A CLASSIFICATION OF ZERO GAUGE NOETHER SYMMETRIES FOR THE
WAVE EQUATION ON CYLINDRICALLY SYMMETRIC STATIC MANIFOLDS

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\textit{Received March 2, 2015}

Symmetry analysis of a wave equation on a class of cylindrical symmetric static
space-times (CSSS) is performed. The results are classified according to the Noether
symmetries and metrics spaces. Symmetry algebras are found in flat metrics and in
non-flat background metrics. It turns out that the wave equation on these metrics admit
strict (zero gauge) Noether symmetry groups of dimension 5, 6, 7, 8 and 12. Conserved
forms for the wave equation are constructed by the application of Noether’s theorem.

\textit{Key words:} Symmetries, wave equations, cylindrical metrics.

\textit{PACS:} 04.20.Fy, 02.30.Xx, 02.30.Jr, 02.40.-k.

A CSSS metric written in generalized cylindrical coordinates, is given by the
line element
\[ ds^2 = e^{\nu(p)} dt^2 - dp^2 - e^{\lambda(p)} a^2 d\theta^2 - e^{\mu(p)} dz^2, \tag{1} \]
where \( a \) has units of length. The usual cylindrically symmetric spaces have an axis
definable by the limit \( p \to 0 \). This is analogous to the spherically symmetric spaces
having an origin as \( r \to 0 \).

The CSSS metric in cosmology has been discussed widely in the literature, in
many different contexts. We note that the concept of symmetries on metrics, in particu-
lar, the CSSS metric, has been looked at in the context of other symmetries such as
homothetic and Killing vectors [1, 2]. A classification of teleparallel conformal sym-
metries in CSSS was done in [3]. It is well known that this metric admits the group
of motions \( \text{SO}(2) \otimes \mathbb{R}^2 \), where \( \text{SO}(2) = (\partial/\partial \theta) \), one \( R = (\partial/\partial z) \) and the other \( R = (\partial/\partial t) \), as the minimal isometry group. Of particular relevance to our work, is the
study done by Qadir et al. [2], wherein the authors provided the various metrics that
we consider here. However, the purpose of their study was to investigate homo-
theties according to special metric functions for \( \lambda, \mu \) and \( \nu \).

Many studies have been devoted to studying differential equations (DEs) in terms of the Lie point symmetries admitted by them [4, 5]. These symmetries play an important role in finding exact analytic solutions of the nonlinear DEs. Noether symmetries are also widely studied and are associated with those DEs which possess a Lagrangian. Emmy Noether [6] discovered the interesting link between symmetries and conservation laws, showing that for every infinitesimal transformation admitted by the action integral of a system, there exists a conservation law. Symmetry classifications have been performed extensively for various equations in engineering and physics. For example, the symmetry classification of geodesic equations of Riemannian spaces [7] or the symmetry classification of the wave/Klein-Gordon equations [8–15]. In this paper, we investigate and classify the Noether symmetries and conservation laws of the wave equation in CSSS.

The paper is organized as follows. We briefly discuss the mathematical tools required to tackle our investigation. In section 2, we classify the Noether symmetries of the wave equation according to some interesting classes of metrics. Section 3 describes some associated conservation laws.

We introduce some preliminaries about the Noether symmetries. Consider an \( r \)-th-order system of PDEs of \( n \) independent variables \( x = (x^1, x^2, \ldots, x^n) \) and \( m \) dependent variables \( u = (u^1, u^2, \ldots, u^m) \)

\[
G^\mu(x, u, u^{(1)}, \ldots, u^{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m},
\]

where \( u^{(1)}, u^{(2)}, \ldots, u^{(r)} \) denote the collections of all first, second, \ldots, \( r \)-th-order partial derivatives.

Suppose \( \mathcal{A} \) is the vector space of differential functions. A symmetry operator is given by

\[
X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_s}},
\]

where \( \xi^i, \eta^\alpha \in \mathcal{A} \) and the additional coefficients are determined uniquely by the prolongation formulae

\[
\begin{align*}
\zeta^\alpha_i &= D_i (W^\alpha) + \xi^j u^\alpha_{ij}, \\
\zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s} (W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1.
\end{align*}
\]

where the total differentiation operator \( D_i \) with respect to \( x^i \) is given by

\[
D_i = \frac{\partial}{\partial x^i} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_{ij}} + \ldots, \quad i = 1, \ldots, n.
\]

and \( W^\alpha \) is the characteristic function given by

\[
W^\alpha = \eta^\alpha - \xi^j u^\alpha_{j}.
\]
In this paper, we will assume that $X$ is a point symmetry operator, i.e., $\xi$ and $\eta$ are functions of $x$ and $u$ and are independent of derivatives of $u$.

The Euler-Lagrange equations, if they exist, associated with (2) is the system

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m,$$

where $\frac{\delta}{\delta u^\alpha}$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \quad (7)$$

Here $L$ is referred to as a Lagrangian and a Noether symmetry operator $X$ of $L$ arises from a study of the invariance properties of the associated functional

$$L = \int L(x, u, u^{(1)}, \ldots, u^{(r)}) dx.$$ 

If we include point dependent gauge terms $f_1, \ldots, f_n$, the Noether symmetries $X$ are given by (see [5])

$$X(L) + LD_i(\xi^i) = D_i(f^i). \quad (8)$$

If $f^i = 0, (i = 1, \ldots, n)$, then $X$ is referred to as a strict Noether symmetry. For more details on Noether symmetries, refer to [16–18].

1. THE WAVE EQUATION

It is well known that the standard wave equation in flat space is a linear equation. Nonlinear equations and, in particular, nonlinear wave equations arise by the model under study or, usually, via the medium that the wave operates on. Also, nonlinearities may be studied by imposing the wave equation and, indeed any equation of physics (where possible), on a curved manifold. To this end, our analysis is now directed at the nonlinear wave equation in CSSS.

The wave equation on (1) is generated by

$$\Box u = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{|-g|} g^{ik} \frac{\partial u}{\partial x^k} \right) = 0, \quad (9)$$

where $\Box$ refers to the d’Alembertian operator.

The wave equation written out explicitly in coordinates is

$$-u_{pp} - \frac{1}{a^2} e^{-\lambda(p)} u_{\theta \theta} - e^{-\mu(p)} u_{zz} + e^{-\nu(p)} u_{tt}$$

$$-\frac{1}{2} (\lambda'(p) + \mu'(p) - \nu'(p)) u_p = 0. \quad (10)$$

The corresponding Lagrangian for (10) is

$$L = -\frac{1}{2} e^{\frac{1}{2} (\lambda(p) + \mu(p) - 3\nu(p))} u_t^2 + \frac{1}{2} e^{\frac{1}{2} (\lambda(p) + \mu(p) - \nu(p))} u_p^2$$

$$+ \frac{1}{2} e^{\frac{1}{2} (\lambda(p) - \mu(p) - \nu(p))} u_z^2 + \frac{1}{2a^2} e^{\frac{1}{2} (\lambda(p) + \mu(p) - \nu(p))} u_\theta^2. \quad (11)$$
Now we can calculate the Noether point symmetries for \( L(t, p, \theta, z, u, u_t, u_p, u_\theta, u_z) \).

Let

\[ X = \tau \partial_t + \xi \partial_p + \eta \partial_\theta + \gamma \partial_z + \phi \partial_u \]

be a Noether point operator that satisfies (8), where \( \tau, \xi, \eta, \gamma \) and \( \phi \) depend on \( t, p, \theta, z, u \). We invoke (3) up to first-order derivatives together with (6)

\[ X^{[1]} = X + \zeta_t \frac{\partial}{\partial u_t} + \zeta_p \frac{\partial}{\partial u_p} + \zeta_\theta \frac{\partial}{\partial u_\theta} + \zeta_z \frac{\partial}{\partial u_z}, \]

(12)

where

\[ \begin{align*}
\zeta_t &= D_t \phi - u_tD_t \tau - u_pD_t \xi - u_\theta D_t \eta - u_z D_t \gamma, \\
\zeta_p &= D_p \phi - u_tD_p \tau - u_pD_p \xi - u_\theta D_p \eta - u_z D_p \gamma, \\
\zeta_\theta &= D_\theta \phi - u_tD_\theta \tau - u_pD_\theta \xi - u_\theta D_\theta \eta - u_z D_\theta \gamma, \\
\zeta_z &= D_z \phi - u_tD_z \tau - u_pD_z \xi - u_\theta D_z \eta - u_z D_z \gamma, \\
D_t &= \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tp} \partial_{u_p} + u_{t\theta} \partial_{u_\theta} + u_{tz} \partial_{u_z} + \ldots, \\
D_p &= \partial_p + u_p \partial_u + u_{tp} \partial_{u_p} + u_{p\theta} \partial_{u_\theta} + u_{pz} \partial_{u_z} + \ldots, \\
D_\theta &= \partial_\theta + u_\theta \partial_u + u_{t\theta} \partial_{u_t} + u_{p\theta} \partial_{u_p} + u_{z\theta} \partial_{u_z} + \ldots, \\
D_z &= \partial_z + u_z \partial_u + u_{tz} \partial_{u_t} + u_{pz} \partial_{u_p} + u_{z\theta} \partial_{u_\theta} + u_{zz} \partial_{u_z} + \ldots.
\end{align*} \]

(13)

from Eqs.(4) and (5). Using (8), the determining equation for the Noether symmetries of the wave equation is

\[ X^{[1]} L + L(D_t \tau + D_p \xi + D_\theta \eta + D_z \gamma) = D_t f^1 + D_p f^2 + D_\theta f^3 + D_z f^4, \]

(14)

where \( f^i = f^i(t, p, \theta, z, u) \) is the gauge function. If the gauge function is zero, we will obtain strict Noether symmetries.

Equation (10) with \( \lambda, \mu \) and \( \nu \) being arbitrary functions of \( p \), admits a 4-dimensional Noether algebra of point symmetry generators with basis (Noether symme-
tries) given by

\[ X_1 = \partial_u, \quad X_2 = \partial_t, \quad X_3 = \partial_z \quad X_4 = \partial_\theta. \]

(15)

The symmetry generators (15) form a principal Noether algebra and is con-
tained in all the cases (a) to (j) considered next. Now we will classify the strict Noether symmetries of the wave equation (10) according to some interesting cases of the CSSS. Below, \( p_0 \) is a constant with units of length and \( m, s \) are arbitrary (real) parameters.

There are three metrics, cases (a) to (c) below, wherein the wave equation (10) will admit a 12-dimensional Noether symmetry algebra. Within these algebras, ten of the zero gauge Noether symmetries form an algebra of isometries, keeping in mind that \( n = 4 \) corresponds to the maximal \( \frac{1}{2}n(n + 1) = 10 \)-dimensional algebra, \( SO(1,4) \).

(a) \( \lambda = \nu = \mu = 0 \).
In this case the metric in eq. (1) reduces to Minkowski space. This metric was called the “wrapped Minkowski space” in the classification of CSSS according to their isometries [19]. In [20, 21], the Minkowski space written in cylindrical coordinates acquired special interest with its application to ‘topological field theories. This metric could alternatively be identified as a CSSS, having a line singularity at \( \theta = 0, 2\pi \), with a topological defect [2].

\[
X_0 = \partial_p, \\
X_1 = -a^2 \theta \partial_p + p \partial_\theta, \\
X_2 = t \partial_p + p \partial_t, \\
X_3 = t \partial_z + z \partial_t, \\
X_4 = -z \partial_p + p \partial_z, \\
X_{10} = -\frac{\alpha}{\pi} \partial_\theta + \theta \partial_z, \\
X_{11} = -p \partial_p - \theta \partial_\theta - t \partial_t + u \partial_u, \\
X_{12} = \frac{\alpha}{\pi} \partial_\theta + \theta \partial_t.
\]

(b) \( \lambda = \nu = 0, \quad \mu = 2 \ln \left( \frac{p}{p_0} \right) \).

\[
X_2 = -\frac{1}{p_0} \cos \left( \frac{\theta}{p_0} \right) \partial_\theta + \frac{1}{p} \sin \left( \frac{\theta}{p_0} \right) \partial_z, \\
X_6 = \frac{\alpha}{p_0} \sin \left( \frac{\theta}{p_0} \right) \partial_\theta + \frac{1}{p} \cos \left( \frac{\theta}{p_0} \right) \partial_z, \\
X_9 = \frac{1}{p_0} \cos \left( \frac{\theta}{p_0} \right) \partial_\theta + \frac{1}{p} \sin \left( \frac{\theta}{p_0} \right) \partial_z + \cos \left( \frac{\theta}{p_0} \right) \partial_t, \\
X_8 = \frac{1}{p_0} \sin \left( \frac{\theta}{p_0} \right) \partial_\theta + \frac{1}{p} \cos \left( \frac{\theta}{p_0} \right) \partial_t, \\
X_{10} = \frac{1}{p_0} \cos \left( \frac{\theta}{p_0} \right) \partial_\theta + \frac{1}{p} \sin \left( \frac{\theta}{p_0} \right) \partial_t, \\
X_{11} = -p \partial_p + \theta \partial_\theta + \theta \partial_z + t \partial_t + u \partial_u, \\
X_{12} = \frac{\alpha}{\pi} \partial_\theta + \theta \partial_t.
\]

(c) \( \lambda = \nu = \mu = 2m \ln \left( \frac{p}{p_0} \right), \quad (m \neq 0, 1) \).

This metric corresponds to the Petrov type D, Segré type [1,(11)1] space and represents a tachyonic fluid and could be reinterpreted as an anisotropic fluid with an appropriately chosen cosmological constant [2].

\[
X_3 = -\frac{p}{m-1} \partial_p - \theta \partial_\theta - \theta \partial_z - t \partial_t + u \partial_u, \\
X_4 = \frac{1}{m-1} \partial_\theta + \theta \partial_t, \\
X_5 = i \partial_z + z \partial_t, \\
X_6 = -\frac{\alpha}{\pi} \partial_\theta + \theta \partial_z.
\]

Eq.(10) constructed on the next metric yields an 8-dimensional Noether symme-
try algebra.

(d) \( \lambda = \mu = 0, \quad \nu = 2 \ln(\frac{p}{p_0}) \).

\[
\begin{align*}
X_5^4 &= -2zp\partial_p - 2\theta z\partial_\theta + (a^2\theta^2 + p^2 - z^2)\partial_z, \\
X_6^4 &= p\theta \partial_p + \frac{a^2\theta^2 - p^2 - z^2}{2a^2} \partial_\theta + \theta z\partial_z, \\
X_7^4 &= p\partial_p + \theta \partial_\theta + z\partial_z, \\
X_8^4 &= -\frac{z}{a^2} \partial_\theta + \theta \partial_z. 
\end{align*}
\]

The two metrics below correspond to Petrov type D, Segré type [1,111] space and the wave equation admits a 7-dimensional Noether symmetry algebra in each case.

(e) \( \lambda = 2\ln(\frac{p}{a}), \quad \mu = 0, \quad \nu = 2\ln(\frac{p}{p_0}) \).

\[
\begin{align*}
X_5^5 &= -2zp\partial_p + (p^2 - z^2)\partial_z, \\
X_6^5 &= \frac{1}{p_0^2} \partial_\theta + \theta \partial_t, \\
X_7^5 &= p\partial_p + z\partial_z. 
\end{align*}
\]

(f) \( \lambda = 0, \quad \mu = \nu = 2\ln(\frac{p}{p_0}) \).

\[
\begin{align*}
X_5^6 &= -2a^2\theta p\partial_p - (a^2\theta^2 - p^2)\partial_\theta, \\
X_6^6 &= t\partial_\theta + z\partial_t, \\
X_7^6 &= p\partial_\theta + \theta \partial_\theta. 
\end{align*}
\]

Cases (g) and (h) each admit a 6-dimensional Noether symmetry algebra for the wave equation.

(g) \( \lambda = \mu = 2\ln(\frac{p}{a}), \quad \nu = 0 \).

This metric is the Petrov type D, Segré type [(1,1)(11)] space and represents a non-null electromagnetic field [2].

\[
\begin{align*}
X_5^7 &= -p\partial_p - t\partial_t + u\partial_u, \\
X_6^7 &= -\frac{1}{a^2} \partial_\theta + \theta \partial_\theta. 
\end{align*}
\]

(h) \( \lambda = \mu = 2s\ln(\frac{p}{p_0}), \quad \nu = 2m\ln(\frac{p}{p_0}) \) (\( m \neq s, \quad m, s \neq 0, 1 \)).

Here, we have a Petrov type D and Segré type [1,(11)1] space. This particular metric represents a perfect fluid for \( s = \frac{m(m-1)}{m+1} \), whereas it represents a non-null electromagnetic field for \( s = m + 1 \), and both conditions are satisfied when \( s = \frac{2}{3} \) and \( m = -\frac{1}{3} \) [2].

\[
\begin{align*}
X_5^8 &= \frac{p}{m-1} \partial_p - \frac{\theta(sl-1)}{m-1} \partial_\theta - \frac{z(s-1)}{m-1} \partial_z - t\partial_t + u\partial_u, \\
X_6^8 &= -\frac{z}{a^2} \partial_\theta + \theta \partial_z. 
\end{align*}
\]
In cases (i) and (j), we find that Eq.(10) admits a 5-dimensional Noether symmetry algebra in each case.

(i) $\lambda = 0, \mu = 2\ln\left(\frac{p}{p_0}\right), \nu = 2m\ln\left(\frac{p}{p_0}\right) \quad (m \neq 0, 1)$.

This case corresponds to Petrov type I and Segré type [1,111] space.

$$X^9_5 = \frac{p}{m-1} \partial_p + \frac{\theta}{m-1} \partial_\theta - t \partial_t + u \partial_u.$$  

(j) $\lambda = 2\ln\left(\frac{p}{a}\right), \mu = 0, \nu = 2m\ln\left(\frac{p}{p_0}\right) \quad (m \neq 0, 1)$.

This case corresponds to Petrov type I and Segré type [1,111].

$$X^{10}_5 = \frac{p}{m-1} \partial_p + \frac{z}{m-1} \partial_z - t \partial_t + u \partial_u.$$  

As a result of this classification, we see that there exists cases of five, six, seven, eight and twelve strict Noether symmetry groups. Each of these symmetry groups contain the isometries of the respective metrics.

2. CONSERVED FORMS

Now that we have determined the Noether symmetries, one is able to determine the respective conserved currents. A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0$$  

along the solutions of (2). Corresponding to each $X$, there exists a conserved vector $T = (T^1, \ldots, T^n)$ that may then be determined by the Noether’s theorem. A symmetry operator $X$ is a Noether symmetry of a Lagrangian $L$ corresponding to an Euler-Lagrange differential equation if and only if the characteristic $W = (W^1, \ldots, W^n)$ of $X$ is also the characteristic of the conservation law (24), where

$$T^i = f^i - N^i(L), \quad i = 1, \ldots, n$$  

of the Euler-Lagrange equation. Here $N^i$ is the Noether operator associated with the symmetry operator $X$ given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s} W^\alpha \frac{\delta}{\delta u^{\alpha}_{i_1 \ldots i_s}},$$  

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by (7).

Below is an example of a conserved form associated with the Noether symmetry $X_2 = \partial_t$ given in (15), where $T = (T^p, T^\theta, T^z, T^t)$ is the conserved vector and these components satisfy the relation from (24)

$$D_i T^i + D_\theta T^p + D_\theta T^\theta + D_z T^z = 0.$$
Thus one can write the components of the Noether conserved vector from (25) as

\[ T^t = f^1 - \tau L - (\phi - \tau u_t - \xi u_p - \eta u_\theta - \gamma u_z) \frac{\partial L}{\partial u_t}, \]
\[ T^p = f^2 - \xi L - (\phi - \tau u_t - \xi u_p - \eta u_\theta - \gamma u_z) \frac{\partial L}{\partial u_p}, \]
\[ T^\theta = f^3 - \eta L - (\phi - \tau u_t - \xi u_p - \eta u_\theta - \gamma u_z) \frac{\partial L}{\partial u_\theta}, \]
\[ T^z = f^4 - \gamma L - (\phi - \tau u_t - \xi u_p - \eta u_\theta - \gamma u_z) \frac{\partial L}{\partial u_z}. \]

(27)

The conserved vector components for \( X_2 \) given in Eq.(15) are

\[ T^p_2 = \frac{1}{2} e^{\frac{1}{2}(\lambda(p) - \nu(p) + \mu(p))} (-u_t u_p + u_{u_p}), \]
\[ T^\theta_2 = \frac{1}{2a^2} e^{\frac{1}{2}(\lambda(p) - \nu(p) + \mu(p))} (-u_\theta u_t + u_{u_\theta}), \]
\[ T^z_2 = \frac{1}{2} e^{\frac{1}{2}(\lambda(p) - \nu(p) + \mu(p))} (-u_z u_t + u_{u_z}), \]
\[ T^t_2 = \frac{1}{4a^2} e^{\frac{1}{2}(\lambda(p) - 3\nu(p) - \mu(p))} (-e^{\nu(p)} u(2a^2 e^{\lambda(x)} u_{zz} + e^\mu(p) (2u_{\theta\theta} + a^2 e^{\lambda(p)} (\lambda' u_p + \mu' u_p - \nu' u_p + 2u_{pp}))) + 2a^2 e^{\lambda(p) + \mu(p)} u^2_t. \]

The symmetry \( X_2 = \partial_t \) is associated with energy conservation. Other conserved vector components for the symmetries listed in this paper can be obtained by using the component relations (27) given above.

Acknowledgements. SJ would like to thank the National Research Foundation of South Africa for financial support.

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