APPLICATION OF FRACTIONAL VARIATIONAL ITERATION METHOD FOR
SOLVING FRACTIONAL FOKKER-PLANCK EQUATION

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In this paper, the fractional variational iteration method is applied to obtain an analytical approximate solution for the time and space fractional Fokker-Planck equation. We illustrated the applicability and efficiency of the FVIM for solving fractional Fokker-Planck equation by two examples. The results from these examples show that the method is both simple and powerful for solving the fractional Fokker-Planck equation involving Jumarie’s modified Riemann-Liouville derivative.

Key words: Fractional Fokker-Planck equation, fractional variational iteration method (FVIM), Jumarie’s modified Riemann-Liouville derivative, Mittag-Leffler function.

1. INTRODUCTION

Fractional calculus and fractional differential equations (FDEs) have played a significant role to describe a variety of problems in many areas of science and engineering such as fluid mechanics, viscoelasticity, physics, control theory, robotics, signal processing, electromagnetism, electrochemistry, etc.[1–4]. Except in a limited number of these equations, it is difficult to find their exact or approximate solutions. Therefore, finding exact or approximate solutions of FDEs is very important and some methods have been proposed to solve them, e.g. Adomian decomposition method [5–7], variational iteration method [8–11], Laplace transform method [12], Fourier method [13], homotopy analysis method [14–16], homotopy perturbation method [17, 18], Haar wavelet method [19], differential transform method [20], finite difference method [21], and local fractional variational iteration method [22, 23], and various collocation methods [24–26].

The variational iteration method (VIM), introduced by He [27], has been widely used to obtain approximate solutions of linear and nonlinear problems arising as ordinary or partial differential equations of integer or fractional order in science and engineering. Recently, fractional variational iteration method (FVIM), which is a modification of VIM for improving the efficiency and accuracy, has been proposed and successful results have been achieved [28–30]. The essential difference between VIM and FVIM is the determination of Lagrange multiplier and construction of the correction functional. In solving FDEs with VIM, the terms with the fractional derivative
are considered as a restricted variation in determination Lagrange multiplier and this multiplier is usually determined by variational theory and some approximations. Lagrange multiplier identified in this approach are not good enough to achieve the approximate solutions of high accuracies. To overcome this problem, FVIM is improved by Wu and Lee [28] and proved to be an efficient tool for solving FDEs. FVIM gives Lagrange multiplier in a more accurate way and establishes the correction functional with the help of this multiplier. So, FVIM provides an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence.

The Fokker-Planck equation (FPE) was first used by Fokker and Planck to investigate Brownian motion and the diffusion model of chemical reactions. FPE arises in various fields such as quantum optics, chemical physics, polymer physics, laser physics, solid-state physics, biophysics, engineering, economics, etc. [32, 33]. The general FPE is described as the following form:

\[
\frac{\partial u(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x)\right] u(x,t), \quad (1)
\]

where \( A(x) \) and \( B(x) \) represent the drift and the diffusion coefficients respectively. The drift and diffusion coefficients may also depend on time. (1) is the equation of motion for the distribution function \( u(x,t) \).

There is a more general form of FPE which is named the nonlinear FPE. This equation has significant applications in engineering and physics [33]. The nonlinear FPE is described as the following form:

\[
\frac{\partial u(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t,u) + \frac{\partial^2}{\partial x^2} B(x,t,u)\right] u(x,t), \quad (2)
\]

In this paper, we extend the application of the fractional variational iteration method to obtain analytical approximate solution the fractional FPE. For this, the following time and space fractional FPE, which is the generalization of (2) by replacing the integer-order derivatives with fractional-order derivatives, has been discussed:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta} A(x,t,u) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(x,t,u)\right] u(x,t), \quad t > 0, \quad 0 < \alpha, \beta \leq 1 \quad (3)
\]

subject to the initial condition

\[
u(x,0) = f(x), \quad x > 0, \quad (4)
\]

where \( \alpha \) and \( \beta \) represent the order of the time and space fractional derivatives respectively.

In recent years, a variety of numerical and analytical methods have been applied to solve the fractional FPE such as Adomian decomposition method [34, 35], variational iteration method [34, 36], finite difference method [37], homotopy perturbation method [38], fractional finite element method [39], fractional implicit trape-
zoidal method [40], Chebyshev spectral collocation method [41], iterative Laplace transform method [42], and local fractional variational iteration method [43].

2. PRELIMINARIES

In this section, some basic definitions and properties of the fractional calculus theory, which could be found in [44, 45], have been reviewed.

**Definition 2.1.** The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (5)$$

**Definition 2.2.** Limit form of the Jumarie’s fractional derivative is defined as:

$$f^{(\alpha)} = \lim_{h \to 0} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha} \quad (6)$$

**Definition 2.3.** The Riemann-Liouville fractional integral operator ($I_\alpha^\alpha$) of order $\alpha > 0$ for a function $f \in C_\mu, \mu \geq -1$ is defined as:

$$0 I_\alpha^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds \quad (7)$$

$I^\alpha$ integral operator provides the following properties for $\alpha, \beta \geq 1$ and $\gamma \geq -1$:

$$I_\alpha^\alpha I_\beta^\beta f(x) = I_\beta^\beta I_\alpha^\alpha f(x) = I_\alpha^\alpha+\beta f(x) \quad (8)$$

$$I_\alpha^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \quad (9)$$

**Definition 2.4.** The Jumarie’s modified Riemann-Liouville fractional derivative of $f(x)$ is defined as:

$$0 D_\alpha^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-s)^{m-\alpha} (f(s) - f(0)) ds \quad (10)$$

where $x \in [0, 1], m - 1 \leq \alpha < m$ and $m \geq 1$.

The modified Riemann-Liouville fractional derivative provides the following properties:

(i) Product rule for fractional derivatives:

$$0 D_\alpha^\alpha (f \cdot g) = 0 D_\alpha^\alpha f \cdot g + f \cdot 0 D_\alpha^\alpha g \quad (11)$$

(ii) Fractional Leibniz Formulation:

$$0 I_\alpha^\alpha 0 D_\alpha^\alpha f(x) = f(x) - f(0), 0 < \alpha \leq 1 \quad (12)$$
(iii) Integration by parts for fractional order:

\[ aI_0^\alpha \left( f^{(\alpha)}g \right) = \left( f \cdot g \right)_a^b - aI_0^\alpha \left( f \cdot g^{(\alpha)} \right) \]  

(13)

**Definition 2.5.** Fractional derivative is defined for compounded functions as follows:

\[ d^\alpha f \equiv \Gamma(1 + \alpha)df, \quad 0 < \alpha \leq 1 \]  

(14)

**Definition 2.6.** The integral with respect to \((dx)^\alpha\) is defined as the solution of the FDE:

\[ dy \equiv f(x)(dx)^\alpha, \quad x \geq 0, \quad y(0) = 0, \quad 0 < \alpha \leq 1 \]  

(15)

**Lemma 2.1.** Let \(f(x)\) represent a continuous function, then the solution of (14) is defined as:

\[ y = \int_0^x f(s)(ds)^\alpha = \alpha \int_0^x (x-s)^{\alpha-1}f(s)ds, \quad 0 < \alpha \leq 1 \]  

(16)

For example, when (16) is applied for function \(f(x) = x^\beta\),

\[ \int_0^x f(s)(ds)^\alpha = \int_0^x s^\beta (ds)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}x^{\alpha+\beta}, \quad 0 < \alpha \leq 1 \]  

(17)

3. FRACTIONAL VARIATIONAL ITERATION METHOD (FVIM)

In order to describe the solution procedure of FVIM, we take (3) as an example. According to VIM, the correction functional for (3) is constructed as follows:

\[ u_{n+1}(x,t) = u_n(x,t) + I^\alpha \left\{ \lambda(x,t) \left( \frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} + \left[ \frac{\partial^\beta}{\partial x^\beta}A(x,t,u_n(x,t)) \right] \bar{u}_n(x,t) \right\} \right. \]  

(18)

where \(\lambda\) is Lagrange multiplier which can be identified optimally via the variational theory. Here, \(\bar{u}_n\) is considered as restricted variations, \(i.e. \delta \bar{u}_n = 0\). Using (7), (18) yields the following equality:

\[ u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(x,s) \left( \frac{\partial^\alpha u_n(x,s)}{\partial s^\alpha} \right. \]  

\[ \left. + \left[ \frac{\partial^\beta}{\partial x^\beta}A(x,s,u_n(x,s)) - \frac{\partial^2\beta}{\partial x^2\beta}B(x,s,u_n(x,s)) \right] \bar{u}_n(x,s) \right) ds \]  

(19)
By combining (16) and (19), the new correction functional is written as follows:

\[
\begin{align*}
    u_{n+1}(x,t) &= u_n(x,t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(x,s) \left( \frac{\partial^\alpha u_n(x,s)}{\partial s^\alpha} \right) \\
    &\quad + \left[ \frac{\partial^\beta}{\partial x^\beta} A(x,s,u_n(x,s)) - \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(x,s,u_n(x,s)) \right] \tilde{u}_n(x,s) (ds)^\alpha
\end{align*}
\]  
(20)

Making the above correction functional stationary,

\[
\begin{align*}
    \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \frac{\delta}{\Gamma(\alpha+1)} \int_0^t \lambda(x,s) \left( \frac{\partial^\alpha u_n(x,s)}{\partial s^\alpha} \right) \\
    &\quad + \left[ \frac{\partial^\beta}{\partial x^\beta} A(x,s,u_n(x,s)) - \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(x,s,u_n(x,s)) \right] \tilde{u}_n(x,s) (ds)^\alpha
\end{align*}
\]  
(21)

or equivalently

\[
\begin{align*}
    \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \lambda \delta u_n(x,s) \big|_{s=t} \\
    &\quad - \frac{1}{\Gamma(\alpha+1)} + \int_0^t \frac{\partial^\alpha \lambda(x,s)}{\partial s^\alpha} u_n(x,s) (ds)^\alpha
\end{align*}
\]  
(22)

with the property from (11) and (13), must satisfy

\[
\frac{\partial^\alpha \lambda(x,s)}{\partial s^\alpha} = 0 , \quad 1 + \lambda(x,s) \big|_{s=t} = 0
\]  
(23)

Therefore, \( \lambda(x,s) \) can be identified as:

\[
\lambda(x,s) = -1
\]  
(24)

Substituting (24) into the correction functional (20), the following iteration formula is obtained:

\[
\begin{align*}
    u_{n+1}(x,t) &= u_n(x,t) - \frac{1}{\Gamma(\alpha+1)} \int_0^t \left( \frac{\partial^\alpha u_n(x,s)}{\partial s^\alpha} \right) \\
    &\quad + \left[ \frac{\partial^\beta}{\partial x^\beta} A(x,s,u_n(x,s)) \right] \tilde{u}_n(x,s) (ds)^\alpha
\end{align*}
\]  
(25)

Taking the initial condition in (4) as initial approximation \( u_0(x,t) \), the successive approximations \( u_n(x,t) \), for \( n \geq 1 \), can be easily achieved. Consequently, the approximate solution of (3) is computed by \( u(x,t) = \lim_{n \to \infty} u_n(x,t) \).

4. ILLUSTRATIVE EXAMPLES

In this section, in order to show the applicability and efficiency of FVIM for solving time and space fractional FPE, the following illustrative examples are given.
Example 4.1. Consider the following nonlinear time-fractional FPE subject to the initial condition [34].

\[
\begin{align*}
\left\{ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \left[ -\frac{\partial}{\partial x} \left( \frac{4u(x,t)}{x} \right) - \frac{x}{3} \right] + \frac{\partial^2}{\partial x^2} u(x,t) \right] u(x,t) \\
u(x,0) &= x^2
\end{align*}
\]

(26)

where \( t > 0, 0 < \alpha, \beta \leq 1 \).

From (25), the corresponding iteration formula for (26) can be derived as:

\[
\begin{align*}
u_{n+1}(x,t) &= u_n(x,t) - \frac{1}{\Gamma(\alpha+1)} \int_0^t \left( \frac{\partial^\alpha u_n(x,s)}{\partial s^\alpha} \right) ds + \left[ \frac{\partial}{\partial x} \left( \frac{4u_n(x,s)}{x} \right) - \frac{x}{3} \right] \frac{\partial^2 u_n(x,s)}{\partial x^2} u_n(x,s) \right) (ds)^\alpha
\end{align*}
\]

(27)

Starting with the initial approximation \( u_0(x,t) = u(x,0) = x^2 \) and using the iteration formula (27), the following successive approximations are obtained:

\[
u_1(x,t) = x^2 \left( 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} \right) \\
u_2(x,t) = x^2 \left( 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) \\
u_3(x,t) = x^2 \left( 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right) \\
\vdots \\
u_n(x,t) = x^2 \left( 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \right)
\]

Consequently, the solution of (26) is obtained as follows:

\[
u(x,t) = \lim_{n \to \infty} u_n(x,t) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^2}{\Gamma(k\alpha + 1)} = x^2 E_\alpha(t^\alpha)
\]

(28)

where \( E_\alpha(t^\alpha) \) is the Mittag-Leffler function.

Example 4.2. Consider the following linear time and space fractional FPE with the initial condition [30, 38].

\[
\begin{align*}
\left\{ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \left[ -\frac{\partial^\beta}{\partial x^\beta} \left( \frac{x}{6} \right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} \left( \frac{x^2}{12} \right) \right] u(x,t) \\
u(x,0) &= x^2
\end{align*}
\]

(29)

where \( t > 0, 0 < \alpha, \beta \leq 1 \).
According to formula (25), the corresponding iteration formula for (29) is given as:

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(\alpha + 1)} \int_0^t \left( \frac{\partial^\alpha u_n(x,s)}{\partial s^\alpha} \right) \frac{1}{\Gamma(\alpha + 1)} \int_0^t ds \left( \frac{x}{6} - \frac{x^2}{12} \right) u_n(x,s) (ds)^\alpha$$

Starting with the initial approximation $u_0(x,t) = u(x,0) = x^2$ and using the formula (30), the following successive approximations are derived:

$$u_1(x,t) = x^2 + \left[ \frac{2(x^4 - 2\beta)}{\Gamma(5 - 2\beta)} - \frac{x^{3-\beta}}{\Gamma(4 - \beta)} \right] \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} \right)$$

$$u_2(x,t) = x^2 + \left[ \frac{2(x^4 - 2\beta)}{\Gamma(5 - 2\beta)} - \frac{x^{3-\beta}}{\Gamma(4 - \beta)} \right] \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) + \left[ \frac{(4-\beta)x^{4-2\beta}}{6\Gamma(5 - 2\beta)} - \frac{(\beta^2 - 17\beta + 40)x^{5-3\beta}}{12\Gamma(6 - 3\beta)} + \frac{(5-2\beta)(3-\beta)x^{6-4\beta}}{3\Gamma(7 - 3\beta)} \right] \left( \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \right)$$

and so on, in the same manner the rest of components of the iteration formula (30) can be obtained. Setting $\alpha = \beta = 1$ in FVIM solution, the approximate solution of (29) is obtained as follows:

$$u(x,t) = x^2 \left( 1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48} + \frac{t^4}{384} + \ldots \right)$$

which converges to the exact solution $u(x,t) = x^2 e^{t/2}$. The numerical results of this application for different values of $\alpha$ and $\beta$ are compared with exact and other methods in literature are given in Table 1. In order to achieve a high level of accuracy by FVIM, many terms can be computed.

5. CONCLUSION

In this paper, the fractional variational iteration method is implemented to obtain analytical approximate solution of the fractional FPE. The obtained results denote that this method is effective, powerful, and convenient in solving the fractional FPE in an easier and more accurate way and it can be considered as an alternative to the other methods in the literature in terms of the purpose of solving this equation. As shown in the implementation of FVIM for solving fractional FPE, we may conclude that this method will be very useful for solving various FDEs.
Table 1

The numerical comparison of (29) for different values of $\alpha$ and $\beta$.

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REFERENCES

38. A. Yildirim, J. King Saud University-Science 22(4), 257-264 (2010).