{\mu-2} \\ 0 & 0 & 1 & \ldots & \varepsilon_{\mu-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$
with \( \varepsilon_k = (k + 1)^{\alpha+1} - 2k^{\alpha+1} + (k - 1)^{\alpha+1} \), for \( k = 1, 2, \ldots, \mu - 1 \), see [26].

By using the operational matrix of fractional integration, linear and nonlinear oscillator equations reduce to systems of algebraic equations, which can be solved to find unknown function. Implementation of this approach is given in the next section.

4. OSCILLATOR EQUATIONS OF FRACTIONAL ORDER AND THEIR NUMERICAL BEHAVIORS

In this section, implementation of the hybrid method on linear and nonlinear fractional oscillator equations is given and numerical behavior of these equations by operational matrices of integration is studied. It should be noted that fractional differential equations can be solved without limitation by the suggested method and hybrid functions have different resolution capability for expanding of different functions.

4.1. VIBRATION EQUATION WITH FRACTIONAL DAMPING

Consider

\[
AD^\alpha y(t) + BD^\beta y(t) + Cy(t) = f(t), \quad y(0) = y_0, \quad y'(0) = y_1,
\]

where \( \alpha \) and \( \beta \) are fractional values and \( 0 < \alpha < 2, 1 < \beta \leq 2, \alpha < \beta \). Let

\[
D^\alpha y(t) = K^T H(\mu(t)),
\]

then applying fractional integration operator to both sides of above equation, we obtain

\[
D^\alpha y(t) = \begin{cases} 
K^T P^\beta H(\mu(t)) + y_1, & 0 < \alpha \leq 1, \\
K^T P^\beta H(\mu(t)), & 1 < \alpha < 2,
\end{cases}
\]

and

\[
y(t) = K^T P^\beta H(\mu(t)) + y_1 t + y_0.
\]

By substituting the two last equations into Eq. (5), we achieve an algebraic equation which can be solved to find unknown vector \( K \) and consequently the solution of Eq. (5). In the following we consider Eq. (5) in two cases:

Case I: Bagley and Torvik formulated the motion of a large thin plate in a Newtonian fluid using fractional differential equation (5) with special order \( \beta = 2 \) and \( \alpha = \frac{3}{2} \). In their formula, \( A = M \) is the mass of thin rigid plate and \( B = 2S \sqrt{\mu \rho} \), wherein \( S \) is area of plate, \( \mu \) and \( \rho \) are viscosity and the fluid density, respectively, also \( C = K \) is the stiffness of the spring. Conforming [12, 13], we study the numerical solutions of the Bagley-Torvik equation subject to initial states: \( y_0 = y_1 = 0, \).
constant coefficients \( A = B = C = 1 \), and in the case of \( f(t) = 8 \). By these assumptions the corresponding algebraic equation to Eq. (5), is as follow

\[
K^T H_\mu(t) + K^T P^{0.5}_{\mu \times \mu} H_\mu(t) + K^T P^{2}_{\mu \times \mu} H_\mu(t) = 8. \tag{6}
\]

In Table 1, we report the numerical solution of the presented method for different values of \( \mu \) together with the results of the Adomian decomposition method (ADM) [13], and the enhanced homotopy perturbation method (EHPM) [12]. In this case the exact solution is available, and as is shown in Table 1, the obtained results by hybrid functions approach almost coincides with the exact solution.

**Table 1**
The hybrid based results in comparison with the ADM, EHPM, and the analytical solution for Bagley-Torvik equation.

<table>
<thead>
<tr>
<th>( t ) Values</th>
<th>ADM ( \mu = 8 )</th>
<th>EHPM ( \mu = 8 )</th>
<th>Hybrid ( \mu = 8 )</th>
<th>Hybrid ( \mu = 64 )</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.036478</td>
<td>0.0381306</td>
<td>0.033953</td>
<td>0.03513</td>
<td>0.033513</td>
</tr>
<tr>
<td>0.2</td>
<td>0.140640</td>
<td>0.1426114</td>
<td>0.125467</td>
<td>0.125224</td>
<td>0.125221</td>
</tr>
<tr>
<td>0.3</td>
<td>0.307485</td>
<td>0.3157922</td>
<td>0.267678</td>
<td>0.267610</td>
<td>0.267609</td>
</tr>
<tr>
<td>0.4</td>
<td>0.532824</td>
<td>0.5391743</td>
<td>0.455387</td>
<td>0.455434</td>
<td>0.455435</td>
</tr>
<tr>
<td>0.5</td>
<td>0.814757</td>
<td>0.8092061</td>
<td>0.684187</td>
<td>0.684332</td>
<td>0.684335</td>
</tr>
<tr>
<td>0.6</td>
<td>1.148840</td>
<td>1.1191803</td>
<td>0.950172</td>
<td>0.950389</td>
<td>0.950393</td>
</tr>
<tr>
<td>0.7</td>
<td>1.532571</td>
<td>1.4627924</td>
<td>1.24967</td>
<td>1.249956</td>
<td>1.249959</td>
</tr>
<tr>
<td>0.8</td>
<td>1.963033</td>
<td>1.8341171</td>
<td>1.579206</td>
<td>1.579552</td>
<td>1.579557</td>
</tr>
<tr>
<td>0.9</td>
<td>2.437333</td>
<td>2.2275953</td>
<td>1.935427</td>
<td>1.935826</td>
<td>1.935832</td>
</tr>
</tbody>
</table>

Case II: Let \( \beta = 1.8, \alpha = 0.75, A = 2, B = 1.5, C = 1, f(t) = 2t \) and \( y_0 = y_1 = 1 \). The corresponding algebraic equation is

\[
2K^T H_\mu(t) + 1.5 \left( K^T P^{1.05}_{\mu \times \mu} H_\mu(t) + 1 \right) + K^T P^{1.8}_{\mu \times \mu} H_\mu(t) + t + 1 = 2t. \tag{7}
\]

The numerical approximations by using hybrid basis for some different values of \( t, \mu \) are given in Table 2.

**Table 2**
Numerical results of hybrid approach for Case II of Eq. (5).

<table>
<thead>
<tr>
<th>( t ) values</th>
<th>Hybrid ( \mu = 4 )</th>
<th>Hybrid ( \mu = 8 )</th>
<th>Hybrid ( \mu = 32 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.085581</td>
<td>1.088458</td>
<td>1.088611</td>
</tr>
<tr>
<td>0.3</td>
<td>1.225146</td>
<td>1.224456</td>
<td>1.224433</td>
</tr>
<tr>
<td>0.5</td>
<td>1.329323</td>
<td>1.327171</td>
<td>1.327030</td>
</tr>
<tr>
<td>0.7</td>
<td>1.413483</td>
<td>1.412674</td>
<td>1.412477</td>
</tr>
<tr>
<td>0.9</td>
<td>1.493428</td>
<td>1.492577</td>
<td>1.492344</td>
</tr>
</tbody>
</table>
4.2. FRACTIONAL VAN DER POL EQUATION

In this part, the fractional Van der Pol equation with fractional damping is considered. The Van der Pol fractional equation is

\[ D_{\mu}^{\alpha+\beta} y(t) + \eta (y^2(t) - 1) D_{\mu}^{\alpha} y(t) + y(t) = a \sin(\omega t), \quad y(0) = 0, \quad y'(0) = 0, \]  

(8)

where \( \eta \) is the damping parameter, \( \alpha, \beta \) are positive fractional numbers and \( \alpha + \beta \leq 2 \).

Assume that \( D_{\mu}^{\alpha+\beta} y(t) = K_T H_\mu(t) \), then we have

\[ y(t) = K_T P_{\mu \times \mu}^{\alpha+\beta} H_\mu(t), \quad D_{\mu}^{\alpha} y(t) = K_T P_{\mu \times \mu}^{\beta} H_\mu(t). \]  

(9)

To examine the behavior of the Van der Pol equation, we put \( a = 1.31 \), and \( \omega = 0.5 \). Accordingly, the corresponding algebraic system for representation of FDE (8), is

\[ K_T H_\mu(t) + \eta \left( (K_T P_{\mu \times \mu}^{\alpha+\beta} H_\mu(t))^2 - 1 \right) K_T P_{\mu \times \mu}^{\beta} H_\mu(t) + K_T P_{\mu \times \mu}^{\alpha+\beta} H_\mu(t) = 1.31 \sin \left( \frac{t}{2} \right). \]

Figure 1 shows the behavior of numerical approximations of \( y(t) \) as the solution of (8) with \( \mu = 12, \alpha + \beta = 2 \) and \( \alpha = 0.1, 0.5, 1, 1.25, 1.5, i.e., \beta = 1.9, 1.5, 1, 0.75, 0.5 \), respectively. Only for \( \alpha = \beta = 1 \) the exact solution of Eq. (8) is available. Table 3 shows the approximate solutions for the Van der Pol equation for different values of \( t, \eta, \mu \) with \( \alpha = \beta = 1 \). From the numerical results in Table 3, it is clear that the approximate solutions are in high agreement with the solutions of the fourth order Runge-Kutta method (using Wolfram Mathematica 9). As it can be seen in Table 3, we achieve a good approximation of the exact solution by using a few terms of approximate function by hybrid basis of block-pulse function and second kind Chebyshev polynomials. Also, this table indicates that the numerical solution is the same as
Table 3

Comparison of numerical solutions with exact solution of the Van der Pol equation with $\alpha = \beta = 1$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\eta = 0.1$</th>
<th>$\eta = 1.02$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu = 6$</td>
<td>$\mu = 12$</td>
</tr>
<tr>
<td>0.1</td>
<td>0.000149</td>
<td>0.000109</td>
</tr>
<tr>
<td>0.2</td>
<td>0.001000</td>
<td>0.000876</td>
</tr>
<tr>
<td>0.3</td>
<td>0.003113</td>
<td>0.002953</td>
</tr>
<tr>
<td>0.4</td>
<td>0.007227</td>
<td>0.006987</td>
</tr>
<tr>
<td>0.5</td>
<td>0.013976</td>
<td>0.013603</td>
</tr>
<tr>
<td>0.6</td>
<td>0.023709</td>
<td>0.023403</td>
</tr>
<tr>
<td>0.7</td>
<td>0.037322</td>
<td>0.036952</td>
</tr>
<tr>
<td>0.8</td>
<td>0.055164</td>
<td>0.054775</td>
</tr>
<tr>
<td>0.9</td>
<td>0.077783</td>
<td>0.077348</td>
</tr>
</tbody>
</table>

in the system without damping when $\eta$ is close to zero. The obtained results in Table 4 confirm that we can improve the accuracy of our approximation by increasing the $\mu$ and that this approach has high accuracy. Finally, via Table 5 we give the numerical solutions obtained by the presented method in this paper for Eq. (8) with various values of $\alpha, \beta$ and $\eta = 1.5, \mu = 10$.

Table 4

Absolute error for the Van der Pol equation with $\alpha = 1, \eta = 1.02$ using hybrid method with $\mu = 120$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0.1$</td>
</tr>
<tr>
<td></td>
<td>$2.36 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 5

Approximate solutions of the Van der Pol equation with $\eta = 1.5$ and hybrid parameter $\mu = 10$, versus the different values of $\alpha, \beta$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\alpha = 0.5, \beta = 1.5$</th>
<th>$\alpha = 0.75, \beta = 1$</th>
<th>$\alpha = 1.25, \beta = 0.5$</th>
<th>$\alpha = 1.5, \beta = 0.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.000131</td>
<td>0.000319</td>
<td>0.000428</td>
<td>0.000912</td>
</tr>
<tr>
<td>0.3</td>
<td>0.003083</td>
<td>0.006177</td>
<td>0.009532</td>
<td>0.026189</td>
</tr>
<tr>
<td>0.5</td>
<td>0.014406</td>
<td>0.026642</td>
<td>0.046089</td>
<td>0.160037</td>
</tr>
<tr>
<td>0.7</td>
<td>0.040402</td>
<td>0.071502</td>
<td>0.135826</td>
<td>0.529188</td>
</tr>
<tr>
<td>0.9</td>
<td>0.088100</td>
<td>0.151490</td>
<td>0.308561</td>
<td>1.007512</td>
</tr>
</tbody>
</table>
4.3. THE RAYLEIGH EQUATION WITH FRACTIONAL DAMPING

Consider the following Rayleigh equation with fractional damping

\[ y''(t) + \gamma (1 - \frac{1}{3}(D^\alpha_\gamma y(t))^2) y'(t) + y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0, \]  

(10)

with \( \gamma \) as the damping parameter and \( 0 < \alpha < 2 \). To approximate the solution of Eq. (10) assume that \( y''(t) = K^T H_\mu(t) \) then \( D^\alpha_\gamma y(t) = K^T P^{2-\alpha}_{\mu} H_\mu(t) \), such as the Van der Pol fractional equation, then we can express Eq. (10) into the corresponding algebraic system as below

\[ K^T H_\mu(t) + \gamma \left(1 - \frac{1}{3}(K^T P^{2-\alpha}_{\mu} H_\mu(t))^2\right) K^T P^1_{\mu} H_\mu(t) \]

\[ + K^T P^2_{\mu} H_\mu(t) + 1 = 0. \]

(11)

We solve Eq. (11) to find a solution of the nonlinear Rayleigh equation. In order to confirm the accuracy of suggested scheme, we can compare the approximate solutions obtained by our method with Runge-Kutta method for \( \mu = 15 \), when \( \alpha = 1 \) in Table 6. The results show that the operational matrix based on hybrid functions provides a numerical solution for nonlinear fractional Rayleigh equation with good accuracy. Also, the computational results for \( y(t) \) by the hybrid method with \( \gamma = 1 \), for various values of \( \alpha \) and \( \mu \) are shown in Table 7. We can see that the error is being rapidly reduced when the time of simulation or numbers of block-pulse functions are increased.

**Table 6**

Comparison of numerical solutions with exact solution of the Rayleigh equation with \( \alpha = 1 \) and \( \mu = 15 \).

<table>
<thead>
<tr>
<th>t</th>
<th>( \gamma = 0.1 )</th>
<th></th>
<th>( \gamma = 0.5 )</th>
<th></th>
<th>( \gamma = 2 )</th>
<th></th>
</tr>
</thead>
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<tr>
<td></td>
<td>Hybrid</td>
<td>R-K</td>
<td>Hybrid</td>
<td>R-K</td>
<td>Hybrid</td>
<td>R-K</td>
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<tr>
<td>0.1</td>
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<td>0.995047</td>
<td>0.995086</td>
<td>0.995299</td>
<td>0.995321</td>
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<td>0.2</td>
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<td>0.980198</td>
<td>0.980681</td>
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<td>0.3</td>
<td>0.955739</td>
<td>0.955775</td>
<td>0.957446</td>
<td>0.957468</td>
<td>0.963031</td>
<td>0.963023</td>
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<tr>
<td>0.4</td>
<td>0.922054</td>
<td>0.922085</td>
<td>0.925977</td>
<td>0.925992</td>
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<tr>
<td>0.5</td>
<td>0.879519</td>
<td>0.879543</td>
<td>0.886046</td>
<td>0.886935</td>
<td>0.909485</td>
<td>0.909463</td>
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<tr>
<td>0.6</td>
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<td>0.828646</td>
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<td>0.877459</td>
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<tr>
<td>0.7</td>
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<td>0.843042</td>
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<tr>
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<td>0.704126</td>
<td>0.731528</td>
<td>0.731510</td>
<td>0.807010</td>
<td>0.806983</td>
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<tr>
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<td>0.631835</td>
<td>0.669382</td>
<td>0.669356</td>
<td>0.769881</td>
<td>0.769855</td>
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</table>
Table 7
Approximate solutions of the Rayleigh equation with coefficient $\gamma = 1$, vs. the different values of $\mu, \alpha$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\alpha = 0.25$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1.9$</th>
<th>$\mu = 10$</th>
<th>$\alpha = 0.25$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1.9$</th>
<th>$\mu = 20$</th>
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<td>0.1</td>
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<td>0.3</td>
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<td>0.895577</td>
<td>0.895577</td>
<td>0.895544</td>
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<td>0.810653</td>
<td>0.805013</td>
<td>0.810780</td>
<td>0.810615</td>
<td>0.804966</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9</td>
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<td>0.703060</td>
<td>0.711684</td>
<td>0.711187</td>
<td>0.702995</td>
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</table>

5. CONCLUSION

In this paper, operational matrix of fractional integration based on hybrid functions of block-pulse functions and second kind Chebyshev polynomials has been demonstrated to solve some applied fractional differential equations. This method has been implemented in Bagely-Torvik equation, fractional Rayleigh equation, and Van der Pol equation with fractional damping. It is shown that by using of property of operational matrix, fractional differential equations can be transformed into algebraic equations with sparse matrices that is convenient for computer programming. Comparison of our obtained results with exact solutions or obtained results by other methods indicates that this method is efficient and has capability for solving different problems arising in various fields of physics and engineering.

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