CNOIDAL SOLUTIONS, SHOCK WAVES, AND SOLITARY WAVE SOLUTIONS OF THE IMPROVED KORTEweg-de VRIES EQUATION

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In this paper we obtain families of exact solutions of the improved Korteweg-de Vries equation with power law nonlinearity. Three integration schemes are applied. They are the travelling wave hypothesis, the ansatz approach, and the semi-inverse variational principle. With the help of soliton perturbation theory, the adiabatic dynamics of soliton parameters was also obtained.

Key words: Solitons; cnoidal waves; shock waves; integrability.
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1. INTRODUCTION

The dynamics of nonlinear waves in both classical and quantum physical systems is one of the recurrent themes of special scientific interest [1-52]. A few of the models used to study shallow water waves along lake shores and beaches are governed by the Korteweg-de Vries (KdV) equation [4, 8], Boussinesq equation [20], Konopelchenko-Dubrovsky equation [21], Benjamin-Bona Mahoney equation [17] [also known as regularized long wave (RLW) equation or Peregrine equation], Rosenau-Kawahara equation [23], and Rosenau-KdV-RLW equation [34, 35], where the last two models typically describe dispersive shallow water waves. For two-layered fluid flows, some important models are Gear-Grimshaw equation [9, 37], Bona-Chen equation [10] and several others. This paper addresses the KdV equation with an additional dispersion term that is of spatio-temporal type. This makes such modified KdV equation an improved KdV model for addressing shallow water dynamics [22].

The integrability aspects of the improved KdV equation with power law nonlinearity will be studied in this paper. Three kinds of integration tools will be applied: traveling wave hypothesis, ansatz approach, and He’s semi-inverse variational principle (SVP). By using the traveling wave hypothesis we will obtain cnoidal-wave solutions and, in the limiting case, the corresponding solitary waves. The ansatz approach will reveal solitary waves, shock waves, and finally singular solitary waves that model rogue waves. There are several integrability conditions, also known as constraint conditions, which will be revealed during the analysis of the improved KdV equation.
Then, He’s SVP will be applied to retrieve the corresponding solitary-wave solutions. Subsequently, the conservation laws will be revisited in this paper. Finally, the soliton perturbation theory will display the adiabatic dynamics of solitary-wave parameters.

2. TRAVELLING WAVE SOLUTIONS

The improved KdV equation is given by [22, 31]:

$$q_t + a q^n q_x + b_1 q_{xxxx} + b_2 q_{xxt} = 0.$$  \(1\)

In (1), the dependent variable $q(x, t)$ represents the wave profile with $x$ and $t$ being the independent variables. Here $x$ is the spatial variable and $t$ is the temporal variable. The two dispersion terms in eq. (1) are indicated by the coefficients $b_j$ for $j = 1, 2$. The inclusion of the $b_2$ term makes this equation an improved KdV equation. In this section we will apply the traveling wave hypothesis to retrieve cnoidal waves and hence in the limiting case, solitary wave solutions to the problem. The parameter $n$ indicates the exponent of the power law nonlinearity. The starting hypothesis is

$$q(x, t) = g(x - vt) = g(s),$$  \(2\)

where

$$s = x - vt$$  \(3\)

We substitute the above relation into eq. (1) to get:

$$-vg' + a g^n g' + (b_1 - b_2 v) g'' = 0.$$  \(4\)

We then integrate once:

$$-vg + \frac{a}{n+1} g^{n+1} + (b_1 - b_2 v) g'' = K_1,$$  \(5\)

where $K_1$ is an integration constant.

We multiply by $g'$ and integrate again:

$$-\frac{v}{2} g^2 + \frac{a}{(n+1)(n+2)} g^{n+2} + \frac{b_1 - b_2 v}{2} (g')^2 = K_1 g + K_2,$$  \(6\)

where $K_2$ is a second integration constant.

Travelling wave solutions will only exist for $n = 1$. Substituting this value for $n$ and introducing a function $G(g)$ leads to:

$$(g')^2 = -\frac{a}{3(b_1 - b_2 v)} g^3 + \frac{v}{b_1 - b_2 v} g^2 + K_1 g + K_2 = G(g).$$  \(7\)
2.1. CNOIDAL WAVES

First, we consider the case with \( G(g) \) in eq. (7) having three real distinct roots such that \( g_3 < g_2 < g_1 \). Then \( G(g) \) can be written as:

\[
G(g) = -\frac{a}{3(b_1-b_2v)}(g-g_1)(g-g_2)(g-g_3) \quad (8)
\]

\[
s = s_3 \pm \int_{g_3}^{g} \frac{df}{\left[-\frac{a}{3(b_1-b_2v)}(f-g_1)(f-g_2)(f-g_3)\right]^{\frac{1}{2}}} \quad (9)
\]

To get a cnoidal wave solution, we use the following substitution:

\[
f = g_3 + (g_2 - g_3)\sin^2 \theta. \quad (10)
\]

Then

\[
s = s_3 \pm 2\sqrt{\frac{3(b_1-b_2v)}{a(g_3-g_1)}} \int_0^\psi \frac{d\psi}{\sqrt{1-m\sin^2 \psi}}, \quad (11)
\]

where

\[
m = \frac{g_2 - g_3}{g_1 - g_3}, \quad (12)
\]

which in this case represents the modulus of the corresponding elliptic function. The order of the distinct roots guarantee that \( 0 < m < 1 \). Hence, the cnoidal-wave solution is given by:

\[
g(s) = g_2 - (g_2 - g_3)\text{cn}^2 \left[\frac{s-s_3}{2} \sqrt{\frac{a(g_3-g_1)}{3(b_1-b_2v)}}m\right]. \quad (13)
\]

2.2. SOLITARY WAVE SOLUTIONS

To get solitary wave solutions, we introduce boundary conditions as follows:

\[
g, g', g'' \to 0 \quad \text{as} \quad |s| \to \infty, \quad (14)
\]

which leads to \( K_1 = K_2 = 0 \). Hence, eq. (7) becomes:

\[
g' = g \sqrt{\frac{v}{b_1-b_2v} - \frac{a}{3(b_1-b_2v)}}g', \quad (15)
\]

whose solution is

\[
\int g \sqrt{\frac{v}{b_1-b_2v} - \frac{a}{3(b_1-b_2v)}}dg = \pm \int ds, \quad (16)
\]

Then the solitary wave solution is given by:

\[
g(x-\nu t) = \frac{3\nu}{a} \text{sech}^2 \left[\frac{1}{2} \sqrt{\frac{\nu}{b_1-b_2v}}(x-\nu t-x_0)\right]. \quad (17)
\]
3. ANSATZ APPROACH

The second method of integrability is the *ansatz* approach. There are some minor disadvantages with traveling wave hypothesis that was studied in the previous section. The speed of the solitary wave was not recovered from the analysis. However, it is important to retrieve the value of that parameter in order to get a complete spectrum of soliton parameters. The ansatz approach enables to circumvent this drawback. Another hindrance of the previous approach is that it is limited to the determination of solitary waves. Shock waves and singular soliton solutions are not recovered by the traveling wave approach.

Therefore, the ansatz approach that was proposed by Biswas *et al.* [7, 8, 11, 12] is a direct method to retrieving soliton solutions to nonlinear evolution equations (NLEEs). Consequently, the popularity of this simple scheme exploded within a very short time frame. In fact, this integration algorithm is applicable to coupled nonlinear evolution systems as well. Additionally, this scheme can be used to study NLEEs that carry time-dependent coefficients.

This method starts with a guess for the soliton solution, depending on the type of soliton or shock wave. This guess is substituted into the NLEE that is under study. The remaining set of algebraic equations leads to the exact solution with corresponding parameter dependencies as well as the speed of the nonlinear wave.

3.1. SOLITARY WAVES

For solitary waves, the starting hypothesis is [31]

\[ q = A \text{sech}^p [B(x - vt)] = A \text{sech}^p \tau, \]  

(18)

where \((A)\) is the soliton amplitude, \((B)\) is the inverse width, and \((v)\) is the velocity of the wave. The unknown exponent \(p\) will be determined in terms of the power law nonlinearity parameter \(n\). Substituting (18) into (1) leads to:

\[
\begin{align*}
[ v &- b_1 p^2 B^2 + b_2 v p^2 B^2 ] \text{sech}^p \tau - a A^n \text{sech}^{n + 1} \tau \\
&+ (b_1 - b_2 v) (p + 1)(p + 2) B^2 \text{sech}^{p + 2} \tau = 0.
\end{align*}
\]  

(19)

By the balancing principle, we equate the exponents \(p(n + 1)\) and \(p + 2\), which gives the relation between the unknown parameter \(p\) and the power law nonlinearity parameter \(n\)

\[ p = \frac{2}{n}. \]  

(20)

The solitary wave solution is given by

\[ q(x, t) = A \text{sech}^{\frac{2}{n}} [B(x - vt)]. \]  

(21)
The relation between the soliton amplitude \(A\) and the inverse width \(B\) is given by
\[
A = \left[ \frac{2(n+1)(n+2)(b_1 - b_2v)B^2}{an^2} \right]^{\frac{1}{n}},
\] (22)
and the velocity \(v\) of the soliton is
\[
v = \frac{4b_1B^2}{n^2 + 4b_2B^2}.
\] (23)

We substitute this expression for the velocity into equation (22) to get the relation between amplitude and width independent from the velocity:
\[
A = \left[ \frac{2(n+1)(n+2)b_1B^2}{a(n^2 + 4b_2B^2)} \right]^{\frac{1}{n}}.
\] (24)

It needs to be noted that this section 3.1 is a revisitation of the results that were obtained earlier [31]. Nevertheless, the above results are just included here in order to paint a complete picture of the model. The details of the derivation are however omitted since they were reported earlier [31].

3.2. SINGULAR SOLITONS

Here the starting hypothesis is taken to be:
\[
q = A\text{csch}^p [B(x - vt)] = A\text{csch}^p \tau,
\] (25)
where \(A\) and \(B\) are free parameters, and \(v\) is the velocity of the wave. The unknown exponent \(p\) will be determined in terms of the power law nonlinearity parameter \(n\). Substituting (25) into (1) leads to:
\[
\left[ v - b_1p^2B^2 + b_2vp^2B^2 \right] \text{csch}^p \tau - aA^p \text{csch}^{p(n+1)} \tau \\
+ (b_1 - b_2v)(p+1)(p+2)B^2 \text{csch}^{p+2} \tau = 0.
\] (26)

By the balancing principle, we equate the exponents \(p(n+1)\) and \(p + 2\), which gives the relation between unknown parameter \(p\) and power law nonlinearity parameter \(n\)
\[
p = \frac{2}{n}.
\] (27)

The singular wave solution is
\[
q(x,t) = A\text{csch}^\frac{2}{n} [B(x - vt)].
\] (28)

The relation between the free parameters \(A\) and \(B\) is given by
\[
A = \left[ \frac{2(n+1)(n+2)(b_1 - b_2v)B^2}{an^2} \right]^{\frac{1}{n}},
\] (29)
and the velocity \((v)\) of the soliton is
\[
v = \frac{4b_1 B^2}{n^2 + 4b_2 B^2}. \quad (30)
\]

We substitute this expression for the velocity into equation \((29)\) to get the relation between the free parameters that is independent from the velocity:
\[
A = \left[ \frac{2(n+1)(n+2)b_1 B^2}{a(n^2 + 4b_2 B^2)} \right]^{\frac{1}{n}}. \quad (31)
\]

### 3.3. SHOCK WAVES

The starting hypothesis is
\[
q = A \tanh^{p} [B(x - vt)] = A \tanh^{p} \tau, \quad (32)
\]
where \((A)\) and \((B)\) are free parameters, and \((v)\) is the velocity of the wave. The unknown exponent \(p\) will be determined in terms of the power law nonlinearity parameter \(n\). Substituting \((32)\) into \((1)\) leads to:
\[
\begin{align*}
&\left[ v + (b_1 - b_2 v)(3p^2 + 3p + 2)B^2 \right] \tanh^{p+1} \tau \\
&- \left[ v + (b_1 - b_2 v)(3p^2 - 3p + 2)B^2 \right] \tanh^{p-1} \tau \\
&+ aA^n \tanh^{p(n+1)-1} \tau - aA^n \tanh^{p(n+1)+1} \tau \\
&- \left[ (b_1 - b_2 v)(p + 1)(p + 2)B^2 \right] \tanh^{p+3} \tau \\
&+ \left[ (b_1 - b_2 v)(p - 1)(p - 2)B^2 \right] \tanh^{p-3} \tau = 0. \quad (33)
\end{align*}
\]

By the balancing principle, we equate the exponents \(p(n+1)\) and \(p + 2\), which gives the relation between the unknown parameter \(p\) and the power law nonlinearity parameter \(n\). We substitute the ansatz into \((15)\), and by the balancing principle we get
\[
np + p + 1 = p + 3 \quad (34)
\]
\[
np + p - 1 = p + 1 \quad (35)
\]
\[
p = \frac{2}{n} \quad (36)
\]

From the coefficient of \(\tanh^{p-3} \tau\) we obtain
\[
(b_1 - b_2 v)AB^3 p(p - 1)(p - 2) = 0. \quad (37)
\]

This leads to two main cases.
3.3.1. CASE-I: \( p = 1, \ n = 2 \)

The topological soliton solution of (1) is given by:

\[
q = A \tanh \left[ B(x - vt) \right],
\]

where the velocity of the soliton is given by:

\[
v = -\frac{2b_1B^2}{1 - 2b_2B^2}
\]

The relation between the free parameters \( A \) and \( B \) is given by:

\[
A = \left[ -\frac{v + 8(b_1 - b_2v)B^2}{a} \right]^{\frac{1}{2}}
\]

or

\[
A = \left[ -\frac{6(b_1 - b_2v)B^2}{a} \right]^{\frac{1}{2}}.
\]

These relations prompt the respective constraints given by

\[
a \left\{ v + 8(b_1 - b_2v)B^2 \right\} < 0
\]

and

\[
a \left\{ (b_1 - b_2v)B^2 \right\} < 0.
\]

Upon equating eq. (40) and eq. (41) we can extract the same expression for the velocity as in (39).

3.3.2. CASE-II: \( p = 2, \ n = 1 \)

The topological soliton solution of (1) is given by:

\[
q = A \tanh^2 \left[ B(x - vt) \right],
\]

where the velocity of the soliton is given by:

\[
v = -\frac{8b_1B^2}{1 - 8b_2B^2}.
\]

The relation between the free parameters \( A \) and \( B \) is given by:

\[
A = -\frac{v + 20(b_1 - b_2v)B^2}{a}
\]

or

\[
A = -\frac{12(b_1 - b_2v)B^2}{a}.
\]

Upon equating these relations we can extract the same expression for the velocity as in (45).
4. CONSERVATION LAWS

The three conserved quantities that (1) possesses are given by mass \( M \), momentum \( P \), and energy \( E \) [20]

\[
M = \int_{-\infty}^{\infty} q \, dx = \frac{A \, \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{n} \right)}{B \, \Gamma \left( \frac{1}{2} + \frac{1}{n} \right)},
\]

\[
P = \int_{-\infty}^{\infty} (q^2 - b_2 q_x^2) \, dx
= \frac{A^2 \, \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2}{n} \right)}{B \, \Gamma \left( \frac{1}{2} + \frac{2}{n} \right)} \frac{\left[ n(n+4) - 4b_2 B^2 \right]}{n(n+4)},
\]

and

\[
E = \int_{-\infty}^{\infty} \left\{ \frac{2a q^{n+2}}{(n+1)(n+2)} - b_1 q_x^2 \right\} \, dx
= \frac{a A^n \left[ b_1 (n+1)(n+2)(4-n) - 8b_2 A^n \right]}{(n+1)(n+2)(n+4) \left[ 2(n+1)(n+2)b_1 - 4b_2 A^n \right]}.
\]

If the second dispersion term with \( b_2 \) is not present in the improved KdV equation (1), the expression for energy shows that the results are consistent with the ones that were reported earlier [4, 15]. Therefore, for \( b_2 = 0 \), one can again conclude that the KdV equation with power law nonlinearity exists provided \( n \neq 4 \).

5. SEMI-INVERSE VARIATIONAL PRINCIPLE

This is the third approach to the integrability aspect of the improved KdV equation. The SVP was first proposed more than a decade ago by He and this method has gained popularity ever since its first appearance [11, 23, 33, 34, 35].

One starts with the traveling wave hypothesis to (1) as

\[
q(x,t) = g(x - vt) = g(s),
\]

where

\[
s = x - vt.
\]

Substituting this traveling wave assumption into (1) and integrating once while taking the integration constant to be zero gives

\[
-v g + \frac{a}{n+1} g^{n+1} + (b_1 - b_2 v) g'' = 0
\]

Now, multiplying both sides of (53) by \( g' \) and integrating leads to

\[
\frac{-v}{2} g^2 + \frac{a}{(n+1)(n+2)} g^{n+2} + \frac{b_1 - b_2 v}{2} (g')^2 = K,
\]
where $K$ is the integration constant. The stationary integral is then defined as
\[
J = \int_{-\infty}^{\infty} K ds
= \int_{-\infty}^{\infty} \left[ -\frac{v}{2} g^2 + \frac{a}{(n+1)(n+2)} g^{n+2} + \frac{b_1 - b_2 v}{2} (g')^2 \right] ds.
\]
(55)

Now, choosing
\[
g(s) = A \text{sech}^{\frac{2}{n}} (Bs)
\]
(56)
as a solution hypothesis for (1), where $A$ and $B$ are still the amplitude and inverse width of the soliton, the stationary integral $J$ reduces to
\[
J = \frac{A^2}{B} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{n}{2} \right)} \times
\left\{ -\frac{v}{2} + \frac{2(b_1 - b_2 v)B^2}{n(n+4)} + \frac{4aA^n}{(n+1)(n+2)(n+4)} \right\}.
\]
(57)

Then, the SVP states that the amplitude ($A$) and the width ($B$) are given by the solution of the coupled system of equations
\[
\frac{\partial J}{\partial A} = 0
\]
(58)
and
\[
\frac{\partial J}{\partial B} = 0
\]
(59)

From (57), the equations (58) and (59) are respectively given by
\[
4(n+1)(b_1 - b_2 v)B^2 - n(n+1)(n+4)v + 4aA^n = 0
\]
(60)
and
\[
4(n+1)(n+2)(b_1 - b_2 v)B^2 + n(n+1)(n+2)(n+4)v - 8aA^n = 0.
\]
(61)

We then eliminate $A^n$ from the above equations to get an expression for the inverse width $B$:
\[
B = \frac{n}{2} \sqrt{-\frac{v}{(b_1 - b_2 v)}}
\]
(62)
with the constraint condition
\[
v(b_1 - b_2 v) < 0.
\]
(63)

Finally, we eliminate $B$ from the coupled equations to get the amplitude $A$:
\[
A = \left[ \frac{(n+1)(n+2)v}{2} \right]^{\frac{n}{2}}.
\]
(64)
6. SOLITON PERTURBATION THEORY

The dynamics of shallow water waves always comes with perturbation terms. The rest of the paper will therefore be devoted to the perturbed improved KdV equation. This section will study the adiabatic dynamics of soliton parameters when perturbation terms are turned on.

6.1. MODIFIED CONSERVATION LAWS

In the presence of perturbation terms, the improved KdV equation is given by
\[ q_t + aq^n q_x + b_1 q_{xxx} + b_2 q_{xxt} = \epsilon R, \]
where \( \epsilon \) represents the perturbation parameter and \( R \) represents the perturbation terms. Here, it is tacitly assumed that \( 0 < \epsilon \ll 1 \). In the presence of perturbation terms, the conservation laws are modified because these conserved quantities undergo adiabatic deformation. Therefore, these modified conservation laws are given by
\[ \frac{dM}{dt} = \epsilon \int_{-\infty}^{\infty} R dx \]
\[ \frac{dP}{dt} = 2\epsilon \int_{-\infty}^{\infty} q R dx \]
\[ \frac{dE}{dt} = 2\epsilon \int_{-\infty}^{\infty} \left[ \frac{aq^{n+1}}{n+1} R - (b_1 q_x + b_2 q_t) \frac{\partial}{\partial x} R \right], \]
which respectively represent the adiabatic variation of the soliton mass, the soliton momentum, and the soliton energy. The slow change in the soliton velocity is given by
\[ v = \frac{4b_1 B^2}{n^2 + 4b_2 B^2} + \epsilon \int_{-\infty}^{\infty} x R dx. \]

6.2. ADIABATIC PARAMETER DYNAMICS

In this paper, the specific perturbation terms that will be taken into consideration are given by [33, 35]
\[ R = \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q q_{xxx} + \nu q q_{xxt} + \sigma q_x^3 + \xi q_{xxx} + \eta q x q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx} \]
To describe perturbation effects: \( \alpha \) - term represents the shoaling, \( \beta \) is the dissipation coefficient, \( \psi \) is the fifth order spatial dispersion coefficient, and \( \rho \) - term is a higher order stabilization. The higher order nonlinear dispersion term is represented by the
\( \delta \)-term, where \( 1 \leq m \leq 4 \). The rest of the perturbation terms are described in the Whitham hierarchy.

So, the perturbed equation considered in this paper is

\[
q_t + aq^n q_x + b_1 q_{xxx} + b_2 q_{xxt} = \epsilon (\alpha q + \beta q_x + \gamma q_x q_x + \delta q^m q_x + \lambda q_{xxx} + \nu q_x q_x q_x + \rho q_{xxxx}) + \sigma q_3 + \xi q_x q_{xxxx} + \eta q_{xxx} q_{xxxx} + \nu q_{xxxx} + \kappa q_{xxxxx}).
\]  

(71)

In the presence of these perturbation terms, the adiabatic change of the mass is given by

\[
\frac{dM}{dt} = \frac{\epsilon \alpha A \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} + \frac{1}{n} \right)}{B \Gamma \left( \frac{1}{2} + \frac{1}{n} \right)} = \epsilon \alpha M,
\]  

(72)

which shows that

\[
M(t) = M_0 e^{\epsilon \alpha t},
\]  

(73)

where \( M_0 \) is the initial mass of the soliton. Hence, in the limiting case

\[
\lim_{t \to \infty} M(t) = 0,
\]  

(74)

for \( \alpha < 0 \). This shows that the mass adiabatically dissipates with time, in the presence of shoaling since this is a dissipative perturbation term. The adiabatic change of the momentum is given by

\[
\frac{dP}{dt} = \frac{2 \epsilon A^2}{Bn^2(n+4)(3n+4)} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{2}{n} \right)} \times \left\{ \alpha (3n+4)(n+4)n^2 - 4 \beta (3n+4)nB^2 - 16 \rho (2n+3)B^4 \right\}.
\]  

(75)

This means that when \( dP/dt = 0 \), the solitons will travel with constant momentum for a stable fixed value of the width given by

\[
\bar{B} = \left[ \frac{-\beta n(3n+4)}{8\rho(2n+3)} \pm \frac{n\sqrt{(3n+4) [\beta^2(3n+4) + 4\rho \alpha (2n+3)(n+4)]}}{8\rho(2n+3)} \right]^{1/2},
\]  

(76)

with the constraint condition

\[
\beta^2(3n+4) \geq -4 \rho \alpha (2n+3)(n+4).
\]  

(77)

The corresponding fixed value of the amplitude, in this case by virtue of (24), is

\[
\bar{A} = \left[ \frac{b_1(n+1)(n+2) [-(3n+4) \beta \pm \sqrt{(3n+4)D_1}]}{2a \left[ 2\rho m(2n+3) + b_2 [-(3n+4) \beta \pm \sqrt{(3n+4)D_1}] \right]} \right]^{1/2},
\]  

(78)

where

\[
D_1 = \beta^2 (3n+4) + 4 \rho \alpha (2n+3)(n+4).
\]  

(79)
The adiabatic change of the energy is given by
\[
\frac{dE}{dt} = 8\epsilon A^2 \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{\beta}{n} \right) \times \left\{ \frac{aA^n D_2}{(n+1)} - \frac{(b_1 - b_2\nu)B^2 D_3}{n^2} \right\},
\]
where
\[
D_2 = \alpha n^4(3n+4)(5n+4) - 4\beta n^3(5n+4)(n+1)B^2 + 16\rho n^2(n+1)(n^2 + 5n + 3)B^4.
\]
and
\[
D_3 = \alpha n^5(3n+4)(5n+4) - 4\beta n^4(5n+4)(2n+3)B^2 - 16\rho(26n^6 + 188n^5 + 518n^4 + 715n^3 + 536n^2 + 208n + 32)B^4.
\]
This means that when \( \frac{dE}{dt} = 0 \), the solitons will travel with constant energy for the given relation between amplitude and inverse width
\[
A = \left[ \frac{B^2(b_1 - b_2\nu)(n+1)D_4}{an^2D_3} \right]^{\frac{1}{n}}.
\]
For these specific perturbation terms, given by (70), the slow change in the velocity of the soliton is
\[
v = \frac{4b_1 B^2}{n^2 + 4b_2 B^2} - \frac{\epsilon A^m B}{(m+1)} \Gamma \left( \frac{1}{2} + \frac{1}{n} \right) \Gamma \left( \frac{m+1}{n} \right) - \frac{4\epsilon A^2 B^3 [n\nu + (n+9)\sigma]}{3n(n+3)(n+6)} \Gamma \left( \frac{1}{2} + \frac{1}{n} \right) \Gamma \left( \frac{3}{n} \right)
\]
\[+ \frac{2\epsilon AB^3}{n(n+4)} \left\{ -\gamma + 3\lambda + \frac{4B^2(2n+3)[3\xi - \eta - 5\kappa]}{n(3n+4)} \right\} \Gamma \left( \frac{1}{2} + \frac{1}{n} \right) \Gamma \left( \frac{3}{n} \right),
\]
by virtue of (69).

### 7. EXACT SOLITARY WAVE SOLUTION

The perturbed improved KdV equation that will be considered in this paper is
\[
q_t + aq^n q_x + b_1 q_{xxx} + b_2 q_{xxt} = R,
\]
where the perturbation terms are given by
\[
R = \gamma q_x q_{xx} + \lambda q q_{xxx} + \nu q q_x q_{xx} + \sigma q_x^3 + \xi q_x q_{xxx} + \eta q_x q_{xxx} + \kappa q_{xxx}.
\]
and the parameter $\epsilon$ is set to 1 so that the perturbation parameters are treated in a strong sense. The starting hypothesis is:

$$q = A \sech^p [B(x - vt)] = A \sech^p \tau.$$  \hfill (87)

We substitute this ansatz into (85):

$$\begin{align*}
&\left[ v - b_1p^2 B^2 + b_2vp^2 B^2 \right] \sech^p \tau - a A^n \sech^{p(n+1)} \tau \\
&+ (b_1 - b_2v)(p+1)(p+2) B^2 \sech^{p+2} \tau \\
&= -p^2 AB^2 \left[ \gamma + \lambda + \nu^2 B^2 (\xi + \eta + \kappa) \right] \sech^{2p} \tau \\
&+ (p+1)AB^2 \left[ \gamma p + \lambda (p+2) \right] \\
&+ 2 \left\{ \eta p^2 (p+1) + (p^2 + 2p + 2) [\xi p + \kappa(p+2)] \right\} B^2 \sech^{2(p+1)} \tau \\
&- (p+1)(p+2) AB^4 \left[ \xi p (p+3) + \eta p(p+1) + \kappa(p+3)(p+4) \right] \sech^{2(p+2)} \tau \\
&- p^2 A^2 B^2 (\nu + \sigma) \sech^{3p} \tau + pA^2 B^2 \left[ \nu(p+1) + \sigma p \right] \sech^{3p+2} \tau.
\end{align*}$$ \hfill (88)

By the balancing principle, we equate the exponents $p(n+1)$ and $p+2$, which leads to

$$p = \frac{2}{n}.$$ \hfill (89)

The solitary wave solution is given by

$$q(x,t) = A \sech^\frac{2}{n} [B(x - vt)].$$ \hfill (90)

The relation between the soliton amplitude $A$ and the inverse width $B$ is given by

$$A = \left[ \frac{2(n+1)(n+2)(b_1 - b_2v)B^2}{an^2} \right]^\frac{1}{n},$$ \hfill (91)

and the velocity $v$ of the soliton is

$$v = \frac{4b_1B^2}{n^2 + 4b_2B^2}.$$ \hfill (92)

We substitute this expression for the velocity into equation (91) to get the relation between amplitude and width independent from the velocity:

$$A = \left[ \frac{2(n+1)(n+2)b_1B^2}{a(n^2 + 4b_2B^2)} \right]^\frac{1}{n}.$$ \hfill (93)

The additional linearly independent functions lead to a set of constraint relations given by

$$\sigma + \nu = 0$$ \hfill (94)

and

$$2\sigma + (n+2)\nu = 0.$$ \hfill (95)
This linear system of equations implies the unique solution

\[
\begin{bmatrix}
\sigma \\
\nu
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\] (96)

The last set of constraint conditions is given by:

\[n^2(\gamma + \lambda) + 4(\xi + \eta + \kappa)B^2 = 0 \] (97)

\[n^2(\gamma + (n+1)\lambda) + 4\left\{ (n^2 + 2n + 2) [\xi + (n+1)\kappa] + (n+2)\eta \right\} B^2 = 0 \] (98)

\[(3n+2)\xi + (n+2)\eta + (3n+2)(2n+1)\kappa = 0. \] (99)

This set of linear equations, given by (97)-(99) gives the solution

\[
\begin{bmatrix}
\gamma \\
\lambda \\
\xi \\
\eta \\
\kappa
\end{bmatrix} = \begin{bmatrix}
\frac{4(n+3)B^2}{n(3n+2)} \\
\frac{4(n+1)B^2}{n(3n+2)} \\
\frac{n(3n+2)}{(3n+2)} \\
\frac{n}{(3n+2)} \\
1
\end{bmatrix} \eta + \begin{bmatrix}
-4B^2 \\
4(n+2)B^2 \\
-(2n+1) \\
0 \\
1
\end{bmatrix} \kappa
\] (100)

Thus, the system of linear equations (97)-(99) for the perturbation coefficients lead to the conclusion that the five perturbation coefficients, $\xi$, $\eta$, $\kappa$, $\gamma$, and $\lambda$ can be solved in terms of two linearly independent parameters, namely $\eta$ and $\kappa$ only.

8. EXACT SINGULAR SOLITON SOLUTION

The starting hypothesis for the exact singular soliton solution is [35]:

\[q = A \text{csch}^p [B(x - vt)] = A \text{csch}^p \tau. \] (101)

Substituting this ansatz into equation (85):

\[
\begin{aligned}
&\left[v - b_1 p^2 B^2 + b_2 v p^2 B^2\right] \text{csch}^p \tau - A A' \text{csch}^{p(n+1)} \tau \\
&\quad + (b_1 - b_2 v)(p+1)(p+2) B^2 \text{csch}^{p+2} \tau \\
&\quad = -p^2 AB^2 \left[\gamma + \lambda + p^2 B^2 (\xi + \eta + \kappa)\right] \text{csch}^{2p} \tau \\
&\quad + (p+1)AB^2 \left[\gamma p + \lambda (p+2)\right] \\
&\quad + 2 \left\{ \eta p^2 (p+1) + (p^2 + 2p + 2) [\xi p + \kappa (p+2)] \right\} B^2 \text{csch}^{2(p+1)} \tau \\
&\quad - (p+1)(p+2) AB^4 \left[\eta (p+3) + \eta p(p+1) + \kappa (p+3)(p+4)\right] \text{csch}^{2(p+2)} \tau \\
&\quad - p^2 A^2 B^2 (\nu + \sigma) \text{csch}^{3p} \tau + p A^2 B^2 [\nu (p+1) + \sigma p] \text{csch}^{3p+2} \tau
\end{aligned}
\] (102)
Then balancing the exponents \( p(n + 1) \) and \( p + 2 \) leads to:

\[
p = \frac{2}{n}.
\] (103)

Finally, the singular soliton solution is given by:

\[
q(x, t) = A \csc h^{\frac{2}{p}} [B(x - vt)].
\] (104)

Note that the same expressions for the free parameter \( A \) and velocity \( v \), which are given by (91)-(93), also apply for the singular soliton solution. The constraint conditions given by (94)-(100) for the perturbation terms considered for this problem hold as well.

9. EXACT SHOCK WAVE SOLUTION

The starting hypothesis for topological soliton solutions is as follows:

\[
q = A \tanh^p [B(x - vt)] = A \tanh^p \tau.
\] (105)

We substitute this ansatz into (85):

\[
\begin{align*}
&[v + (b_1 - b_2 v)(3p^2 + 3p + 2)B^2] \tanh^{p+1} \tau \\
&- [v + (b_1 - b_2 v)(3p^2 - 3p + 2)B^2] \tanh^{p-1} \tau \\
&+ aA^{n} \tanh^{p(n+1)-1} \tau - aA^{n} \tanh^{p(n+1)+1} \tau \\
&- [(b_1 - b_2 v)(p + 1)(p + 2)B^2] \tanh^{p+3} \tau \\
&+ [(b_1 - b_2 v)(p - 1)(p - 2)B^2] \tanh^{p-3} \tau \\
&= (p - 1)(p - 2)AB^4 [\xi(p - 3) + np(p - 1) + \kappa(p - 3)(p - 4)] \tanh^{2p-5} \tau \\
&- (p + 1)(p + 2)AB^4 [\xi(p + 3) + np(p + 1) + \kappa(p + 3)(p + 4)] \tanh^{2p+5} \tau \\
&- (p + 1)AB^2 [\gamma p + \lambda(p + 2) - \xi p(5p^2 + 13p + 14)] B^2 \\
&- np(p + 1)(5p + 2)B^2 - 5\kappa(p + 2)(p^2 + 3p + 4)B^2 \tanh^{2p+3} \tau \\
&+ (p - 1)AB^2 [\gamma p + \lambda(p - 2) - \xi p(5p^2 - 13p + 14)] B^2 \\
&- np(p - 1)(5p - 2)B^2 - 5\kappa(p - 2)(p^2 - 3p + 4)B^2 \tanh^{2p-3} \tau \\
&- AB^2 [\gamma p(3p - 1) + \lambda(3p^2 - 3p + 2) - 2\xi p(5p^3 - 6p^2 + 13p + 4)] B^2 \\
&+ 2np^2(5p^2 - 4p + 1)B^2 - 2\kappa(5p^4 - 10p^3 + 25p^2 - 20p + 8)B^2 \tanh^{2p-1} \tau \\
&+ AB^2 [\gamma p(3p + 1) + \lambda(3p^3 + 3p + 2) - 2\xi p(5p^3 + 6p^2 + 13p + 4)] B^2 \\
&- 2np^2(5p^2 + 4p + 1)B^2 - 2\kappa(5p^4 + 10p^3 + 25p^2 + 20p + 8)B^2 \tanh^{2p+1} \tau.
\end{align*}
\] (106)

By the balancing principle we get

\[
np + p + 1 = p + 3
\] (107)
\[ np + p - 1 = p + 1 \quad (108) \]
\[ p = \frac{2}{n} \quad (109) \]

From the coefficient of \( \tanh^{p-3} \tau \)
\[ (b_1 - b_2 v)AB^3p(p-1)(p-2) = 0 \quad (110) \]
we can get two main cases.

9.1. CASE-I: \( p = 1, n = 2 \)

The topological soliton solution of (85) is given by:
\[ q = A \tanh [B(x - vt)]. \quad (111) \]
The velocity of the soliton is given by:
\[ v = -\frac{2b_1B^2}{1-2b_2B^2}. \quad (112) \]
The relation between the free parameters \( A \) and \( B \) is given by:
\[ A = \left[ -\frac{v + 8(b_1 - b_2 v)B^2}{a} \right]^{\frac{1}{2}} \quad (113) \]
or
\[ A = \left[ -\frac{6(b_1 - b_2 v)B^2}{a} \right]^{\frac{1}{2}} \quad (114) \]
These relations prompt the respective constraints given by
\[ a \left\{ v + 8(b_1 - b_2 v)B^2 \right\} < 0 \quad (115) \]
and
\[ a \left\{ (b_1 - b_2 v)B^2 \right\} < 0 \quad (116) \]
Upon equating eq. (113) and eq. (114) we can extract the same expression for the velocity as in (112).
The additional linearly independent functions lead to a set of constraint relations given by:
\[ \sigma = 0 \quad (117) \]
\[ 2\nu + 3\sigma = 0 \quad (118) \]
\[ 4\nu + 3\sigma = 0 \quad (119) \]
\[ 2\nu + \sigma = 0 \]  
\[ \gamma + \lambda - 2B^2(4\xi + \eta + 4\kappa) = 0 \]  
\[ \gamma + 2\lambda - B^2(14\xi + 5\eta + 34\kappa) = 0 \]  
\[ \gamma + 3\lambda - 2B^2(16\xi + 7\eta + 60\kappa) = 0 \]  
\[ 2\xi + \eta + 10\kappa = 0. \]

Equations (117)-(120) lead to a unique solution:
\[
\begin{bmatrix}
\sigma \\
\nu
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

The system of linear equations (121)-(124) for the five perturbation coefficients, \( \xi, \eta, \kappa, \gamma, \) and \( \lambda \) can be solved in terms of only two linearly independent parameters \( \eta \) and \( \kappa \).

\[
\begin{bmatrix}
\gamma \\
\lambda \\
\xi \\
\eta \\
\kappa
\end{bmatrix} = \begin{bmatrix}
-2B^2 \\
0 \\
-\frac{1}{2} \\
1 \\
0
\end{bmatrix} \eta + \begin{bmatrix}
-28B^2 \\
-4B^2 \\
-5 \\
0 \\
1
\end{bmatrix} \kappa.
\]

9.2. CASE-II: \( p = 2, n = 1 \)

The topological soliton solution of (85) is given by:
\[
q = A \tanh^2 [B(x - vt)].
\]

The velocity of the soliton is given by:
\[
v = \frac{-8b_1B^2}{1 - 8b_2B^2}
\]

The relation between the free parameters \( A \) and \( B \) is given by:
\[
A = -\frac{v + 20(b_1 - b_2v)B^2}{a}
\]
or
\[
A = -\frac{12(b_1 - b_2v)B^2}{a}.
\]
Upon equating these expressions we can extract the same expression for the velocity as in (128).
The additional linearly independent functions lead to a set of constraint relations given by:

\[ \nu + \sigma = 0 \]  \hspace{1cm} (131)

\[ 5\nu + 6\sigma = 0 \]  \hspace{1cm} (132)

\[ 7\nu + 6\sigma = 0 \]  \hspace{1cm} (133)

\[ 3\nu + 2\sigma = 0 \]  \hspace{1cm} (134)

\[ \gamma - 2B^2(4\xi + 9\eta) = 0 \]  \hspace{1cm} (135)

\[ 5\gamma + 4\lambda - 2B^2(38\xi + 26\eta + 17\kappa) = 0 \]  \hspace{1cm} (136)

\[ 7\gamma + 10\lambda - 2B^2(94\xi + 29\eta + 77\kappa) = 0 \]  \hspace{1cm} (137)

\[ \gamma + 2\lambda - 4B^2(15\xi + 9\eta + 35\kappa) = 0 \]  \hspace{1cm} (138)

\[ 5\xi + 3\eta + 15\kappa = 0. \]  \hspace{1cm} (139)

Equations (131)-(134) lead to a unique solution:

\[
\begin{bmatrix}
\sigma \\
\nu
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]  \hspace{1cm} (140)

The system of linear equations (135)-(139) for the five perturbation coefficients, \( \xi, \eta, \kappa, \gamma \) and \( \lambda \) can be solved for a unique solution:

\[
\begin{bmatrix}
\gamma \\
\lambda \\
\xi \\
\eta \\
\kappa
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]  \hspace{1cm} (141)

10. THE SEMI-INVERSE VARIATIONAL PRINCIPLE (THE PERTURBED CASE)

This section will focus on the integrability aspect of the perturbed improved KdV equation, where the perturbation terms are all of Hamiltonian type in order to permit integrability:

\[ q_t + aq_x + bq_{xxx} + cq_{xxxxx} + k(q^n)_x = \delta q^m q_x + \psi q_{xxxxx}. \]  \hspace{1cm} (142)
The traveling wave hypothesis is taken to be
\[ q(x, t) = g(x - vt) = g(s), \]  

where
\[ s = x - vt. \]  

Substituting this traveling wave assumption into (142) and integrating once while taking the integration constant to be zero gives
\[ -vg + \frac{a}{n+1}g^{n+1} + \frac{\delta}{m+1}g^{m+1} + (b_1 - b_2v)g'' + \psi g''' = 0. \]  

Now, multiplying both sides of (145) by \( g' \) and integrating leads to
\[ -\frac{v}{2}g^2 + \frac{a}{(n+1)(n+2)}g^{n+2} + \frac{b_1 - b_2v}{2}(g')^2 \]
\[ + \frac{\delta}{(m+1)(m+2)}g^{m+2} + \psi \left[ g'g''' - \frac{1}{2}(g'')^2 \right] = K, \]

where \( K \) is the integration constant. The stationary integral is then defined as
\[ J = \int_{-\infty}^{\infty} K ds \]
\[ = \int_{-\infty}^{\infty} \left[ -\frac{v}{2}g^2 + \frac{a}{(n+1)(n+2)}g^{n+2} + \frac{b_1 - b_2v}{2}(g')^2 \right. \]
\[ \left. + \frac{\delta}{(m+1)(m+2)}g^{m+2} + \psi \left[ g'g''' - \frac{1}{2}(g'')^2 \right] \right] ds. \]  

Now, choosing
\[ g(s) = A \text{ sech}^2\left(\frac{B}{B}\right) \]  

as a solution hypothesis for (142), where \( A \) and \( B \) are still the amplitude and inverse width of the soliton, the stationary integral \( J \) reduces to
\[ J = \frac{A^2}{2B} \frac{\Gamma \left( \frac{1}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{2}{n} \right)} \left\{ \frac{8aA^n}{(n+1)(n+2)(n+4)} - v + \frac{4(b_1 - b_2v)B^2}{(n+4)n} \right. \]
\[ + \frac{16\psi(2n+3)B^4}{(n+4)(3n+4)n^2} \left. \right\} + \frac{\delta A^{m+2}}{(m+1)(m+2)B} \frac{\Gamma \left( \frac{1}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{m+2}{n} \right)}. \]  

Then SVP gives
\[ \frac{\partial J}{\partial A} = \frac{A}{B} \frac{\Gamma \left( \frac{1}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{2}{n} \right)} \left\{ \frac{4aA^n}{(n+1)(n+4)} - v + \frac{4(b_1 - b_2v)B^2}{(n+4)n} \right. \]
\[ + \frac{16\psi(2n+3)B^4}{(n+4)(3n+4)n^2} \left. \right\} + \frac{\delta A^{m+1}}{(m+1)B} \frac{\Gamma \left( \frac{1}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{m+2}{n} \right)} = 0 \]  

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\[
\frac{\partial J}{\partial B} = \frac{A^2}{2} \gamma \left( \frac{3}{2} + \frac{1}{n} \right) \left\{ \frac{-8aA^n}{(n+1)(n+2)(n+4)B^2} + \frac{v}{B^2} + \frac{4(b_1 - b_2v)}{(n+4)n} \right\} + \frac{48\psi(2n+3)B^2}{(n+4)(3n+4)n^2} - \frac{\delta A^{m+2}}{(m+1)(m+2)B^2} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{m+2}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{m+2}{n}\right)} = 0,
\]
(151)

which leads to the amplitude-width relationship given by

\[
A = \left[ -\frac{(n+1)(n+2)}{8a(3n+4)n^2(n-m)} \times \left\{ m\psi(n+4)(3n+4)n^2 + 4(b_1 - b_2v)(3n+4)n(m+4)B^2 \right. \\
+ \left. 16\psi(2n+3)(3m+8)B^4 \right\} \right]^\frac{1}{m}.
\]
(152)

11. CONCLUSIONS

In this paper we have obtained exact solitary-wave, shock-wave, and cnoidal-wave solutions of the improved Korteweg-de Vries equation with power law nonlinearity. Additionally, the soliton perturbation theory recovered adiabatic dynamics of soliton parameters. These results will be further generalized. The improved Korteweg-de Vries equation with time-dependent coefficients will be studied. The stochastic perturbation terms will be applied and the dynamics of soliton parameters, in the presence of random perturbations will be investigated. Those new results will be reported elsewhere.

REFERENCES