EXACT SOLUTIONS OF THE GENERALIZED POCHHAMMER-CHREE EQUATION WITH SIXTH-ORDER DISPERSION

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Nonlinear waves described by the generalized Pochhammer–Chree equation are analytically investigated. The addition of the sixth-order dispersion term, which will drastically change the characteristics of the equation, is examined. We employ the sine-cosine method to derive a variety of exact solutions of distinct physical structures including solitons, compactons, periodic and solitary pattern solutions for the adopted model, in the presence of the sixth-order dispersion term. The solitary wave ansatz method is used as well to obtain bright, singular, and dark soliton solutions. Parametric conditions for the existence of the exact solutions are given. The obtained results show that the generalized Pochhammer–Chree equation with sixth order dispersion reveals the richness of explicit soliton and periodic solutions. It should be noted that the study of a new model admitting soliton-type solutions is very important and these solutions will be useful for future research work.

Key words: Pochhammer-Chree equation; dispersion; solitons.

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1. INTRODUCTION

Fully nonlinear evolution equations (NLEEs) have become more interesting and more important in various physics problems like the analysis of patterns in liquid drops [1, 2], weakly nonlinear ion-acoustic waves dynamics in a plasma [3, 4] and many others [5–13]. Interestingly, the dynamics of nonlinear waves described by these models have richer phenomena than the regular case, since various physical structures may be found as candidates for such propagating waves.

In 1993, Rosenau and Hyman [1] introduced the so-called K(m,n) equations to understand the role played by the nonlinear dispersion in the formation of patterns in liquid drops. This first class of fully NLEEs is a straightforward generalization of the well known Korteweg–de Vries equation. Subsequently, many other NLEEs were generalized, offering a rich knowledge on the wave dynamics in nonlinear systems of all kinds. Examples of the many equations that have been generalized are the fully Boussinesq equation B(m,n) [14, 15], the Zakharov–Kuznetsov equation ZK(m,n,k)

Investigation of exact solutions of fully NLEEs is important from many points of view (e.g., for the calculation of certain physical quantities as well as serving as diagnostics for numerical simulations). Additionally, finding the explicit exact solutions for generalized NLEEs is particularly interesting in the study of nonlinear physical phenomena.

Recently, many numerical and analytical methods such as the sine-cosine methods [19–21], the subsidiary ordinary differential equation method [22–24], Hirota’s method [25], the Petrov–Galerkin method [26], the collocation method [27], the solitary wave ansatz method [28, 29] and many others, have been successfully applied to exactly solve NLEEs and their generalizations.

The generalized Pochhammer–Chree (GPC) equation given by [30]

\[
  u_{tt} - u_{ttxx} - \left( \alpha u + \beta u^{n+1} + \gamma u^{2n+1} \right)_{xx} = 0,
\]

where \( \alpha, \beta, \) and \( \gamma \) are constants, represents a nonlinear model of longitudinal wave propagation of elastic rods [31–43].

Considering the higher-order dispersion terms in a given NLEE can significantly change its characteristics. In this work we extend the studies of the nonlinearly dispersive partial differential GPC equation to consider higher order dispersive effects. In particular, we focus our attention on the role of the sixth order dispersion term on the integrability of the GPC equation within the framework of the following model:

\[
  u_{tt} - \varepsilon u_{ttxx} - \left( \alpha u + \kappa u^{n+1} + \gamma u^{2n+1} \right)_{xx} - \delta u_{xxxxxx} = 0,
\]

which can be considered as a natural extension of the generalized Pochhammer-Chree equation (1). Here in Eq. (2), \( \varepsilon, \alpha, \kappa, \gamma, \) and \( \delta \) are nonzero arbitrary constants, while the exponent \( n (> 1) \) is the power law nonlinearity parameter. In the case \( \delta = 0 \), namely, in the absence of sixth order dispersion effect and for \( \varepsilon = 1 \), Eq. (2) reduces to the usual GPC equation (1).

To the best of our knowledge, the above sixth order GPC (6GPC) equation (2) has not been previously considered in literature. The purpose of the present work is to establish fundamental analytical results regarding Eq. (2). Interestingly, many new families of exact travelling wave solutions of the adopted model are successfully obtained using the sine-cosine and the solitary wave ansatz methods.

2. EXACT SOLUTIONS

In this section, we will use the sine-cosine and the solitary wave ansatz methods to develop exact travelling wave solutions to the 6GPC equation (2) with arbitrary constant coefficients.
First, the wave variable \( \xi = x - ct \) converts the NLEE (2) into a nonlinear ordinary differential equation (ODE):

\[
e^2 u_{\xi\xi} - \varepsilon e^2 u_{\xi\xi\xi\xi\xi} = \left( \alpha u + \kappa u^{n+1} + \gamma u^{2n+1} \right)_{\xi\xi} - \delta u_{\xi\xi\xi\xi\xi\xi} = 0, \tag{3}
\]

where \( u_\xi \) denotes \( du/d\xi \).

After integrating twice, Eq. (3) becomes

\[
(c^2 - \alpha) u - \varepsilon c^2 u_{\xi\xi} - \kappa u^{n+1} - \gamma u^{2n+1} - \delta u_{\xi\xi\xi\xi} = 0, \tag{4}
\]

where the integration constants are considered to be equal to zero.

2.1. THE SINE-COSINE METHOD

The sine-cosine ansatz admits the use of the following assumption for solving the reduced ODE Eq. (4) [20]

\[
 u(\xi) = \begin{cases} 
 \{ \lambda \cos^\beta (\mu \xi) \}, & |\xi| \leq \frac{\pi}{2\mu}, \\
 0, & \text{otherwise}
\end{cases} \tag{5}
\]

or the assumption

\[
 u(\xi) = \begin{cases} 
 \{ \lambda \sin^\beta (\mu \xi) \}, & |\xi| \leq \frac{\pi}{\mu}, \\
 0, & \text{otherwise}
\end{cases} \tag{6}
\]

where \( \lambda, \mu, \) and \( \beta \) are parameters that will be determined, \( \mu \) and \( c \) are the wave number and the wave speed, respectively. The exponent \( \beta \) will be determined as a function of \( n \).

The assumption (5) gives

\[
 u^{n+1}(\xi) = \lambda^{n+1} \cos^{\beta(n+1)} (\mu \xi), \tag{7}
\]

\[
 u^{2n+1}(\xi) = \lambda^{2n+1} \cos^{\beta(2n+1)} (\mu \xi), \tag{8}
\]

\[
 u_{\xi\xi} = -\mu^2 \beta^2 \lambda \cos^\beta (\mu \xi) + \mu^2 \lambda \beta (\beta - 1) \cos^{\beta-2} (\mu \xi), \tag{9}
\]

\[
 u_{\xi\xi\xi\xi} = \mu^4 \beta^4 \lambda \cos^\beta (\mu \xi) - 2\mu^4 \lambda \beta (\beta - 1) (\beta^2 - 2\beta + 2) \cos^{\beta-2} (\mu \xi) + \mu^4 \lambda \beta (\beta - 1) (\beta - 2) (\beta - 3) \cos^{\beta-4} (\mu \xi), \tag{10}
\]

and the derivatives of (6) become

\[
 u^{n+1}(\xi) = \lambda^{n+1} \sin^{\beta(n+1)} (\mu \xi), \tag{11}
\]
\[ u^{2n+1}(\xi) = \lambda^{2n+1} \sin^\beta(2n+1) (\mu \xi), \]  
(12)

\[ u_{\xi\xi} = -\mu^2 \beta^2 \lambda \sin^\beta (\mu \xi) + \mu^2 \lambda \beta (\beta - 1) \sin^\beta (\mu \xi), \]  
(13)

\[ u_{\xi\xi\xi\xi} = \mu^4 \beta^4 \lambda \sin^\beta (\mu \xi) - 2\mu^4 \lambda \beta (\beta - 1) (\beta^2 - 2\beta + 2) \sin^\beta (\mu \xi) \]
\[ + \mu^4 \lambda \beta (\beta - 1) (\beta - 2) (\beta - 3) \sin^\beta (\mu \xi), \]  
(14)

Substituting Eqs. (7)-(10) into the reduced ODE (4) gives

\[ (c^2 - \alpha) \lambda \cos^\beta (\mu \xi) + \varepsilon c^2 \mu^2 \beta^2 \lambda \cos^\beta (\mu \xi) - \varepsilon c^2 \mu^2 \lambda \beta (\beta - 1) \cos^\beta (\mu \xi) \]
\[ - \kappa \lambda^{n+1} \cos^\beta (n+1) (\mu \xi) - \gamma \lambda^{2n+1} \cos^\beta (2n+1) (\mu \xi) - \delta \mu^4 \beta^4 \lambda \cos^\beta (\mu \xi) \]
\[ + 2\delta \mu^4 \lambda \beta (\beta - 1) (\beta^2 - 2\beta + 2) \cos^\beta (\mu \xi) \]
\[- \delta \mu^4 \lambda \beta (\beta - 1) (\beta - 2) (\beta - 3) \cos^\beta (\mu \xi) = 0. \]  
(15)

Equating the exponents and the coefficients of like powers of cosine function in Eq. (15) leads to

\[ \beta (\beta - 1) (\beta - 2) (\beta - 3) \neq 0, \]  
(16)

\[ \beta - 4 = \beta (2n + 1), \]  
(17)

\[ (c^2 - \alpha) \lambda + \varepsilon c^2 \mu^2 \beta^2 \lambda - \delta \mu^4 \beta^4 \lambda = 0, \]  
(18)

\[ -\varepsilon c^2 \mu^2 \lambda \beta (\beta - 1) - \kappa \lambda^{n+1} + 2\delta \mu^4 \lambda \beta (\beta - 1) (\beta^2 - 2\beta + 2) = 0, \]  
(19)

\[ -\gamma \lambda^{2n+1} - \delta \mu^4 \lambda \beta (\beta - 1) (\beta - 2) (\beta - 3) = 0. \]  
(20)

Solving this system yields

\[ \beta \neq 0, 1, 2, 3, \]  
(21)

\[ \beta = -\frac{2}{n}, \]  
(22)

\[ \mu = \frac{n}{2} \sqrt{\varepsilon c^2 + \frac{\sqrt{\varepsilon^2 c^4 + 4\delta (c^2 - \alpha)}}{2\delta}}, \]  
(23)

\[ \mu = \frac{n}{2} \sqrt{\frac{1}{\delta (n^2 + 2n + 2)} \left( \frac{\varepsilon c^2 + \kappa}{\frac{\delta (n + 1) (3n + 2)}{\gamma (n + 2)}} \right)}, \]  
(24)
Exact solutions of generalized Pochhammer-Chree eq. with 6th-order dispersion

\[ \lambda = \left\{ \frac{1}{2\delta (n^2 + 2n + 2)} \lambda_1(n, \delta) \right\}^{\frac{1}{n}}, \tag{25} \]

where

\[ \lambda_1(n, \delta) = \left( \varepsilon c^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) \sqrt{-\frac{\delta (n+2)(n+1)(3n+2)}{\gamma}}. \tag{26} \]

Equating the two values of \( \mu \) from Eqs. (23) and (24) gives

\[ n(n+2)\varepsilon c^2 + (n^2+2n+2)\sqrt{\varepsilon^2 c^4 + 4\delta (c^2 - \alpha)} = 2\kappa \sqrt{-\frac{\delta(n+1)(3n+2)}{\gamma(n+2)}}, \tag{27} \]

which serves as a constraint relation between the model coefficients and the wave speed \( c \) provided that \( \varepsilon^2 c^4 + 4\delta (c^2 - \alpha) > 0 \) and \( \delta \gamma < 0 \).

2.2. THE PERIODIC SOLUTIONS

By inserting the previous results (22)-(25) in (5) and (6), we obtain the following periodic solutions \( u(x, t) \):

\[
\begin{align*}
    u(x, t) & = \left\{ \frac{1}{2\delta (n^2 + 2n + 2)} \Gamma(n, \delta) \right\}^{\frac{1}{n}} \\
    & \times \sec^{\frac{n}{2}} \left[ n \sqrt{\frac{1}{\delta (n^2 + 2n + 2)}} \left( \varepsilon c^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) (x - ct) \right],
\end{align*}
\]

and

\[
\begin{align*}
    u(x, t) & = \left\{ \frac{1}{2\delta (n^2 + 2n + 2)} \Gamma(n, \delta) \right\}^{\frac{1}{n}} \\
    & \times \csc^{\frac{n}{2}} \left[ n \sqrt{\frac{1}{\delta (n^2 + 2n + 2)}} \left( \varepsilon c^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) (x - ct) \right],
\end{align*}
\]

where

\[ \Gamma(n, \delta) = \left( \varepsilon c^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) \sqrt{-\frac{\delta (n+2)(n+1)(3n+2)}{\gamma}}. \tag{30} \]

Note that the solutions (28) and (29) exist provided that

\[ \delta \left( \varepsilon c^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) > 0. \tag{31} \]
2.3. SOLITON SOLUTIONS

However, for $\delta \left( \varepsilon c^2 \mp \kappa \sqrt{-\frac{\delta(n+1)(3n+2)}{\gamma(n+2)}} \right) < 0$, the soliton solutions for $u(x,t)$

\[
\begin{align*}
\Gamma_1(n,\delta) = \left( \varepsilon c^2 - \kappa \sqrt{-\frac{\delta(n+1)(3n+2)}{(n+2)\gamma}} \right) \sqrt{-\frac{\delta(n+2)(n+1)(3n+2)}{\gamma}}. 
\end{align*}
\]

are readily obtained, where

\[
\begin{align*}
& \frac{n}{2} \sqrt{-\frac{1}{\delta(n^2 + 2n + 2)} \left( \varepsilon c^2 - \kappa \sqrt{-\frac{\delta(n+1)(3n+2)}{\gamma(n+2)}} \right)} (x - ct), 
\end{align*}
\]

(32)

and

\[
\begin{align*}
& \frac{n}{2} \sqrt{-\frac{1}{\delta(n^2 + 2n + 2)} \left( \varepsilon c^2 + \kappa \sqrt{-\frac{\delta(n+1)(3n+2)}{\gamma(n+2)}} \right)} (x - ct), 
\end{align*}
\]

(33)

2.4. COMPACTON SOLUTIONS

It is obvious that the periodic and soliton solutions are obtained for $n > 0$. However, for negative exponent, we set $n = -m$, where $m$ is an integer, $m > 0$. In this case, Eqs. (17)-(20) become

\[
\begin{align*}
& (c^2 - \alpha) \lambda + \varepsilon c^2 \mu^2 \beta^2 \lambda - \delta \mu^4 \beta^4 \lambda = 0, 
\end{align*}
\]

(35)

\[
\begin{align*}
& -\varepsilon c^2 \mu^2 \lambda \beta (\beta - 1) - \kappa \lambda^{-m+1} + 2 \delta \mu^4 \beta (\beta - 1) (\beta^2 - 2\beta + 2) = 0, 
\end{align*}
\]

(36)

\[
\begin{align*}
& -\gamma \lambda^{-2m+1} - \delta \mu^4 \lambda \beta (\beta - 1) (\beta - 2) (\beta - 3) = 0, 
\end{align*}
\]

(37)

\[
\begin{align*}
& -\gamma \lambda^{-2m+1} - \delta \mu^4 \lambda \beta (\beta - 1) (\beta - 2) (\beta - 3) = 0, 
\end{align*}
\]

(38)
Solving the above system we get
\[ \beta = \frac{2}{m}, \]  
\[ \mu = \frac{m}{2} \sqrt{\varepsilon \omega^2 + \frac{\varepsilon^2 \omega^4 + 4\delta (\omega^2 - \alpha)}{2\delta}}, \]  
\[ \mu = \frac{m}{2} \sqrt{\frac{1}{\delta (m^2 - 2m + 2)}} \left( \varepsilon \omega^2 + \kappa \sqrt{- \frac{\delta (1 - m)(2 - 3m)}{\gamma (2 - m)}} \right), \]  
\[ \lambda = \left\{ \frac{1}{2\delta (m^2 - 2m + 2)} \Gamma_2(m, \delta) \right\}^{\frac{1}{m}}, \]  
Now substituting (35)-(38) into (5) and (6) we obtain a family of compacton solutions given by
\[ u = \begin{cases} \frac{1}{2\delta (m^2 - 2m + 2)} \Gamma_2(m, \delta) \right\}^{\frac{1}{m}} \cos \frac{2}{m} [\mu(x - ct)], & |\mu \xi| \leq \frac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases} \]  
and
\[ u = \begin{cases} \frac{1}{2\delta (m^2 - 2m + 2)} \Gamma_2(m, \delta) \right\}^{\frac{1}{m}} \sin \frac{2}{m} [\mu(x - ct)], & |\mu \xi| \leq \pi, \\ 0, & \text{otherwise}, \end{cases} \]  
where
\[ \Gamma_2(m, \delta) = \left( \varepsilon \omega^2 + \kappa \sqrt{- \frac{\delta (1 - m)(2 - 3m)}{\gamma (2 - m)}} \right) \sqrt{- \frac{\delta (2 - m)(1 - m)(2 - 3m)}{\gamma}}. \]  
and the wave speed \( c \) is obtained from equating the two values of the wave number \( \mu \) from (40) and (41) as
\[ \varepsilon c^2 m (m - 2) + (m^2 - 2m + 2) \sqrt{\varepsilon^2 c^4 + 4\delta (c^2 - \alpha)} \]
\[ = 2\kappa \sqrt{- \frac{\delta (1 - m)(2 - 3m)}{\gamma (2 - m)}}. \]  
Note that the solutions (43) and (44) exist provided that
\[ \delta \left( \varepsilon \omega^2 + \kappa \sqrt{- \frac{\delta (1 - m)(2 - 3m)}{\gamma (2 - m)}} \right) > 0 \]  
and
\[ \delta \gamma < 0. \]
2.5. THE SOLITARY PATTERN SOLUTIONS

For $\delta \left( \varepsilon^2 \mp \kappa \sqrt{-\frac{\delta (1-m)(2-3m)}{\gamma (2-m)}} \right) < 0$, we obtain a family of solitary pattern solutions given by

$$u(x, t) = \left\{ -\frac{1}{2\delta (m^2 - 2m + 2)} \mu_3(\delta, m) \right\}^{-\frac{1}{m}} \times \cosh \frac{m}{2} \left[ \frac{1}{\delta(m^2 - 2m + 2)} \left( \varepsilon^2 - \kappa \sqrt{-\frac{\delta (1-m)(2-3m)}{\gamma (2-m)}} \right) (x - ct) \right], \quad (49)$$

and

$$u(x, t) = \left\{ -\frac{1}{2\delta (m^2 - 2m + 2)} \mu_4(\delta, m) \right\}^{-\frac{1}{m}} \times \sinh \frac{m}{2} \left[ \frac{1}{\delta(m^2 - 2m + 2)} \left( \varepsilon^2 + \kappa \sqrt{-\frac{\delta (1-m)(2-3m)}{\gamma (2-m)}} \right) (x - ct) \right], \quad (50)$$

where

$$\mu_3(\delta, m) = \left( \varepsilon^2 \mp \kappa \sqrt{-\frac{\delta (1-m)(2-3m)}{\gamma (2-m)}} \right) \sqrt{-\frac{\delta (2-m)(1-m)(2-3m)}{\gamma}}, \quad (51)$$

and

$$\mu_4(\delta, m) = \left( \varepsilon^2 \mp \kappa \sqrt{-\frac{\delta (1-m)(2-3m)}{\gamma (2-m)}} \right) \sqrt{-\frac{\delta (2-m)(1-m)(2-3m)}{\gamma}}, \quad (52)$$

where $m > 0$ and the wave speed $c$ can be determined using Eq. (46).

3. THE SOLITARY WAVE ANSATZ METHOD

It is always useful and desirable to construct exact solutions, especially soliton-type envelope solutions for the understanding of most nonlinear physical phenomena. In this section we will apply the solitary wave ansatz method to obtain the bright, singular, and dark soliton solutions of the GPC equation in presence of the sixth-order dispersion term.
3.1. BRIGHT SOLUTIONS

To obtain the bright soliton solution of Eq. (4), we adopt a soliton ansatz of the form \[28, 29\]

\[ u(\xi) = \frac{A}{\cosh^p(\mu \xi)} \]  

(53)

where \( \xi = x - ct \) and \( p > 0 \) for solitons to exist. Here, \( A \) is the amplitude of the soliton, \( c \) is the wave velocity and \( \mu \) is the inverse width of the soliton. The exponent \( p \) is unknown at this point and its value will fall out in the process of deriving the solution of this equation. Thus from (53), we get

\[ u^{n+1}(\xi) = \frac{A^{n+1}}{\cosh^{p(n+1)}(\mu \xi)}, \]  

(54)

\[ u^{2n+1}(\xi) = \frac{A^{2n+1}}{\cosh^{p(2n+1)}(\mu \xi)}, \]  

(55)

\[ u_{\xi\xi} = \frac{p^2 A \mu^2}{\cosh^p(\mu \xi)} - \frac{p(p+1)A \mu^2}{\cosh^{p+2}(\mu \xi)}, \]  

(56)

\[ u_{\xi\xi\xi\xi} = \frac{A p^4 \mu^4}{\cosh^p(\mu \xi)} - \frac{A \mu^4 p(p+1) \left\{ p^2 + (p+2)^2 \right\}}{\cosh^{p+2}(\mu \xi)} + \frac{A \mu^4 p(p+1)(p+2)(p+3)}{\cosh^{p+4}(\mu \xi)}, \]  

(57)

Substituting Eqs. (54)-(57) into Eq. (4) yields

\[ \left( c^2 - \alpha \right) A \cosh^p(\mu \xi) - \frac{\varepsilon e^2 p^2 A \mu^2}{\cosh^p(\mu \xi)} + \frac{\varepsilon e^2 p(p+1)A \mu^2}{\cosh^{p+2}(\mu \xi)} \]

\[- \frac{\kappa A^{n+1}}{\cosh^{p(n+1)}(\mu \xi)} \frac{\gamma A^{2n+1}}{\cosh^{p(2n+1)}(\mu \xi)} - \frac{\delta A p^4 \mu^4}{\cosh^p(\mu \xi)} + \frac{\delta A \mu^4 p(p+1) \left\{ p^2 + (p+2)^2 \right\}}{\cosh^{p+2}(\mu \xi)} - \frac{\delta A \mu^4 p(p+1)(p+2)(p+3)}{\cosh^{p+4}(\mu \xi)} = 0, \]  

(58)

Now, from Eq. (58), equating the exponents \( p(2n+1) \) and \( p + 4 \) leads to

\[ p(2n+1) = p + 4 \]  

(59)

which gives

\[ p = \frac{2}{n} \]  

(60)
It is clear that the same value of $p$ can be obtained if the exponents $p(n+1)$ and $p+2$ are equated. Therefore, from Eq. (58), the linearly independent functions are $1/\cosh^{p+j} \tau$, where $j = 0, 2, 4$. Now setting the coefficients of the linearly independent functions to zero, we get the following expressions:

$$
\mu = \frac{n}{2} \sqrt{-\varepsilon c^2 + \sqrt{\varepsilon^2 c^4 + 4\delta (c^2 - \alpha)}}
$$

(61)

$$
\mu = \frac{n}{2} \sqrt{-\frac{1}{\delta (n^2 + 2n + 2)} \left( \varepsilon c^2 - \kappa \right) \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}}}
$$

(62)

$$
A^n = \frac{-1}{2\delta (n^2 + 2n + 2)} \left( \varepsilon c^2 - \kappa \right) \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}}
\times \sqrt{-\delta(n+2)(n+1)(3n+2)}
\gamma
$$

(63)

Therefore we obtain from Eqs. (61) and (62) the following expression:

$$
-n(n+2)\varepsilon c^2 + (n^2 + 2n + 2) \sqrt{\varepsilon^2 c^4 + 4\delta (c^2 - \alpha)} =
2\kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}}.
$$

(64)

Having obtained the expressions for the pulse parameters, we construct a family of bright solitary wave solutions for the wave equation (2) as follows

$$
u(x,t) = \left\{-\frac{1}{2\delta (n^2 + 2n + 2)} \mu_5(\delta, n) \right\}^{1/2} \frac{1}{n} \sech \frac{\mu}{2} [-i\mu (x-ct)],
$$

(65)

where

$$
\mu_5(\delta, n) = \left( \varepsilon c^2 - \kappa \right) \sqrt{-\frac{\delta (n+1)(3n+2)}{(n+2)\gamma}} \sqrt{-\frac{\delta(n+2)(n+1)(3n+2)}{\gamma}}
$$

(66)

which exist provided that $\gamma \delta < 0$ and $\delta \left( \varepsilon c^2 - \kappa \right) \sqrt{-\frac{\delta(n+1)(3n+2)}{(n+2)\gamma}} < 0$.

### 3.2. SINGULAR SOLUTIONS

To find the singular soliton solution for the reduced ODE (4), we adopt an ansatz solution in the form

$$
u(\xi) = A \text{csch}^p (\mu \xi),
$$

(67)
where $A$ and $\mu$ are the amplitude and the inverse width of the soliton, respectively, while $\xi = x - ct$ with $c$ is the velocity of the wave. The unknown exponent $p > 0$ will be determined in term of the nonlinearity exponent $n$.

From Eq. (67), it is possible to obtain

$$u^{n+1}(\xi) = A^{n+1} \text{csch}^{p(n+1)}(\mu \xi),$$

(68)

$$u^{2n+1}(\xi) = A^{2n+1} \text{csch}^{p(2n+1)}(\mu \xi),$$

(69)

$$u_{\xi\xi} = p^2 A \mu^2 \text{csch}^p(\mu \xi) + p(p+1) A \mu^2 \text{csch}^{p+2}(\mu \xi),$$

(70)

and

$$u_{\xi\xi\xi\xi} = Ap^4 \mu^4 \text{csch}^p(\mu \xi) + A \mu^4 p(p+1) \left\{p^2 + (p+2)^2\right\} \text{csch}^{p+2}(\mu \xi)$$

$$+ A \mu^4 p(p+1)(p+2)(p+3) \text{csch}^{p+4}(\mu \xi)$$

(71)

Substituting Eqs. (68)-(71) into Eq. (4), gives

$$(c^2 - \alpha) A \text{csch}^p(\mu \xi) - \varepsilon c^2 p \mu^2 \text{csch}^p(\mu \xi)$$

$$- \varepsilon c^2 p(p+1) A \mu^2 \text{csch}^{p+2}(\mu \xi)$$

$$- \kappa A^{n+1} \text{csch}^{p(n+1)}(\mu \xi) - \gamma A^{2n+1} \text{csch}^{p(2n+1)}(\mu \xi) - \delta A^p A \mu^4 \text{csch}^p(\mu \xi)$$

(72)

$$- \delta A^p A \mu^4 p(p+1) \left\{p^2 + (p+2)^2\right\} \text{csch}^{p+2}(\mu \xi)$$

$$- \delta A^p A \mu^4 p(p+1)(p+2)(p+3) \text{csch}^{p+4}(\mu \xi) = 0,$$

Now, from Eq. (72), equating the exponents $p(2n+1)$ and $p+4$ leads to

$$p(2n+1) = p+4$$

(73)

so that

$$p = \frac{2}{n}$$

(74)

which is also obtained by equating the exponents $p(n+1)$ and $p+2$.

Again from Eq. (72) the linearly independent functions are $\text{csch}^{p+j}$ for $j = 0, 2, 4$. Hence setting their respective coefficients to zero yields

$$\mu = \frac{n}{2} \sqrt{\frac{-\varepsilon c^2 + \sqrt{\varepsilon^2 c^4 + 4 \delta (c^2 - \alpha)}}{2 \delta}},$$

(75)

$$\mu = \frac{n}{2} \sqrt{\frac{1}{\delta (n^2 + 2n + 2)} \left(\varepsilon c^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma (n+2)}}\right)},$$

(76)
A = \left\{ -\frac{1}{2\delta (n^2 + 2n + 2)} \left( \varepsilon^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) \alpha(n, \delta) \right\} \frac{1}{n} \quad (77)

Finally, inserting the above expressions into Eq. (67) leads to the exact singular soliton solution of Eq. (2):

\[ u(x,t) = \left\{ -\frac{1}{2\delta (n^2 + 2n + 2)} \left( \varepsilon^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) \alpha(n, \delta) \right\} \frac{1}{n} \times \text{csch} \left( \frac{n}{2} \sqrt{-\frac{1}{\delta (n^2 + 2n + 2)} \left( \varepsilon^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right)} (x - ct) \right), \]

where

\[ \alpha(n, \delta) = \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma}} \]

which exist provided that \( \delta \gamma < 0 \) and \( \delta \left( \varepsilon^2 + \kappa \sqrt{-\frac{\delta (n+1)(3n+2)}{\gamma(n+2)}} \right) < 0 \). We observe that the soliton solutions (56) and (68) are the same as obtained above by the sine-cosine method.

3.3. DARK SOLUTIONS

The dark solitons that are also known as topological solitons are also supported by the NLEEs. In this case it will be seen that for the case of the GPC equation with sixth-order dispersion term the dark solitons will exist under certain restrictions.

Let us begin the analysis by assuming an ansatz solution of the from [29]

\[ u(\xi) = A \tanh^p (\mu \xi), \quad (78) \]

where \( A \) and \( \mu \) are free parameters, and \( \xi = x - ct \) with \( c \) is the velocity of the wave. Also, the unknown exponent \( p > 0 \) will be determined during the course of the derivation of the soliton solution to Eq. (2).

From the ansatz (78), we get

\[ u^{n+1}(\xi) = A^{n+1} \tanh^{p(n+1)} (\mu \xi), \quad (79) \]

\[ u^{2n+1}(\xi) = A^{2n+1} \tanh^{p(2n+1)} (\mu \xi), \quad (80) \]

\[ u_{\xi\xi} = pA \mu^2 \left[ (p-1) \tanh^{p-2} (\mu \xi) - 2p \tanh^p (\mu \xi) + (p+1) \tanh^{p+2} (\mu \xi) \right], \quad (81) \]
and
\[ u_{\xi^5} = pA\mu^4 ((p - 1)(p - 2)(p - 3) \tanh^{p-4}(\mu \xi)) \]
\[ + pA\mu^4 (p + 1)(p + 2)(p + 3) \tanh^{p+4}(\mu \xi) \]
\[ - 2pA\mu^4 \{ p^2 + (p - 2)^2 \} \{ p - 1 \} \tanh^{p-2}(\mu \xi) \]
\[ - 2pA\mu^4 \{ p^2 + (p + 2)^2 \} \{ p + 1 \} \tanh^{p+2}(\mu \xi) \]
\[ + pA\mu^4 \{ 4p^3 + (p - 1)^2(p - 2) + (p + 1)^2(p + 2) \} \tanh^p(\mu \xi) \].

Substituting the previous results into Eq. (4), gives
\[ (c^2 - \alpha) A \tanh^p(\mu \xi) \]
\[ - \varepsilon^2 pA\mu^2 [(p - 1) \tanh^{p-2}(\mu \xi) - 2p \tanh^p(\mu \xi) + (p + 1) \tanh^{p+2}(\mu \xi)] \]
\[ - \kappa A^{n+1} \tanh^{p(n+1)}(\mu \xi) - \gamma A^{2n+1} \tanh^{p(2n+1)}(\mu \xi) \]
\[ - \delta pA\mu^4 (p - 1)(p - 2)(p - 3) \tanh^{p-4}(\mu \xi) \]
\[ - \delta pA\mu^4 (p + 1)(p + 2)(p + 3) \tanh^{p+4}(\mu \xi) \]
\[ + 2\delta pA\mu^4 \{ p^2 + (p - 2)^2 \} \{ p - 1 \} \tanh^{p-2}(\mu \xi) \]
\[ + 2\delta pA\mu^4 \{ p^2 + (p + 2)^2 \} \{ p + 1 \} \tanh^{p+2}(\mu \xi) \]
\[ - \delta pA\mu^4 \{ 4p^3 + (p - 1)^2(p - 2) + (p + 1)^2(p + 2) \} \tanh^p(\mu \xi) = 0. \] (83)

From Eq. (83), equating the exponents \( p(2n + 1) \) and \( p + 4 \) gives
\[ p(2n + 1) = p + 4 \] (84)
so that
\[ p = \frac{2}{n} \] (85)

Again this same value of \( p \) is obtained on equating the exponents \( p(n + 1) \) and \( p + 2 \). Now from Eq. (83) the linearly independent functions are \( \tanh^{p+j} \tau \) for \( j = 0, \pm 2, \pm 4 \). Hence setting their respective coefficients to zero yields
\[ A \left[ c^2 - \alpha + 2p^2 \varepsilon^2 \mu^2 - \delta pA^4 \{ 4p^3 + (p - 1)^2(p - 2) + (p + 1)^2(p + 2) \} \right] = 0 \] (86)
\[ A(p - 1) \left[ - \varepsilon^2 p\mu^2 + 2\delta p\mu^4 \{ p^2 + (p - 2)^2 \} \right] = 0 \] (87)
\[ -\varepsilon^2 pA\mu^2 (p + 1) - \kappa A^{n+1} + 2\delta pA\mu^4 \{ p^2 + (p + 2)^2 \} \{ p + 1 \} = 0 \] (88)
\[ -\gamma A^{2n+1} - \delta pA\mu^4 (p + 1)(p + 2)(p + 3) = 0 \] (89)
\[ -\delta pA\mu^4 (p - 1)(p - 2)(p - 3) = 0 \] (90)
Solving both of Eqs. (87) and (90) yields

\[ p = 1 \]  

(91)

Thus, from Eqs. (80) and (91) we obtain

\[ n = 2 \]  

(92)

This shows that topological 1-soliton solutions of the GPC equation with sixth-order dispersion term will exist for \( n = 2 \). There is no other value of the power law non-linearity parameter \( n \) for which the topological solitons will exist. This is a very important observation.

In this case, equation (2) becomes

\[ u_{tt} - \varepsilon u_{ttxx} - (\alpha u + \kappa u^3 + \gamma u^5)_{xx} - \delta u_{xxxxxx} = 0, \]  

(93)

Substituting \( p = 1 \) into (86)-(90) gives

\[ \mu = \frac{1}{4} \left[ \frac{\varepsilon c^2 + \sqrt{\varepsilon^2 c^4 + 16\delta (c^2 - \alpha)}}{\delta} \right] \]  

(94)

\[ \mu = \sqrt{\frac{1}{20\delta} \left( \varepsilon c^2 + \kappa \sqrt{-\frac{6\delta}{\gamma}} \right)} \]  

(95)

\[ A = \left[ \frac{1}{10\delta} \left( \varepsilon c^2 + \kappa \sqrt{-\frac{6\delta}{\gamma}} \right) \sqrt{-\frac{6\delta}{\gamma}} \right]^{\frac{1}{2}} \]  

(96)

Equating the two values of \( \mu \) from Eqs. (94) and (95) gives the condition:

\[ \varepsilon c^2 + 5\sqrt{\varepsilon^2 c^4 + 16\delta (c^2 - \alpha)} = 4\kappa \sqrt{-\frac{6\delta}{\gamma}} \]  

(97)

The latter serves as a constraint relation between the coefficients and the soliton velocity.

Substituting the previous results in Eq. (78) leads to the following topological 1-soliton solution to the 6GPC equation (2):

\[ u = \left[ \frac{1}{10\delta} \left( \varepsilon c^2 + \kappa \sqrt{-\frac{6\delta}{\gamma}} \right) \sqrt{-\frac{6\delta}{\gamma}} \right]^{\frac{1}{2}} \tanh \left( \sqrt{\frac{1}{20\delta} \left( \varepsilon c^2 + \kappa \sqrt{-\frac{6\delta}{\gamma}} \right)} (x - ct) \right) \]  

(98)

which exist provided that \( \delta \gamma < 0 \) and \( \delta \left( \varepsilon c^2 + \kappa \sqrt{-\frac{6\delta}{\gamma}} \right) > 0 \).
4. CONCLUSION

We have considered a generalized Pochhammer–Chree equation with sixth-order dispersion term, which is an extension of the regular generalized Pochhammer–Chree equation. By means of sine-cosine method, we have obtained a variety of distinct physical structures such as periodic solutions, soliton solutions, compacton solutions, and solitary pattern solutions. The solitary wave ansatz method is also used to obtain bright, singular, and dark soliton solutions. Also, the parametric conditions for the existence of the exact solutions are given. These solutions may be of significant importance for the explanation of some special physical phenomena arising in dynamical systems modeled by the generalized Pochhammer–Chree equation with higher-order dispersion effects. This study highlights the power of these methods for the determination of exact solutions to fully nonlinear evolution equations with arbitrary constant coefficients.

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