SOLITARY WAVES AND OTHER SOLUTIONS TO KADOMTSEV-PETVIASHVILI EQUATION WITH SPATIO-TEMPORAL DISPERSION

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This paper studies Kadomtsev-Petviashvili equation with spatio-temporal dispersion and power law nonlinearity. There are several approaches adopted to integrate the equation and display a large spectrum of solutions. These integrability approaches are modified $F$-expansion, exp-function method, $G'/G$-expansion method, functional variable algorithm, first integral approach, traveling wave hypothesis, and semi-inverse variational principle. The solutions that are recovered by using these approaches carry respective constraint conditions that must remain valid for the solutions to exist.

Key words: solitons; integrability; constraints.

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1. INTRODUCTION

Nonlinear evolution equations (NLEEs) play a very important role in several areas of pure and applied sciences such as mathematical physics, theoretical physics, applied mathematics, and engineering sciences [1-70]. In addition, NLEEs appear in mathematical biosciences, nuclear physics, meson physics, plasma physics, fluid dynamics, and nonlinear optics. Therefore it is extremely important to study these equations from all possible angles in order to get a deeper understanding of such equations along with their mathematical structure as well as their integrability aspects. This will lead to several forms of nonlinear wave solutions to fall out of such NLEEs.

Nonlinear waves play a major role in several areas of applied sciences. Because of their complex structure, such wave equations are not integrable with the aid of
an unified integration mechanism. This lead to the proposal of several methods of integration that are available nowadays. A few of these integration mechanisms are inverse method, $F$-expansion method, Jacobi’s elliptic function expansion method, Hirota bilinear method, Wronskian determinant technique and several others. This paper will employ a few of these tools to extract soliton as well as other solutions to a proposed NLEE that will display nonlinear wave solutions. This will lead to several forms of nonlinear wave solutions to these equations. A few of these examples are plain waves, solitary waves, singular solitons, singular periodic waves and several others.

This paper will adopt several integration techniques to extract nonlinear wave solutions of a particular form of a NLEE in $(2+1)$-dimensions. This NLEE is the Kadomtsev-Petviashvili (KP) equation with spatio-temporal dispersion (STD) and power law nonlinearity.

2. GOVERNING EQUATION

The $(2+1)$-dimensional generalizations of the Korteweg-de Vries (KdV) equation is the Kadomtsev-Petviashvili (KP) equation [33] that appears in the form

$$\left( u_t + \alpha uu_x + u_{xxx} \right)_x + u_{yy} = 0,$$

which is used to model shallow water waves with weakly nonlinear restoring forces and waves in ferromagnetic media [1, 6], where $\alpha$ is a constant parameter. The KP equation with STD is given by

$$u_t + \alpha (u^3)_x - \beta u_{xxt} = 0,$$

which appears in many physical applications [24, 36]. We consider the generalized form KP equation with STD

$$(u_t + \alpha (u^n)_x + \beta u_{xxt})_x + ru_{yy} = 0,$$

where $n$ represents the power law nonlinearity parameter. This equation will be studied in the next few sections where its integrability aspect will be addressed. The tools of integration that will be employed in this paper are modified $F$-expansion method, exp-function approach, $G'/G$-expansion scheme, functional variable method, first integral approach, traveling wave hypothesis, and finally the application of semi-inverse variational (SVP) principle.

3. MODIFIED $F$-EXPANSION METHOD

In this section, we apply the modified $F$-expansion method [12, 39, 40] to construct the traveling wave solutions for the $(2+1)$-dimensional generalized KP equation with STD.
3.1. OVERVIEW OF THE METHOD

We describe the modified F-expansion method. Consider a given nonlinear partial differential equation with independent variables \( X = (x, y, t) \) and dependent variable \( u \) as

\[
\phi(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{xt}, u_{yt}, \ldots) = 0. \tag{4}
\]

We look for traveling wave solution of eq. (4) by taking

\[
u(x, y, t) = U(\xi), \quad \xi = kx + ly + ct, \tag{5}
\]

where \( c, l \) and \( k \) are unknown constants to be determined later. Inserting (5) into (4) yields an ODE for \( U(\xi) \) as

\[
P(U, U', U'', U''', \ldots) = 0. \tag{6}
\]

We seek the solutions in the form of

\[
U(\xi) = \sum_{i=-N}^{N} a_i F_i(\xi), \quad (a_N \neq 0) \tag{7}
\]

where \( a_i (i = -N, \ldots, N) \) are arbitrary constants and \( F(\xi) \) satisfies Riccati equation

\[
F'(\xi) = A + BF(\xi) + CF^2(\xi), \quad (C \neq 0) \tag{8}
\]

and hence holds for \( F(\xi) \)

\[
F''(\xi) = AB + (B^2 + 2AC)F(\xi) + 3BCF^2(\xi) + 2C^2F^3(\xi), \tag{9}
\]

where \( A, B \) and \( C \) are all parameters, the prime denotes \( d/d\xi \). Given different values of \( A, B, \) and \( C \), the different Riccati function solution \( F(\xi) \) can be obtained from eq. (8). To determine \( U \) explicitly, we take the following four steps:

Step-1: Determine the integer \( N \) by balancing the highest order nonlinear terms and the highest order derivative of \( W \) in equation (6).

Step-2: Substituting (7) into (6), and using (8) and (9), the left-hand side of (6) can be converted into a finite series in \( F^i(\xi) (i = -N, \ldots, N) \). Equating each coefficient of \( F^i(\xi) \) to zero yields a system of algebraic equations for \( a_i (i = -N, \ldots, N), k, c, l, \) and \( L \).

Step-3: Solve the system of algebraic equations, with the aid of Maple, \( a_i (i = -N, \ldots, N), k, c, l, \) and \( L \) be expressed by \( A, B, C, \) then substitute these results into (7).

Step-4: With the aid of Appendix, from the general form of traveling wave solutions, we can give a series of soliton-like solutions, trigonometric function solutions and rational solutions of eq. (4).
3.2. APPLICATION TO KP EQUATION

We apply the F-expansion method to construct the traveling wave solutions of eq. (3), so we suppose

\[ u(x, y, t) = U(\xi), \quad \xi = ky + tz + ct, \]  

(10)

where \( c, k \) and \( l \) are constants that to be determined later. We substitute eq. (10) into eq. (3). We integrate the obtained equation twice with respect to \( \xi \) and we take both constants of integration to be zero. Then we obtain the following ODE:

\[ (kc + rl^2)U + \alpha k^2 U'^\alpha + ck^3 \beta U'' = 0. \]  

(11)

We multiply eq. (11) by \( U' \) and integrate it a third time with respect to \( \xi \). So we can write

\[ \frac{ck + rl^2}{2} U^2 + \frac{\alpha k^2}{n+1} U^{(n+1)} + \frac{ck^3}{2} U'^2 + L = 0, \]  

(12)

where \( L \) is a constant of integration to be determined later. We make transformation \( U = W^{\frac{2}{n-1}} \). Thus eq. (12) is transformed into the following ordinary differential equation (ODE)

\[ (n + 1)(n - 1)^2 (ck + rl^2) W^2(\xi) + 2\alpha k^2 (n - 1)^2 W^4(\xi) + 4ck^3 \beta (n + 1) W'^2(\xi) + L = 0. \]  

(13)

Balancing \( W^4 \) with \( W'^2 \) yields \( N = 1 \). Therefore according to the modified F-expansion method, we have

\[ W(\xi) = \frac{a_{-1}}{F(\xi)} + a_0 + a_1 F(\xi), \]  

(14)

where \( F(\xi) \) satisfies eq. (8). Substituting (14) into eq. (13), and using eq. (8) simultaneously, the left-hand side of eq. (13) can be converted into a finite series in \( F^j(\xi) (j = -4, \cdots, -1, 0, 1, \cdots, 4) \), then setting each coefficient to zero, we get a set of over algebraic equations for \( a_{-1}, a_0, a_1, c, l, k, \) and \( L \). Solving the obtained algebraic equations, gets the following solutions:

**Case-1.** When \( A = 0 \), yields

\[ a_0 = \frac{Ba_1}{2C}, \quad a_1 = a_1, \quad k = \pm \frac{\sqrt{2(n - 1)}}{2\sqrt{\beta B}}, \]  

\[ L = \frac{(a_1^2 \alpha n^2 - 2a_1^2 \alpha n - 2rl^2 \beta n C^2 - 2rl^2 \beta C^2 + a_1^2 \alpha)(n^2 - 2n + 1)a_1^2 B^2}{16C^4 \beta}, \]  

(15)

\[ a_{-1} = 0, \quad c = \pm \frac{\sqrt{2B(n - 1)} a_1^2 \alpha}{2(n + 1)C^2 \sqrt{\beta}}, \quad l = l. \]
Case-2. When $B = 0$, yields
\[ a_1 = -\frac{a_{-1}C}{A}, \quad a_0 = 0, \quad a_{-1} = a_{-1}, \quad l = l, \]
\[ k = \pm \frac{\sqrt{2}a_{-1}^2(n - 1)}{\sqrt{2}A}, \quad L = \frac{(n - 1)^2D}{\beta A}, \]
\[ c = \frac{2a_{-1}^2(n - 1)^2\sqrt{2}C}{\sqrt{2}A}, \]
where $D = -a_{-1}^2a_{-1}n^2 + 2a_{-1}^2\alpha + 2r^2nA^2\beta - a_{-1}^2\alpha + 2r^2A^2\beta$.

Case-3. When $A = B = 0$, yields
\[ a_1 = a_1, \quad a_0 = a_{-1} = 0, \quad l = \pm \frac{\sqrt{2}r(1 + n)\alpha(n - 1)a_1}{2r\beta(1 + n)C}, \]
\[ k = k, \quad c = \frac{a_1^2\alpha(2n - 1 - n^2)}{2\beta C^2(1 + n)k}, \quad L = 0. \]

Substituting over solutions into eq. (14), and by using Appendix we have:

**Solution-1.** When $A = 0$, $B = 1$, $C = -1$ and $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$ from appendix, by using Case-1, we have
\[ W_1(\xi) = \frac{a_1}{2}\tanh(\frac{\xi}{2}), \]
so
\[ u_1(x, y, t) = \left\{ \frac{a_1}{2}\tanh(\frac{\xi}{2}) \right\}^2, \]

**Solution-2.** When $A = 0$, $B = -1$, $C = 1$ from appendix, then $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$, by using Case-1, we obtain
\[ W_2(\xi) = -\frac{a_1}{2}\coth(\frac{\xi}{2}), \]
so
\[ u_2(x, y, t) = \left\{ -\frac{a_1}{2}\coth(\frac{\xi}{2}) \right\}^2, \]
where $\xi = \pm(\frac{\sqrt{2}(n - 1)}{2\sqrt{2}B})x + ly \mp (\frac{\sqrt{2}B(n - 1)a_1^2n}{2(1 + n)C^2\sqrt{2}B})t$. 

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Solution-3. When $B = 0$, $A = \frac{1}{2}$, $C = \frac{-1}{2}$ and $F(\xi) = \coth(\xi) \pm \text{csch}(\xi)$ from Appendix, by using Case-2, we have

$$W_3(\xi) = \frac{2\cosh(\xi)a_{-1}}{\sinh(\xi)},$$

and choosing $F(\xi) = \tanh(\xi) \pm \text{sech}(\xi)$, gives

$$W_4(\xi) = \frac{2a_{-1}\sinh(\xi)}{\cosh(\xi)},$$

respectively, we have

$$u_3(x,y,t) = \left\{\frac{2\cosh(\xi)a_{-1}}{\sinh(\xi)}\right\}^{\frac{2}{n-1}},$$

$$u_4(x,y,t) = \left\{\frac{2a_{-1}\sinh(\xi)}{\cosh(\xi)}\right\}^{\frac{2}{n-1}},$$

Solution-4. When $A = 1$, $B = 0$, $C = -1$ and $F(\xi) = \tanh(\xi)$ from Appendix, by using Case-2, we obtain

$$W_5(\xi) = \frac{a_{-1}(1 + \tanh^2(\xi))}{\tanh(\xi)},$$

and $F(\xi) = \coth(\xi)$, gets

$$W_6(\xi) = \frac{a_{-1}(1 + \coth^2(\xi))}{\coth(\xi)},$$

respectively, we obtain

$$u_5(x,y,t) = \left\{\frac{a_{-1}(1 + \tanh^2(\xi))}{\tanh(\xi)}\right\}^{\frac{2}{n-1}},$$

$$u_6(x,y,t) = \left\{\frac{a_{-1}(1 + \coth^2(\xi))}{\coth(\xi)}\right\}^{\frac{2}{n-1}},$$

Solution-5. By choosing $A = C = \frac{1}{2}$, $B = 0$ and $F(\xi) = \sec(\xi) + \tan(\xi)$ from Appendix, by using Case-2, yields

$$W_7(\xi) = \frac{2a_{-1}\sin(\xi)}{\cos(\xi)},$$

and $F(\xi) = \csc(\xi) - \cot(\xi)$, gives

$$W_8(\xi) = \frac{2\cos(\xi)a_{-1}}{\sin(\xi)},$$
respectively, yields

\[ u_7(x, y, t) = \left\{ -\frac{2a_{-1}\sin(\xi)}{\cos(\xi)} \right\}^{\frac{2}{n-1}}, \]

\[ u_8(x, y, t) = \left\{ \frac{2\cos(\xi)a_{-1}}{\sin(\xi)} \right\}^{\frac{2}{n-1}}, \]

**Solution-6.** We select \( A = C = -\frac{1}{2}, \ B = 0 \) and \( F(\xi) = \csc(\xi) + \cot(\xi) \) from appendix, by using Case-2, we have

\[ W_9(\xi) = -\frac{2\cos(\xi)a_{-1}}{\sin(\xi)}, \]

and \( F(\xi) = \sec(\xi) - \tan(\xi) \), gets

\[ W_{10}(\xi) = \frac{2a_{-1}\sin(\xi)}{\cos(\xi)}, \]

so we obtain

\[ u_9(x, y, t) = \left\{ -\frac{2\cos(\xi)a_{-1}}{\sin(\xi)} \right\}^{\frac{2}{n-1}}, \]

\[ u_{10}(x, y, t) = \left\{ \frac{2a_{-1}\sin(\xi)}{\cos(\xi)} \right\}^{\frac{2}{n-1}}, \]

**Solution-7.** When \( A = C = 1(-1), \ B = 0 \) and \( F(\xi) = \tan(\xi) \) or \( \cot(\xi) \) from appendix and Case-2, we have

\[ W_{11}(\xi) = \frac{a_{-1}(1 - \tan^2(\xi))}{\tan(\xi)}, \]

\[ W_{12}(\xi) = \frac{a_{-1}(1 - \cot^2(\xi))}{\cot(\xi)}, \]

then we obtain

\[ u_{11}(x, y, t) = \left\{ \frac{a_{-1}(1 - \tan^2(\xi))}{\tan(\xi)} \right\}^{\frac{2}{n-1}}, \]

\[ u_{12}(x, y, t) = \left\{ \frac{a_{-1}(1 - \cot^2(\xi))}{\cot(\xi)} \right\}^{\frac{2}{n-1}}, \]

where \( \xi = \pm \left( \frac{\sqrt{2\alpha\beta AC}}{\sqrt{2\alpha\beta ACa_{-1}}} \right)x + ty + \left( \frac{2a_{-1}a^2(n-1)^2\sqrt{2C}}{\sqrt{\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta}} \right)t \) and \( D = -a_{-1}a1^n2 + 2a_{-1}a1^n2 + 2rl^2nA^2\beta - a^2_{-1}\alpha + 2rl^2A^2\beta. \)
Solution-8. When \( A = B = 0, C \neq 0 \) and \( F(\xi) = \frac{-1}{C_\xi + \lambda} \) (\( \lambda \) is an arbitrary constant) from Appendix, by using Case-3, yields
\[
W_{13}(\xi) = -\frac{a_1}{C_\xi + \lambda},
\]
so
\[
u_{13}(x, y, t) = \left\{ -\frac{a_1}{C_\xi + \lambda} \right\}^{\frac{2}{n-1}} y + \left\{ \frac{a_1^2(2n-1-n^2)}{2\beta C^2(1+n)k} \right\}^\xi t.
\]

4. EXP-FUNCTION METHOD

In this section, we first give the details of the exponential function method [73, 74], then apply it to KP equation with STD.

4.1. DETAILS OF THE METHOD

We consider the general nonlinear PDE of the type
\[
P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{xt}, u_{yt}, \cdots) = 0.
\]
We making use the traveling wave transformation
\[
u(x, y, t) = U(\xi), \quad \xi = kx + ly + ct,
\]
therefore eq. (3) reduces to an ordinary differential equation
\[
F(U, U', U'', \cdots) = 0,
\]
where the prime denotes the derivation with respect to \( \xi \). Also \( c, k, l \) are constants. The exp-function method is based on the assumption that the traveling wave solutions can be expressed in the following form
\[
U(\xi) = \sum_{n=-w}^d a_n \exp(n\xi) + \sum_{m=-p}^q b_m \exp(m\xi),
\]
where \( w, d, p \) and \( q \) are positive integers which are known to be determined further, \( a_n \) and \( b_m \) are unknown constants. Equation (21) can be rewritten in an alternative form
\[
U(\xi) = a_{-w} \exp(-w\xi) + \cdots + a_d \exp(d\xi) + b_{-p} \exp(-p\xi) + \cdots + b_q \exp(q\xi).
\]
To determine the values of \( d \) and \( q \), we balance the linear term of highest order in eq. (20) with the highest order nonlinear term. Similarly, to determine the values of \( c \) and \( p \), we balance the linear term of lowest order in eq. (20) with the lowest order nonlinear term.
4.2. APPLICATION TO KP EQUATION

We consider eq. (3) in the form

\[ (n+1)(n-1)^2(ck+rl^2)W^2(\xi) + 2\alpha k^2(n-1)^2W^4(\xi) + 4ck^3\beta(n+1)W'^2(\xi) = 0, \]

(23)

with the transformation

\[ U(\xi = kx + ly + ct) = W^{2n-1}(\xi). \]

(24)

We balance the linear term of highest order in eq. (23) with the highest order nonlinear term. By simple calculation, we have

\[ W'^2 = \frac{c_1 \exp[(2w+4p)\xi] + \cdots}{c_2 \exp[6p\xi] + \cdots}, \]

(25)

and

\[ W^4 = \frac{c_3 \exp[(4w+2p)\xi] + \cdots}{c_4 \exp[6p\xi] + \cdots}, \]

(26)

where \( c_i \) are determined coefficients only for simplicity. Balancing the highest order of exp-function in (25) and (26), we have

\[ 2w + 4p = 4w + 2p, \]

(27)

which leads to the result

\[ w = p. \]

(28)

Similarly to determine values of \( d \) and \( q \), we balance the linear term of lowest order in eq. (23)

\[ W'^2 = \frac{\cdots + d_1 \exp[-(4q+2d)\xi]}{\cdots + d_2 \exp[-6q\xi]}, \]

(29)

and

\[ W^4 = \frac{\cdots + d_3 \exp[-(2q+4d)\xi]}{\cdots + d_4 \exp[-6q\xi]}, \]

(30)

where \( d_i \) are determined coefficients. Therefore, we can obtain the following relation

\[ -(4q + 2d) = -(2q + 4d), \]

(31)

which gives

\[ q = d. \]

(32)

We can freely choose the values of \( w \) and \( d \), but the final solution does not strongly depend upon the choice of values of \( w \) and \( d \). For simplicity, we set \( w = p = 1 \) and \( d = q = 1 \), then from eq. (22), we have

\[ W(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \]

(33)
Substituting eq. (33) into eq. (23), equating the coefficients of all powers of \( \exp(i\xi) \), \( i = 0, \pm 1, \pm 2, \cdots, \pm 6 \) to zero yields a set of algebraic equations for \( a_0, b_0, a_1, a_{-1}, b_{-1}, b_1, c, l \) and \( k \). Solving the system of algebraic equations with the aid of Maple, we obtain:

\[
\begin{align*}
    a_0 &= a_0, \quad b_0 = 0, \quad b_{-1} = \frac{(2n - n^2 - 1 - 4\beta k^2)a_0^2\alpha}{8(n + 1)rl^2b_1\beta}, a_1 = a_{-1} = 0 \\
    b_1 &= b_1, \quad l = l, \quad c = -\frac{rl^2(n-1)^2}{k(n^2 - 2n + 1 + 4\beta k^2)}, \quad k = k.
\end{align*}
\] (34)

Substituting eqs. (34) into eq. (33) yields

\[
W(\xi) = \frac{a_0}{b_1e^\xi - \frac{a_0^2\alpha(n^2-2n+4\beta k^2+1)e^{-\xi}}{8(n+1)rl^2b_1\beta}}, \] (35)

from transformation (24), we obtain the following general solution

\[
u(x, y, t) = \left\{ \frac{a_0}{b_1e^\xi - \frac{a_0^2\alpha(n^2-2n+4\beta k^2+1)e^{-\xi}}{8(n+1)rl^2b_1\beta}} \right\}^{\frac{2}{n+1}}, \] (36)

where \( \xi = kx + ly - \left( \frac{rl^2(n-1)^2}{k(n^2 - 2n + 1 + 4\beta k^2)} \right) t \).

5. THE \( G'/G \)-EXPANSION METHOD

We describe the \( G'/G \)-expansion method [71–87] to find traveling wave solutions of the (2+1)-dimensional generalized KP equation with STD.

5.1. DETAILS OF THE METHOD

We have the following nonlinear partial differential equation

\[
P(u, u_x, u_t, u_{xx}, u_{xt}, \cdots) = 0, \] (37)

where \( u = u(x, t) \) is an unknown function, \( P \) is a polynomial in \( u(x, t) \) and its various partial derivatives, which includes nonlinear terms and the highest order derivatives. We take the following four steps:

Step-1: Suppose that

\[
u(x, y, t) = U(\xi), \xi = kx + ly + ct, \] (38)

which reduce eq. (37) to an ODE for \( u = U(\xi) \) in the form

\[
F(U, U', U'', \cdots) = 0, \] (39)
where prime denotes the derivative with respect to $\xi$.

Step-2: Suppose that the solution of ODE (39) can be expressed by a polynomial in $G'/G$ as follows:

$$U(\xi) = a_m \left( \frac{G'(\xi)}{G(\xi)} \right)^m + \cdots,$$

(40)

where $G = G(\xi)$ satisfies the second order linear ODE in the form

$$G'' + \lambda G' + \mu G = 0,$$

(41)

where $a_m, \cdots, a_0, \lambda$ and $\mu$ are constants to be determined later, $a_m \neq 0$, and the positive integer $m$ can be determined by balancing the highest order derivative with the highest nonlinear terms in (39).

Step-3: Substitute (40) into (39) and using (41), collect all terms with the same power of $G'/G$ together, and then equate each coefficient of each powers of $G'/G$ to zero.

Step-4: Solve the system of algebraic equations obtained from step 3, for $a_m, \cdots, c, l, k, \lambda$, and $\mu$ by using Maple.

Step-5: Depending on the sign of the discriminant $\Delta = \lambda^2 - 4 \mu$, we get solutions of eq. (39). So we can obtain exact solutions of the given system of equations for eq. (37).

5.2. APPLICATION TO KP EQUATION

We consider the generalized extended (2+1)-dimensional KP equation with STD in the form of eq. (13) with the transformation

$$u(x, y, t) = U(\xi = kx + ly + ct) = W^{\frac{2}{n-1}}(\xi).$$

(42)

Balancing the highest order nonlinear term with a supreme derivative, we can get $m = 1$, so

$$W(\xi) = a_0 + a_1 \left( \frac{G'(\xi)}{G(\xi)} \right), \quad a_1 \neq 0,$$

(43)

where $a_0$ and $a_1$ are constants to be determined later.

We substitute eq. (24) and eq. (42) into eq. (3). Then with the help of the symbolic software Maple we equate the coefficients of this polynomial in $G'/G^i(i = 0, \ldots, 4)$ to zero; thus we can get the following solutions:
\[ a_0 = a_0, \quad \mu = \frac{w + 8\alpha k^2 a_0^2 \beta}{8a_1^2 \alpha k^2 \beta}, \quad k = k, \]

\[ l = l, \quad \lambda = \frac{2a_0}{a_1}, \quad c = \frac{a_1^2 \alpha (2n - n^2 - 1)}{2k\beta (n + 1)}, \]

\[ L = \frac{(n - 1)^2 w^2}{32\beta^2 \alpha k^2}, \quad a_1 = a_1, \]

where \( w = 2na_1^2 \alpha - a_1^2 \alpha - a_1^2 \alpha n^2 + 2rl^2 \beta + 2nr l^2 \beta. \) Since in (44), \( \Delta = \lambda^2 - 4\mu > 0, \) so we obtain only rational function solution

\[ W(\xi) = \frac{a_1 s(c_1 \cosh(s\xi) + c_2 \sinh(s\xi))}{c_2 \cosh(s\xi) + c_1 \sinh(s\xi)}, \]

so

\[ u(x, y, t) = \left\{ \frac{a_1 s(c_1 \cosh(s\xi) + c_2 \sinh(s\xi))}{c_2 \cosh(s\xi) + c_1 \sinh(s\xi)} \right\}^{\frac{1}{n-1}}, \]

where \( \xi = kx + ly + \left( \frac{a_1^2 \alpha (2n - n^2 - 1)}{2k\beta (n + 1)} \right) t \) and \( s = \sqrt{\frac{a_1^2 \alpha + a_1^2 \alpha n^2 - 2na_1^2 \alpha - 2rl^2 \beta - 2nr l^2 \beta}{8a_1^2 \alpha k^2 \beta}}. \]

6. FUNCTIONAL VARIABLE METHOD

The functional variable method, which is a direct and effective algebraic method for the computation of compactons, solitons, solitary patterns and periodic solutions, was first proposed by Zerarka et al. [41]. This method was further developed by many authors [5, 35, 42].

6.1. DETAILS OF THE METHOD

We now summarize the functional variable method, established by Zerarka et al. [41], the details of which can be found in [5, 35, 42] among many others. Consider a general nonlinear PDE in the form

\[ P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \]

where \( P \) is a polynomial in \( u \) and its partial derivatives.

To find the traveling wave solution of eq. (47) we introduce the wave variable \( \xi = x - ct \) so that

\[ u(x, t) = U(\xi). \]

The nonlinear partial differential equation can be converted to an ODE as

\[ Q(U, U', U'', \ldots) = 0, \]
where $Q$ is a polynomial in $U$ and its total derivatives and $' = \frac{d}{d\xi}$.

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_\xi = F(U)$$

and some successively derivatives of $U$ are

$$U_{\xi\xi} = \frac{1}{2}(F^2)',$$

$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2},$$

$$U_{\xi\xi\xi\xi} = \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'],$$

where $' = \frac{d}{dU}$.

The ODE (49) can be reduced in terms of $U$, $F$ and its derivatives. Upon using the expressions of eq. (51) into eq. (49) gives

$$R(U, F, F', F'', F''', ...) = 0. (52)$$

The key idea of this particular form eq. (52) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, eq. (52) provides the expression of $F$, and this in turn together with eq. (50) give the relevant solutions to the original problem.

**Remark-1:** The functional variable method definitely can be applied to nonlinear PDEs which can be converted to second-order ordinary differential equations through the travelling wave transformation.

### 6.2. APPLICATION TO KP EQUATION

In this subsection, we apply the functional variable method to construct the exact 1-soliton solutions of the (2+1)-dimensional generalized KP equation with STD. Substituting the traveling wave transformation

$$u(x, y, t) = U(\xi), \quad \xi = B_1x + B_2y - vt + \xi_0,$$

into eq. (3) and integrating the resultant equation twice with zero constants, we have:

$$(rB_2^2 - vB_1)U + \alpha B_1^2 U'' - vB_1^3 \beta U''' = 0. (54)$$

According to this method, we use the transformation

$$U'(\xi) = F(U(\xi))$$

(55)
that will convert eq. (54) to

\[(rB_2^2 - vB_1)U + \alpha B_1^2 U^n - \frac{vB_1^3 \beta}{2} (F^2(U))' = 0. \tag{56}\]

Thus, we get from eq. (56) the expression of the function \( F(U) \), which reads

\[ F(V) = \sqrt{\frac{rB_2^2 - vB_1}{vB_1^3 \beta}} U \sqrt{1 - \frac{2\alpha B_1^2}{(vB_1 - rB_2^2)(n+1)}} U^{n-1}. \tag{57}\]

After making the change of variables

\[ Z = \frac{2\alpha B_1^2}{(vB_1 - rB_2^2)(n+1)} U^{n-1}, \tag{58}\]

and using the relation \( U' = F(U) \), the solution of the eq. (54) is in the following form

\[ U(\xi) = \left\{ \frac{(rB_2^2 - vB_1)(n+1)}{2\alpha B_1^2} \text{csch}^2 \left( \frac{n-1}{2} \sqrt{\frac{rB_2^2 - vB_1}{vB_1^3 \beta}} \right) \right\}^{\frac{1}{n-1}}. \tag{59}\]

Using the transformation (55), we can obtain the following soliton solutions of eq. (3):

\[ u_1(x, y, t) = \left\{ \frac{(rB_2^2 - vB_1)(n+1)}{2\alpha B_1^2} \text{csch}^2 \left( \frac{n-1}{2} \sqrt{\frac{rB_2^2 - vB_1}{vB_1^3 \beta}} (B_1 x + B_2 y - vt + \xi_0) \right) \right\}^{\frac{1}{n-1}} \tag{60}\]

and

\[ u_2(x, y, t) = \left\{ \frac{(vB_1 - rB_2^2)(n+1)}{2\alpha B_1^2} \text{sech}^2 \left( \frac{n-1}{2} \sqrt{\frac{rB_2^2 - vB_1}{vB_1^3 \beta}} (B_1 x + B_2 y - vt + \xi_0) \right) \right\}^{\frac{1}{n-1}} \tag{61}\]

for

\[(rB_2^2 - vB_1)vB_1 \beta > 0.\]
It is easy to see that solutions (60) and (61) can reduce to singular periodic solutions as follows:

\[ u_3(x, y, t) = \left\{ \frac{(v B_1 - r B_2^2)(n + 1)}{2 \alpha B_1^2} \sec^2 \left( \frac{n - 1}{2} \sqrt{\frac{v B_1 - r B_2^2}{v B_1^2 \beta}} (B_1 x + B_2 y - vt + \xi_0) \right) \right\}^{\frac{1}{n-1}} \]

and

\[ u_4(x, y, t) = \left\{ \frac{(v B_1 - r B_2^2)(n + 1)}{2 \alpha B_1^2} \csc^2 \left( \frac{n - 1}{2} \sqrt{\frac{v B_1 - r B_2^2}{v B_1^2 \beta}} (B_1 x + B_2 y - vt + \xi_0) \right) \right\}^{\frac{1}{n-1}} \]

for

\[(r B_2^2 - v B_1) v B_1 \beta < 0.\]

These solutions given by (60)-(63) remain valid as long as \( n \neq 1 \).

7. FIRST INTEGRAL APPROACH

One of the most effective direct methods to develop the traveling wave solution of NLEEs is the first integral method [24]. This method has been successfully applied to obtain exact solutions for a variety of NLEEs [2–4, 34, 35]. Different from other traditional methods, the first integral method has many advantages, which is mainly embodied in that it could avoid a great deal of complicated and tedious calculations and provide more exact and explicit traveling solitary solutions with high accuracy.

7.1. DETAILS OF THE METHOD

Bekir et al. [4] summarized the main steps for using the first integral method, as follows:

Step-1: Suppose a NLEE

\[ P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \]

can be converted to an ODE

\[ Q(U, -\omega U', kU', \omega^2 U'', -k\omega U''', k^2 U''', \ldots) = 0, \]

using a traveling wave variable \( u(x, t) = U(\xi), \ \xi = kx - \omega t, \) where the prime denotes the derivation with respect to \( \xi \). If all terms contain derivatives, then Eq. (65)
is integrated where integration constants are considered zeros.

Step-2: Suppose that the solution of ODE (65) can be written as follows:

\[ u(x, t) = U(\xi) = f(\xi). \]  

(66)

Step-3: We introduce a new independent variable

\[ X(\xi) = f(\xi), \quad Y(\xi) = f'(\xi), \]  

(67)

which leads a system of equations

\[ X'(\xi) = Y(\xi), \]  

\[ Y'(\xi) = F(X(\xi), Y(\xi)). \]  

(68)

Step-4: By using the Division Theorem for two variables in the complex domain \( C \) which is based on the Hilbert-Nullstellensatz Theorem [10], we can obtain one first integral to Eq. (68), which can reduce Eq. (65) to a first-order integrable ordinary differential equation. An exact solution to Eq. (64) is then obtained by solving this equation directly.

**Division Theorem:** Suppose that \( P(w, z) \) and \( Q(w, z) \) are polynomials in \( C[w, z] \); and \( P(w, z) \) is irreducible in \( C[w, v] \). If \( Q(w, z) \) vanishes at all zero points of \( P(w, z) \), then there exists a polynomial \( G(w, z) \) in \( C[w, z] \) such that

\[ Q(w, z) = P(w, z)G(w, z). \]

7.2. APPLICATION TO KP EQUATION

In this subsection, we would like to extend the first integral method to solve the (2+1)-dimensional modified KP equation with STD

\[ (u_t + \alpha (u^3)_x + \beta u_{xxt})_x + rv_{yy} = 0. \]  

(69)

Substituting the traveling wave transformation

\[ u(x, y, t) = U(\xi), \quad \xi = B_1 x + B_2 y - vt, \]  

(70)

into eq. (70) and integrating the resultant equation twice with zero constants, we have:

\[ (r B_2^2 - v B_1) U + \alpha B_1^2 U^3 - v B_1^3 \beta U'' = 0. \]  

(71)

If we let \( X(\xi) = U(\xi), \quad Y(\xi) = \frac{d U(\xi)}{d \xi} \), eq. (71) is equivalent to the two dimensional autonomous system

\[ X'(\xi) = Y(\xi), \]  

\[ Y'(\xi) = \left( \frac{v B_1 - r B_2^2}{v B_1^3 \beta} \right) X(\xi) - \frac{\alpha}{v B_1^3 \beta} X^3(\xi). \]  

(72)
Now, we apply the above Division Theorem to look for the first integral of system (72). Suppose that \( X(\xi) \) and \( Y(\xi) \) are nontrivial solutions to system (72), and \( Q(X,Y) = \sum_{l=0}^{m} a_l(X)Y^l \) is an irreducible polynomial in the complex domain \( C \) such that

\[
Q(X(\xi),Y(\xi)) = \sum_{l=0}^{m} a_l(X(\xi))Y^l(\xi) = 0,
\]

where \( a_l(X)(l = 0,1,\ldots,m) \) are polynomials of \( X \) and \( a_m(X) \neq 0 \). Eq. (73) is a first integral of system (72). We note that \( \frac{dQ}{d\xi} \) is a polynomial of \( X \) and \( Y \), and \( Q(X(\xi),Y(\xi)) = 0 \) implies that \( \frac{dQ}{d\xi} |_{(21)} = 0 \). According to the Division Theorem, there exists a polynomial \( T(X,Y) = g(X) + h(X)Y \) in the complex domain \( C \) such that

\[
\frac{dQ}{d\xi} = \frac{dQ}{dX} \frac{dX}{d\xi} + \frac{dQ}{dY} \frac{dY}{d\xi} = (g(X) + h(X)Y) \sum_{l=0}^{m} a_l(X)Y^l.
\]

We assume that \( m = 1 \) in eq. (73). Taking eqs. (72) and (74) into account, we get

\[
\sum_{l=0}^{1} a'_l(X)Y^{l+1} + \left( \frac{vB_1 - rB_2^2}{vB_1^2 \beta} \right) X - \frac{\alpha}{vB_1 \beta} X^3 \left( \sum_{l=0}^{1} la_l(X)Y^{l-1} \right)
= (g(X) + h(X)Y) \sum_{l=0}^{1} a_l(X)Y^l,
\]

where the primes denote derivatives with respect to \( X \). Equating the coefficients of \( Y^l(l = 2,1,0) \) in eq. (75) leads to the system

\[
a'_1(X) = h(X)a_1(X),
\]

\[
a'_0(X) = g(X)a_1(X) + h(X)a_0(X),
\]

\[
a_1(X) \left\{ \left( \frac{vB_1 - rB_2^2}{vB_1^2 \beta} \right) X - \frac{\alpha}{vB_1 \beta} X^3 \right\} = g(X)a_0(X).
\]

Since \( a_l(X)(l = 0,1) \) are polynomials, then from eq. (76) we deduce that \( a_1(X) \) is constant and \( h(X) = 0 \). For simplicity, take \( a_1(X) = 1 \). Balancing the degrees of \( g(X) \) and \( a_0(X) \), we conclude that \( \text{deg}(g(X)) = 1 \) only. Suppose that \( g(X) = A_1X + B_0 \), then we find \( a_0(X) \).

\[
a_0(X) = A_0 + B_0X + \frac{A_1}{2} X^2,
\]
where $A_0$ is arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into eq. (78) and setting all the coefficients of powers $X$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0 = 0, \quad A_1 = \pm \frac{\sqrt{-2vB_1^3}}{vB_1^3}, \quad A_0 = \pm \frac{rB_1^2 - vB_1}{B_1^2\sqrt{-2vB_1^3}}$$  \hspace{1cm} (80)

where $B_1$, $B_2$, and $v$ are arbitrary constants.

Using the conditions (80) in eq. (73), we obtain

$$Y(\xi) = \pm \left( \frac{rB_1^2 - vB_1}{B_1^2\sqrt{-2vB_1^3}} - \frac{-2vB_1^3\alpha}{2vB_1^3} X^2(\xi) \right).$$  \hspace{1cm} (81)

Combining (81) with (72), we obtain the exact solution to eq. (71) and then exact solutions for the eq. (69) can be written as:

**Type-1:** When $vB_1\beta (rB_2^2 - vB_1) > 0$, we have

1. Topological 1-soliton solution

   $$u_1(x, y, t) = \mp \sqrt{\frac{vB_1 - rB_2^2}{\alpha B_1^2}} \tanh \left( \sqrt{\frac{rB_1^2 - vB_1}{2vB_1^3\beta}} (B_1 x + B_2 y - vt + \xi_0) \right).$$  \hspace{1cm} (82)

2. Singular 1-soliton solution

   $$u_2(x, y, t) = \mp \sqrt{\frac{vB_1 - rB_2^2}{\alpha B_1^2}} \coth \left( \sqrt{\frac{rB_1^2 - vB_1}{2vB_1^3\beta}} (B_1 x + B_2 y - vt + \xi_0) \right).$$  \hspace{1cm} (83)

**Type-2:** When $vB_1\beta (rB_2^2 - vB_1) < 0$, we have the following singular periodic solutions

$$u_3(x, y, t) = \pm \sqrt{\frac{vB_1 - rB_2^2}{\alpha B_1^2}} \tan \left( \sqrt{\frac{vB_1 - rB_2^2}{2vB_1^3\beta}} (B_1 x + B_2 y - vt + \xi_0) \right).$$  \hspace{1cm} (84)

and

$$u_4(x, y, t) = \pm \sqrt{\frac{vB_1 - rB_2^2}{\alpha B_1^2}} \cot \left( \sqrt{\frac{vB_1 - rB_2^2}{2vB_1^3\beta}} (B_1 x + B_2 y - vt + \xi_0) \right).$$  \hspace{1cm} (85)

**Type-3:** When $B_2 = \pm \sqrt{\frac{rB_1}{\alpha v}}$, we can obtain the following continuous wave solution
The rational soliton solution given by (86) exists for $vB_1\alpha < 0$.

Remark-2: For $m = 2, 3, 4$, we have the exact solutions given by eqs. (82)-(86) to eq. (69) which have been obtained in case $m = 1$. Also, there is no need of discussion for the case $m \geq 5$ due to the fact that polynomial equation with the degree greater than or equal to 5 is generally not solvable.

8. TRAVELING WAVE HYPOTHESIS

This section will employ one of the most fundamental algorithms to solve the KP equation with STD and power law nonlinearity. The search will be for a solitary wave solution. The equation is rewritten as

$$q_t - a q^n q_x + b q_{xx} + c q_{yy} = 0.$$  (87)

The starting hypothesis is given by [11]

$$q(x, y, t) = g(B_1 x + B_2 y - vt)$$  (88)

where $g(s)$ represents the wave profile with $B_j$ for $j = 1, 2$ being the inverse widths along $x$ and $y$ directions, respectively, with $v$ being the speed of the wave. Also,

$$s = B_1 x + B_2 y - vt.$$  (89)

Substituting the hypothesis (88) into (87) and integrating once gives

$$(vB_1 - c B_2^2) g' + a B_2^2 g^n g' + vB_1 B_2^2 g'' = 0,$$  (90)

where the notations $g' = dg/ds, d^2 g/ds^2 = g''$ and so on are adopted. Multiplying both sides of (90) by $g'$ and integrating, while choosing the integration constant to be zero leads to

$$vB_1 B_2^2 g'' = (c B_2^2 - vB_1) g - \frac{aB_1^2}{n+1} g^{n+1}. $$  (91)

Integrating one last time leads to the 1-soliton solution

$$q(x, y, t) = g(B_1 x + B_2 y - vt) = A \text{sech}^\frac{2}{n} [B (B_1 x + B_2 y - vt)]$$  (92)

where the amplitude $A$ of the solitary wave is given by

$$A = \left[ \frac{(n+1)(n+2) (c B_2^2 - vB_1)}{2aB_1^2} \right]^{\frac{1}{2}} $$  (93)
and the newly introduced parameter $B$ is
\[
B = \frac{n}{2B_2} \sqrt{\frac{cB_2^2 - vB_1}{bvB_1}}. \tag{94}
\]
These parameters poses constraint relations
\[
a (cB_2^2 - vB_1) > 0, \tag{95}
\]
whenever $n$ is even, and
\[
bvB_1 (cB_2^2 - vB_1) > 0. \tag{96}
\]

9. SEMI-INVERSE VARIATIONAL PRINCIPLE

This is yet another analytical approach to retrieve the soliton solution of the KP equation with STD and power law nonlinearity. This approach does not lead to an exact solution. However, the semi-inverse variational principle (SVP) is an inverse problem mechanism. The starting point of this method stays the same as in the traveling wave hypothesis given by (88). Next, multiplying both sides of (91) by $g'$ and integrating with respect to $s$ gives
\[
\frac{vbB_1B_2^2 (g')^2}{2} - (cB_2 - vB_1) g^2 - \frac{2aB_1^2}{(n+1)(n+2)} g^{n+2} = K \tag{97}
\]
where $K$ is the integration constant. The stationary integral is then defined as [14–17, 33, 38]
\[
J = \int_{-\infty}^{\infty} K ds = \int_{-\infty}^{\infty} \left[ \frac{vbB_1B_2^2 (g')^2}{2} - (cB_2 - vB_1) g^2 - \frac{2aB_1^2}{(n+1)(n+2)} g^{n+2} \right] ds. \tag{98}
\]
Now, choose [11]
\[
g(s) = A \text{sech}^2 \left( \frac{s}{B} \right) \tag{99}
\]
and substitute into (98) so that the stationary integral reduces to
\[
J = \left[ \frac{4vbA^2 BB_1B_2^2}{n^2} - \frac{(cB_2 - vB_1)A^2}{B} - \frac{16vbA^2BB_1B_2^2}{n^2(n+4)} \right] - \frac{8aA^{n+2}B_1^2}{(n+1)(n+2)(n+4)B_2} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} + \frac{3}{4} \right). \tag{100}
\]
The SVP states that, the parameters $A$ and $B$ can be obtained from the coupled system of equations given by
\[
\frac{\partial J}{\partial A} = 0 \tag{101}
\]
and
\[
\frac{\partial J}{\partial B} = 0. \tag{102}
\]
From (100), equations (101) and (102) reduce to
\[
\frac{4vbB^2B_1B_2^2}{n(n+2)} - (cB_2^2 - vB_1) - \frac{4aA^nB_1^2}{(n+1)(n+4)} = 0 \tag{103}
\]
and
\[
\frac{4vbB^2B_1B_2^2}{n(n+2)} + (cB_2^2 - vB_1) - \frac{4aA^nB_1^2}{(n+1)(n+2)(n+4)} = 0. \tag{104}
\]
Upon uncoupling equations (103) and (104), leads to
\[
A = \left[ \frac{(n+1)(n+2)(vB_1 - cB_2^2)}{2aB_1^2} \right]^{\frac{1}{2}} \tag{105}
\]
and
\[
B = \frac{n}{2B_2} \sqrt{\frac{vB_1 - cB_2^2}{bvB_1}}. \tag{106}
\]
Additionally, the relation between the amplitude \(A\) and the parameter \(B\) is given by
\[
B = \frac{1}{B_2} \sqrt{\frac{anB_1}{2(n+1)(n+2)bvA^2}}. \tag{107}
\]
These relations introduces the following constraints whenever \(n\) is even,
\[
a(cB_2^2 - vB_1) < 0, \tag{108}
\]
\[
bcB_1(cB_2^2 - vB_1) < 0, \tag{109}
\]
and
\[
abcB_1 > 0. \tag{110}
\]
Therefore, the 1-soliton solution to KP equation with the STD, obtained from the SVP, is given by (92), with the parameter definitions given by (105)-(107) along with the respective constraints located in (108)-(110).

10. CONCLUSIONS

This paper addressed the KP equation with STD and power law nonlinearity. There are several integration algorithms that are employed in this paper to exhibit a plethora of solutions. These solutions are nonlinear waves in several forms such as solitons, singular solitons, singular periodic waves and several others. There are constraint conditions that naturally emerged from the solution structures. These solutions will be immensely useful in several areas of applications such as oceanography and other areas in mathematical physics.
The future of this problem looks pretty shinny. There are several additional aspects to this equation that can be addressed in future. These are soliton perturbation theory, time-dependent coefficients, fractional temporal evolution and several others. The results of these research are awaited at this time and will be reported in future.

APPENDIX

Relations between values of \((A, B, C)\) and corresponding \(F(\xi)\) in Riccati equation

\[
F'(\xi) = A + BF(\xi) + CF^2(\xi), \quad (C \neq 0)
\]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>(F(\xi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>(\frac{1}{2} + \frac{1}{2} \tanh(\frac{1}{2} \xi))</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>(\frac{1}{2} - \frac{1}{2} \coth(\frac{1}{2} \xi))</td>
</tr>
<tr>
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<td>0</td>
<td>(\frac{-1}{2})</td>
<td>(\coth(\xi) \pm \text{csch}(\xi), \tanh(\xi) \pm i \text{sech}(\xi))</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>(\tanh(\xi), \coth(\xi))</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>(\frac{-1}{2})</td>
<td>(\sec(\xi) + \tan(\xi), \csc(\xi) - \cot(\xi))</td>
</tr>
<tr>
<td>(\frac{-1}{2})</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>(\csc(\xi) + \cot(\xi), \sec(\xi) - \tan(\xi))</td>
</tr>
<tr>
<td>((-1))</td>
<td>0</td>
<td>((-1))</td>
<td>(\tan(\xi)(\cot(\xi)))</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(\neq 0)</td>
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<td>0</td>
<td>0</td>
<td>(\frac{A}{\xi})</td>
</tr>
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<td>arbitrary constant</td>
<td>(\neq 0)</td>
<td>0</td>
<td>(\exp(\frac{B}{2}) - A)</td>
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REFERENCES