We investigate a coupled system of partial differential equations (PDEs) describing the carbon nanotubes conveying fluid using the generalized symmetry analysis, the multiplier approach, and a new conservation theorem. The symmetries and the conservation laws of the coupled system of PDEs are given.

Key words: Carbon nanotube; Symmetry analysis; Conservation laws.


1. INTRODUCTION

The carbon nanotube (CNT) filled with fluids plays a key role in a surprising number of scientific disciplines, such as nanomechanical [1, 2] and nanobiological [3] applications, optics [4] and so on. Yoon et al. [5] investigated the vibration analysis of CNTs conveying fluid. They considered the natural frequencies of the fluid-conveying single-walled carbon nanotube (SWCNT). In Ref. [6], the authors studied the flexural vibration responses of the fluid conveying SWCNT using Timoshenko beam theory. The authors in Ref. [7] using the Hamilton principle, first investigated a double-walled carbon nanotube (DWCNT) conveying fluid with the nonlinear van der Waals forces. They deduced both uncoupling and coupling between the longitudinal and transverse displacements. In Ref. [8], the in-plane vibration analysis of curved CNTs conveying fluid embedded in viscoelastic medium was considered. The finite element method was employed to discretize the equation of motion and...
the frequencies, and the authors also presented some numerical results. In Ref. [9], on the basis of the framework of the nonlocal Euler-Bernoulli beam theory, using the modal superposition and Newmark’s direct integration methods, dynamic analysis of a simply SWCNT carrying a moving harmonic load was performed. In Ref. [10], the authors investigated some systems of carbon nanotubes conveying fluid using the classical Lie group method; they also reported some power series solutions.

In order to provide more information on the system of carbon nanotubes conveying fluid, we use the group method [10-26], the multiplier approach, and the new conservation theorem to study the SWCNT [10]

\[
\begin{align*}
    u_t + \alpha (v - u_x) + 2\beta u_{xt} + \beta u_x &= 0, \\
    v_t - \lambda v_{xx} + \mu (v - u_x) &= 0,
\end{align*}
\]

where \(\alpha, \beta, \lambda, \) and \(\mu\) are arbitrary constants, \(u = u(x, t)\) means the transverse displacement of the flexural vibration, and \(v = v(x, t)\) expresses the rotation angle of the cross section perpendicular to the longitudinal axis.

The main purpose of this paper is to investigate the symmetries and conservation laws of a system of coupled partial differential equations (PDEs) describing the carbon nanotubes conveying fluid. The paper is organized as follows. In Sec. 2, the group method is used to deal with the system of coupled PDEs. In Sec. 3, the generalized symmetries are employed to handle those equations. In Sec. 4, the conservation laws are constructed using the direct method and a new conservation theorem. Finally, our conclusions and some remarks are given in Sec. 5.

2. EQUIVALENCE TRANSFORMATIONS

In this section, we will use the improved direct reduction method [22] to deal with the coupled system of equations (1).

We look for the equivalence transformations between the system (1) and the following coupled equations

\[
\begin{align*}
    U_\tau + \alpha (V - U_\xi) + 2\beta U_{\xi\tau} + \beta U_\xi &= 0, \\
    V_\tau - \lambda V_{\xi\xi} + \mu (V - U_\xi) &= 0,
\end{align*}
\]

Let us suppose that Eq. (1) has the following solutions

\[
\begin{align*}
    u(x, t) &= \alpha_1 + \beta_1 U(\xi, \eta) + \gamma_1 V(\xi, \eta), \\
    v(x, t) &= \alpha_2 + \beta_2 V(\xi, \eta) + \gamma_2 U(\xi, \eta),
\end{align*}
\]

where \(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \xi = \xi(x, t), \tau = \tau(x, t)\) are functions to be further fixed. Also \(U(\xi, \tau), V(\xi, \tau)\) satisfies the same equations as \(u = u(x, t)\) and \(v = v(x, t)\) with the transformation \(\{u, v, x, t\} \rightarrow \{U, V, \xi, \tau\}\). That is to say, we restrict \(U(\xi, \tau)\) and \(V(\xi, \tau)\) to satisfy Eq. (2).
Then we insert (3) into (1) and require \( U(\xi, \tau), V(\xi, \tau) \) to satisfy Eq. (2). Letting the coefficients of \( U, V \) and their derivatives to be zero, one can get
\[
\begin{align*}
\tau_1 \beta_2 \lambda - \beta_2 \lambda c^2_x & = 0, \\
\xi_x \beta_1 \mu - \tau_1 \beta_2 \mu & = 0, \\
\beta_2 \xi_t - \lambda \beta_2 \xi_x - 2 \lambda \beta_2 \xi_x & = 0, \\
\mu \beta_2 + \beta_2 \tau_2 - \beta_2 \tau_1 \mu & = 0, \\
\gamma_1 = \gamma_2 = 0, \\
\beta \beta_1 x - \alpha \beta_1 x + 2 \beta \beta_1 t x + \beta_1 t & = 0, \\
2 \beta \beta_1 x \tau_t + 2 \beta \beta_1 x \tau_t - \alpha \beta_1 \xi^2_t & = 0, \\
\beta_1 \tau_t + 2 \beta \beta_1 t x \left( \frac{(2 \beta \beta_1 \xi \tau_t - \alpha \beta_1 \xi^2_t)}{\alpha} \right) & = 0, \\
\alpha \beta_2 \xi_x + 2 \beta \beta_1 \xi \xi_t - \alpha \beta_1 \xi^2_t & = 0, \\
\beta_1 \xi_t - \alpha \beta_1 \xi_x + \beta \xi \xi_1 + 2 \beta \beta_1 \xi x + 2 \beta \beta_1 \xi t + 2 \beta \beta_1 \xi t \\
+ \frac{\beta (2 \beta \beta_1 \xi t - \alpha \beta_1 \xi^2_t)}{\alpha} & = 0, \\
\alpha_1 t + \alpha \alpha_2 x - \alpha \alpha_1 x x + 2 \beta \alpha_1 x t + \beta \alpha_1 x & = 0, \\
\beta_1 x = 0, \beta_2 x = 0, \\
\alpha_2 t - \lambda \alpha_2 x x + \mu \alpha_2 - \mu \alpha_1 x & = 0.
\end{align*}
\]
Solving these equations, one gets
\[
\begin{align*}
\xi & = x + c_2, \tau = t + c_1, \beta_1 = \beta_2 = c_3, \\
\alpha_1 t + \alpha \alpha_2 x - \alpha \alpha_1 x x + 2 \beta \alpha_1 x t + \beta \alpha_1 x & = 0, \\
\alpha_2 t - \lambda \alpha_2 x x + \mu \alpha_2 - \mu \alpha_1 x & = 0,
\end{align*}
\]
where \( \alpha_1 = \alpha_1(x, t) \) and \( \alpha_2 = \alpha_2(x, t) \) satisfy the system of PDEs \( \alpha_1 t + \alpha \alpha_2 x - \alpha \alpha_1 x x + 2 \beta \alpha_1 x t + \beta \alpha_1 x = 0, \alpha_2 t - \lambda \alpha_2 x x + \mu \alpha_2 - \mu \alpha_1 x = 0. \) That is to say, the two functions \( \alpha_1(x, t) \) and \( \alpha_2(x, t) \) are solutions to the system (1). In this way, one obtains

**Theorem 1** If \( U(\xi, \tau), V(\xi, \tau) \) are solutions of Eqs. (1), then
\[
\begin{align*}
\alpha_1 + \alpha \alpha_2 x - \alpha \alpha_1 x x + 2 \beta \alpha_1 x t + \beta \alpha_1 x = 0, \\
\alpha_2 t - \lambda \alpha_2 x x + \mu \alpha_2 - \mu \alpha_1 x & = 0,
\end{align*}
\]
where \( \alpha_1 = \alpha_1(x, t) \) and \( \alpha_2 = \alpha_2(x, t) \) satisfy the system of PDEs \( \alpha_1 t + \alpha \alpha_2 x - \alpha \alpha_1 x x + 2 \beta \alpha_1 x t + \beta \alpha_1 x = 0, \alpha_2 t - \lambda \alpha_2 x x + \mu \alpha_2 - \mu \alpha_1 x = 0. \) That is to say, the two functions \( \alpha_1(x, t) \) and \( \alpha_2(x, t) \) are solutions to the system (1). In this way, one obtains

**Theorem 1** If \( U(\xi, \tau), V(\xi, \tau) \) are solutions of Eqs. (1), then
\[
\begin{align*}
u(x, t) = \alpha_1 + \beta_1 U(\xi, \eta) + \gamma_1 V(\xi, \eta), \\
u(x, t) = \alpha_2 + \beta_2 V(\xi, \eta) + \gamma_2 U(\xi, \eta),
\end{align*}
\]
are also solutions of Eq. (1), where \( \alpha_1, \beta_1, \alpha_2, \beta_2, \xi, \tau \) are given by (5).

### 3. GENERALIZED SYMMETRIES OF (1)

In this section, we consider the symmetries of Eqs. (1) using the generalized symmetry method [20].

Given a nonlinear evolution equation
\[
F(t, x, u, u_t, u_{xx}, \ldots) = 0,
\]
a function $\sigma$ is a symmetry of Eq. (7), if it satisfies the following equation
\[ F'(u)\sigma = 0, \]  
with
\[ F'(u)\sigma = \frac{\partial F}{\partial u}\sigma + \frac{\partial F}{\partial u_t}\sigma_t + \frac{\partial F}{\partial u_x}\sigma_x + \frac{\partial F}{\partial u_{tx}}\sigma_{tx} + \frac{\partial F}{\partial u_{xx}}\sigma_{xx} + \cdots \]  
Thus, we can derive the symmetry equations of (1) as follows
\[ \sigma_1 t + \alpha(\sigma_2 x - \sigma_1 xx) + 2\beta \sigma_1 xt + \beta \sigma_1 x = 0, \]  
\[ \sigma_2 t - \lambda \sigma_2 xx + \mu(\sigma_2 - \sigma_1 x) = 0. \]  
In order to find the symmetry of the coupled system (1), we set
\[ \sigma_1 = au_t + bu_x + eu + g, \]  
\[ \sigma_2 = av_t + bv_x + mv + h, \]  
where $a, b, e, g, m$ and $h$ are functions of $t, x$ to be further fixed.

Plugging (11) into (10) with the help of (1), one can get the associated determining equations,
\[ b_x = e_x = e_t = m_x = 0, \]  
\[ -e\alpha + m\alpha - \alpha a_x + 2\beta a_t = 0, \]  
\[ \frac{2\beta}{\alpha} a_t + a_t + 2\beta a_{tx} - \beta a_x - \alpha a_{xx} = 0, \]  
\[ a_t - \lambda a_{xx} = 0, \]  
\[ b_t - 2a_x = 0, \]  
\[ \mu b_t - mt = 0, \]  
\[ -\mu e + \mu m - \mu a_x + \mu b_t = 0, \]  
\[ g_t + \alpha(h - g_x)x + 2\beta g_{xt} + \beta g_x = 0, \]  
\[ h_t - \lambda h_{xx} + \mu(h - g_x) = 0. \]  
And then solving them, one obtains
\[ a = c_1, b = c_2, e = m = c_3, g = g(x,t), h = h(x,t), \]  
where $c_i (i = 1, 2, 3)$ are arbitrary constants and $g(x,t)$ and $h(x,t)$ are also solutions to system (1). Thus, one gets
\[ \sigma = c_1 u_x + c_2 u_t + (c_3 + g(x,t))u + (c_3 + h(x,t))v. \]  
We can get the equivalent vector expression as follows
\[ V = c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial t}(-c_3 u + g(x,t)) \frac{\partial}{\partial u} + (-c_3 v + h(x,t)) \frac{\partial}{\partial v}. \]
or
\[
V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t}, V_3 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, V_g = g \frac{\partial}{\partial u}, V_h = h \frac{\partial}{\partial v},
\]
which coincide precisely with the vector field \( V \) are obtained in Ref. [10].

4. CONSERVATION LAWS

In this section, using the multipliers method [14-17] and the new conservation theorem [23], we will handle the conservation laws of the system.

4.1. BASIC CONCEPTS

Generally speaking, we consider a system of \( r \) PDEs of \( k \)-order with \( x \) and \( u \), given by [14-17]
\[
F_\iota(x,u,u_i(1),u_i(2),\ldots,u_i(k)) = 0, \quad \iota = 1,2,\ldots,r,
\]
where \( u_i(1) = \{u_i^j\}, u_i(2) = \{u_i^{ij}\}, \ldots \), and \( u_i^\iota = \frac{\partial u_i}{\partial x_i}, u_i^{ij} = \frac{\partial^2 u_i}{\partial x_i \partial x_j}, \ldots \).

1) The total derivative operators \( D_i \) are defined by
\[
D_i = \frac{\partial}{\partial x_i} + u_i^\iota \frac{\partial}{\partial u_i^\iota} + u_i^{ij} \frac{\partial}{\partial u_i^{ij}} + u_i^{ijk} \frac{\partial}{\partial u_i^{ijk}} + \cdots,
\]
where \( i,j,k,\ldots = 1,2,\ldots,n \) and \( \iota = 1,2,\ldots,m \).

2) The multipliers for the PDE system (17) are undetermined functions \( \{\Lambda^\iota[U]\} \) such that
\[
\Lambda^\iota[U]F_\iota[U] = D_i T^i[U],
\]
for some appropriate functions \( \{T^i[U]\} \) [14-17].

If \( U^\sigma = u^\sigma(x) \) is a solution of the PDE system (17), then one can get the conservation laws of (17) as follows
\[
D_i T^i[U] = 0,
\]
where \( T^i[U] \) means the associated conserved densities.

3) The Euler operators [14-17] are defined by
\[
E_{u^j} = \frac{\partial}{\partial u^j} - D_1 \frac{\partial}{\partial u^1} - \cdots + (-1)^s D_1 \cdots D_s \frac{\partial}{\partial u^{i_1 \ldots i_s}} + \cdots,
\]
for each \( j = 1,2,\ldots,m \).

For the conservation laws of the system (17), there is a set of multipliers constructed by using
\[
E_{u^j}(\Lambda^\iota[U]F_\iota[U]) \equiv 0, \quad j = 1,\ldots,N.
\]
4) **Theorem.** Every Lie point, Lie-Bäcklund, and non-local symmetry of Eq. (1) gives a conservation law for this equation and the adjoint equation [23]. Then the elements of the conservation vector \((C^1, C^2)\) are given by

\[
C^i = \xi^i L + W^\alpha \left[ \frac{\partial L}{\partial u_0^\alpha} - D_j \left( \frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_k \left( \frac{\partial L}{\partial u_{ijk}^\alpha} \right) - \cdots \right] \\
+ D_j(W^\alpha) \left[ \frac{\partial L}{\partial u_{ij}^\alpha} - D_k \left( \frac{\partial L}{\partial u_{ijk}^\alpha} \right) + \cdots \right] + D_j D_k(W^\alpha) \left[ \frac{\partial L}{\partial u_{ijk}^\alpha} - \cdots \right], \tag{23}
\]

where \(W^\alpha = \eta^\alpha - \xi^i u_0^\alpha\).

4.2. CONSERVATION LAWS OF (1) USING THE MULTIPLIERS METHOD

From the above, we suppose the conservation law is given by \(D_x P^x + D_t P^t = 0\), on the solutions of (1).

We seek the conservation law multipliers of the following form:

\[
Q_1 = \xi(x, t), Q_2 = \tau(x, t). \tag{24}
\]

The multiplier is \(Q = (Q_1, Q_2)\) and the conservation law is \(D_x P^x + D_t P^t = 0\), along the solutions of the equation. In this case, one can get

(i) \(\alpha = \left(\frac{\mu - \lambda - 1}{2(\lambda + 1)}\right), \beta = \frac{1}{2}\); we obtain \(Q = (2e^x e^t(1 + \lambda), e^x e^t)\)

\[
T^x = e^{1+x} (-1 + \lambda)u + (-1 + \mu) v + u_t + \lambda u_t + u_x + \lambda u_x - \mu u_x - \lambda v_x), \\
T^t = e^{1+x} ((1 + \lambda)u + v + (1 + \lambda)u_x). \tag{25}
\]

(ii) \(\alpha = \frac{2\mu - 2\lambda - 1}{2(2\lambda + 1)}\); we obtain \(Q = (e^x e^{\frac{1}{2}t}(1 + 2\lambda), e^x e^{\frac{1}{2}t})\)

\[
T^x = \frac{1}{2} e^{\frac{1}{2}+x} (-1 + \beta)(1 + 2\lambda)u \\
+ (-1 + \lambda)u_x + \beta (1 + 2\lambda)x_t + (1 + 2\lambda - 2\beta)u_x - 2\lambda v_x, \\
T^t = e^{\frac{1}{2}+x} (v + (1 + 2\lambda) - (1 + \beta)u + \beta u_x). \tag{26}
\]

(iii) \(\alpha = \mu - \lambda - 1, \beta = -\lambda\); we get \(Q = (e^x e^t, e^x e^t)\)

\[
T^x = e^{1+x} (-1 + \lambda)u + (-1 + \mu) v - \lambda u_t + u_x + \lambda u_x - \mu u_x - \lambda v_x), \\
T^t = e^{1+x} ((1 + \lambda)u + v - \lambda u_x). \tag{27}
\]

4.3. CONSERVATION LAWS OF (1) USING NEW CONSERVATION THEOREM

For Eq. (1), the adjoint equation has the form

\[
F_1 = -\theta_{1t} - \alpha \theta_{1x} v_x - \alpha \theta_{1xx} + 2 \beta \theta_{1xt} - \beta \theta_{1x} + \mu \theta_{2x} = 0, \\
F_2 = -\theta_{2t} - \lambda \theta_{2xx} + \mu \theta_{2x} + \mu \theta_{2x} u_x - \alpha \theta_{1x} = 0, \tag{28}
\]
The corresponding operator $V$ is given by

$$V = \xi^1(x,t,u,v) \frac{\partial}{\partial t} + \xi^2(x,t,u,v) \frac{\partial}{\partial x} + \eta^1(x,t,u,v) \frac{\partial}{\partial u} + \eta^2(x,t,u,v) \frac{\partial}{\partial v}. \quad (30)$$

The operator $V$ yields the conservation law $D_t(C^1) + D_x(C^2) = 0$, where the conserved vector is given by Eq. (23). Its components are given by

$$C^1 = \xi^1 L + W^1 \left[ \frac{\partial L}{\partial u_t} - D_x \left( \frac{\partial L}{\partial u_{xt}} \right) \right] + W^2 \left[ \frac{\partial L}{\partial v_t} + D_x(W^1) \left[ \frac{\partial L}{\partial u_{xt}} \right] \right], \quad (31)$$

$$C^2 = \xi^2 L + W^1 \left[ \frac{\partial L}{\partial u_x} - D_x \left( \frac{\partial L}{\partial u_{xx}} \right) - D_t \left( \frac{\partial L}{\partial u_{xt}} \right) \right] + W^2 \left[ \frac{\partial L}{\partial v_x} - D_x(W^1) \left[ \frac{\partial L}{\partial u_{xx}} \right] \right] + D_t(W^1) \left[ \frac{\partial L}{\partial u_{xt}} \right] + D_x(W^2) \left[ \frac{\partial L}{\partial u_{xx}} \right]. \quad (32)$$

We consider the following operators $V = \partial_t$ and $V = \partial_x$. For the operator $V = \partial_t$, we have

$$\xi^2 = 1, \xi^1 = \eta^1 = \eta^2 = 0, W^1 = -u_t, W_2 = -v_t. \quad (33)$$

In this case, using Eq. (23), we provide the following components:

$$C^1 = \left( \theta_1(u_t + \alpha(v - u_x)x + 2\beta u_{xt} + \beta u_x) + \theta_2(v_t - \lambda v_{xx} + \mu(v - u_x)) \right)$$

$$- u_t(\theta_1 - 2\beta \theta_1 u_x) - v_t \theta_2 - 2\beta \theta_1 u_{xt} \quad (34)$$

$$= \left( \theta_1(\alpha(v - u_x)x + \beta u_x) + \theta_2(-\lambda v_{xx} + \mu(v - u_x)) + 2\beta u_t \theta_1x \right),$$

$$C^2 = -u_t[\beta \theta_1 + D_x(\alpha \theta_1) - D_t(2\beta \theta_1)] - v_t[\alpha \theta_1 + D_x(\lambda \theta_2)]$$

$$+ D_x(-u_t)[2\beta \theta_1] + D_t(-u_t)[2\beta \theta_1] + D_x(-u_t)[-\alpha \theta_1], \quad (35)$$

$$+ D_x(-v_t)[-\lambda \theta_2] = (-\beta u_t - \alpha v_t - 2\beta u_{xt} - 2\beta u_t + \alpha u_x) \theta_1$$

$$- \alpha u_t \theta_1 x + 2\beta u_t \theta_1 x - \lambda v_t \theta_2 x + \lambda v_x \theta_2.$$

For the operator $V = \partial_x$, one gets

$$\xi^2 = 1, \xi^1 = \eta^1 = \eta^2 = 0, W^1 = -u_x, W_2 = -v_x. \quad (36)$$

For this case, we get the following components:

$$C^1 = -u_x \theta_1 + 2\beta u_x \theta_1 x - v_x \theta_2 - 2\beta u_{xx} \theta_1, \quad (37)$$
$C^2 = \left( \theta_1(u_t + \alpha(v - u_x)_x + 2\beta u_{xt} + \beta u_x) + \theta_2(v_t - \lambda v_{xx} + \mu(v - u_x)) \right)
- u_x[\beta \theta_1 + D_x(\alpha \theta_1) - D_t(2\beta \theta_1)] - v_x[\alpha \theta_1 + D_x(\lambda \theta_2)]
+ D_x(-u_x)[2\beta \theta_1] + D_t(-u_x)[2\beta \theta_1] + D_x(-u_x)(-\alpha \theta_1) + D_x(-v_x)[-\lambda \theta_2]
= \left( \theta_1(u_t + \alpha(v - u_x)_x + 2\beta u_{xt} + \beta u_x) + \theta_2(v_t - \lambda v_{xx} + \mu(v - u_x)) \right)
- \beta u_x \theta_1 - \alpha u_x \theta_1 x + 2\beta u_x \theta_1 t - \alpha v_x \theta_1 x - \lambda v_x \theta_2 x - 2\beta u_x \theta_1
+ \alpha u_x \theta_1 x + \lambda v_x \theta_2 x
= (u_t - 2\beta u_{xx})\theta_1 + [v_t + \mu(v - u_x)]\theta_2 - \alpha u_x \theta_1 x + 2\beta u_x \theta_1 t - \lambda v_x \theta_2 x.

This vector involves arbitrary solutions $\theta_1, \theta_2$ of the adjoint equation (28), so we can get an infinite number of conservation laws.

**Remark.** For others generators, we can also get the corresponding conservation laws; however, we do not list them here.

**5. CONCLUSION**

In this paper, we have investigated a coupled partial differential system of equations describing the carbon nanotubes conveying fluid. All the symmetries of the system are constructed. Finally, the conservation laws are presented. To the best of our knowledge, these conservation laws were not reported before. The obtained results are important for the understanding of some practical physical problems.

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