SYMMETRY ANALYSIS AND CONSERVATION LAWS OF THE QUANTUM ZAKHAROV EQUATIONS FOR PLASMAS

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In this paper the Lie symmetry analysis is performed on the quantum Zakharov equations that describe the interaction between Langmuir waves and ion-acoustic waves in an electron-ion dense quantum plasma. The similarity reductions and some exact solutions with the aid of sub-equation method are obtained based on the optimal system of one-dimensional subalgebras for the quantum Zakharov equations. In addition, the conservation laws of the quantum Zakharov equations are also constructed using the direct (multiplier) approach.

\textit{Key words}: Quantum Zakharov equations, Lie symmetry method, Conservation laws, Sub-equation method, Solitary wave solutions.

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1. INTRODUCTION

It is well known that nonlinear evolution equations (NLEEs) are related to nonlinear phenomena in various branches of natural science, applied science, social science, economics and business problems. To further explain these physical phenomena by seeking the exact solutions of NLEEs is of paramount importance and has become one of the most exciting and extremely active areas in the nonlinear mathematical physics. The investigation of the exact solutions of NLEEs is a challenging task and only in certain special cases one can write down the solutions explicitly. Over the years, mathematicians and physicists focused their attention to obtain exact solutions of NLEEs by developing several methods, for example, the Hirota’s bilinear method [1], inverse scattering transform method [2], Bäcklund transformation [3], Darboux transformation [4], truncated Painlevé expansion method [5], Lie symmetries method [6-8], homogenous balance method [9], variational iteration method [10] and other methods [11].

The Lie symmetry analysis has proved to be a versatile tool for solving nonlinear differential equations. It is based upon the study of the invariance under one
parameter Lie group of point transformations [6-8,12,13], and it systematically unify and extend well known ad hoc techniques to construct explicit solutions for differential equations, especially for nonlinear partial differential equations (PDEs).

The conservation laws are important in the solutions and reductions of PDEs. In the past few decades, active research efforts have been made on the derivation of conservation laws for PDEs. Many significant methods have been developed for the construction of conservation laws, such as the Nöether’s theorem [14] for variational problems, the Laplace’s direct method [2], the characteristic form introduced by Stuebel [3], the multiplier approach [8,15], the symmetry action on a known conservation law [16], the partial Nöether approach [17], the new conservation method [18], etc.

In this work, we consider a one-dimensional (1D) quantum Zakharov (qZ) system of equations [19] in the form

\[ iE_t + E_{x2} - H^2 E_{x4} - nE = 0, \]
\[ n_{t2} - n_{x2} + H^2 n_{x4} - (|E|^2)_{x2} = 0, \]  

which model the nonlinear interaction between quantum Langmuir waves and quantum ion-acoustic waves in an electron-ion dense quantum plasma. In the dimensionless equations (1), $E = E(x,t)$ is the Langmuir envelope electric field, $n = n(x,t)$ is the density fluctuation, $H = \hbar \omega_i / \kappa B T_e$ ($\hbar$ is the Planck constant divided by $2\pi$, $\kappa B$ is the Boltzmann constant, $\omega_i$ is the ion plasma frequency, and $T_e$ is the electron temperature) is the quantum parameter representing the ratio between the ion plasmon energy and the electron thermal energy. The effect of this quantum correction is to introduce higher-order dispersion. The classical limit $H \equiv 0$ leads the above quantum system to the original classical Zakharov equations [20].

Recently, several aspects related to the qZ system have been studied. The dynamical behavior of the nonlinear interaction of quantum Langmuir waves and quantum ion-acoustic waves, and the coexistence of temporal chaos and spatiotemporal chaos were studied by Misra and Shukla [19]. In Ref. [21], the nonlinear interaction of quantum Langmuir waves and quantum ion-acoustic waves of the qZ system is analyzed in terms of a superposition of three interacting wave modes in Fourier space. Misra et al. [22] have used a Galerkin type approximation to reduce the qZ system to a simplified system of nonlinear ordinary differential equations (ODEs) that governs the temporal behaviors of the slowly varying envelope of the high-frequency electric field and the low frequency density fluctuation. Tang et al. have considered the Lie point symmetries and similarity reductions of the qZ system in Ref. [23]. Based on the Exp-function method and an improved tanh function method, more new exact solutions of the qZ system were obtained by Abdoua et al. [24,25]. The qZ system was analyzed through a time-dependent Gaussian trial function method for an associated Lagrangian formalism in Ref. [26]. Yang et al. [27] have investigated
the qZ system and the existence of quantum solitons in the fully nonlinear quantum wave. Beyond that several authors [28-32] have investigated the different methods for some Zakharov type equations, which govern the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field.

In order to study the Lie symmetry reductions, exact solutions and conservation laws of the qZ system (1), we express the complex electric field as

$$E(x,t) = u(x,t) + iv(x,t),$$

with real fields $u(x,t)$ and $v(x,t)$. By substituting it into Eqs. (1) and separating the imaginary and real parts, we obtain a new qZ system

$$u_t + v_{xx} - H^2 v_{xx} - nv = 0,$$
$$v_t - u_{xx} + H^2 u_{xx} + nu = 0,$$
$$n_{tt} - n_{xx} + H^2 n_{xx} - (u^2 + v^2)_{xx} = 0.$$

The layout of the paper is as follows. In the next section, we obtain symmetry reductions of the qZ system (2) using the Lie group analysis based on the optimal system of one-dimensional subalgebras for the qZ system. Also some exact solutions of Eqs. (1) are obtained by using the sub-equation methods. In Sec. 3, the conservation laws of qZ system (2) are constructed by exploiting the direct (multiplier) method. Finally, the conclusions are summarized in Sec. 4.

2. LIE SYMMETRY ANALYSIS OF qZ SYSTEM

The symmetry group of the qZ system (2) is generated by a vector field of the form

$$\Gamma = \sum_{i=1}^{2} \xi_i \frac{\partial}{\partial x^i} + \sum_{j=1}^{3} \eta_j \frac{\partial}{\partial u^j},$$

where the infinitesimals $\xi_i = \xi_i(x^\alpha, u^\beta)$, $\eta_j = \eta_j(x^\alpha, u^\beta)$ and $(x^1, x^2, u^1, u^2, u^3) = (x, t, u, v, n)$. Applying the second extension $\Gamma^{(2)}$ [6-8] to Eq. (2) and solving the resultant overdetermined system of linear partial differential equations with the aid of Wu’s method [33], one obtains the following infinitesimal functions

$$\xi_1(x, t, u, v, n) = C_1,$$
$$\xi_2(x, t, u, v, n) = C_2,$$
$$\eta_1(x, t, u, v, n) = \frac{1}{2} C_5 t^2 + C_4 t + C_3 v,$$
$$\eta_2(x, t, u, v, n) = -\frac{1}{2} C_5 t^2 + C_4 t + C_3 u,$$
$$\eta_3(x, t, u, v, n) = C_5 t + C_4,$$

where $C_i, i = 1, \ldots, 5$ are five arbitrary constants. Hence the infinitesimal symmetries of Eq. (2) form a five dimensional Lie algebra spanned by the following linearly
independent operators (Lie point symmetries)

\[
\begin{align*}
\Gamma_1 &= \frac{\partial}{\partial x}, \\
\Gamma_2 &= \frac{\partial}{\partial t}, \\
\Gamma_3 &= v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}, \\
\Gamma_4 &= tv \frac{\partial}{\partial u} - tu \frac{\partial}{\partial v} + \frac{\partial}{\partial n}, \\
\Gamma_5 &= \frac{1}{2} t^2 v \frac{\partial}{\partial u} - \frac{1}{2} t^2 u \frac{\partial}{\partial v} + t \frac{\partial}{\partial n}.
\end{align*}
\]

(5)

The operators \( \Gamma_1 \) and \( \Gamma_2 \) are related to the space and time translation, respectively. The symmetry vector field \( \Gamma_3 \) is related to a rotation in the \( u-v \) space. It is seen that the vector field (5) is exactly the same as that of the classical Zakharov equations. Consequently, the quantum corrections (the entrance of higher-order dispersion-the \( H \) terms in equations (1)) do not have any effect on the symmetries of the underlying system [23].

2.1. ONE-DIMENSIONAL OPTIMAL SYSTEM OF SUBALGEBRAS

In this subsection we present the optimal system of one-dimensional subalgebras for the Lie algebra for the point symmetries (5) of the \( qZ \) system (2) to obtain the corresponding optimal set of group-invariant solutions; see Ref. [8] for the method used here for obtaining the one-dimensional optimal system of subalgebras. The adjoint transformations are given by

\[
\text{Ad}(\exp(\varepsilon X_i))X_j = X_j - \varepsilon [X_i, X_j] + \frac{1}{2} \varepsilon^2 [X_i, [X_i, X_j]] - \cdots.
\]

(6)

Here \([X_i, X_j]\) is the commutator given by

\[
[X_i, X_j] = X_i X_j - X_j X_i.
\]

(7)

The commutator table of the Lie point symmetries of the \( qZ \) system (2) and the adjoint representations of the symmetry group of Eq. (2) on its Lie algebra are given in Tables 1 and 2, respectively. Tables 1 and 2 are used to construct the optimal system of one-dimensional subalgebras for system (2).

From Tables 1 and 2 one can obtain an optimal system of one-dimensional subalgebras given by \( \{c_1 \Gamma_1 + \Gamma_2, c_1 \Gamma_1 + \Gamma_2 + c_3 \Gamma_3, c_1 \Gamma_1 + \Gamma_2 + c_4 \Gamma_4 + c_5 \Gamma_5, c_1 \Gamma_1 + \Gamma_2 + c_3 \Gamma_3 + c_4 \Gamma_4 + c_5 \Gamma_5\} \), where \( c_i = \frac{C_i}{C_2}, i = 1, 2, 3, 4, 5 \) and \( C_2 \neq 0 \).
Table 1

Commutator table of the Lie algebra of the qZ system (2)

<table>
<thead>
<tr>
<th>[X_i, X_j]</th>
<th>Γ_1</th>
<th>Γ_2</th>
<th>Γ_3</th>
<th>Γ_4</th>
<th>Γ_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Γ_1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Γ_2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2Γ_3</td>
<td>Γ_4</td>
</tr>
<tr>
<td>Γ_3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Γ_4</td>
<td>0</td>
<td>−2Γ_3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Γ_5</td>
<td>0</td>
<td>−Γ_4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2

Adjoint table of the Lie algebra of the qZ system (2)

<table>
<thead>
<tr>
<th>Ad</th>
<th>Γ_1</th>
<th>Γ_2</th>
<th>Γ_3</th>
<th>Γ_4</th>
<th>Γ_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Γ_1</td>
<td>Γ_1</td>
<td>Γ_2</td>
<td>Γ_3</td>
<td>Γ_4</td>
<td>Γ_5</td>
</tr>
<tr>
<td>Γ_2</td>
<td>Γ_1</td>
<td>Γ_2</td>
<td>Γ_3</td>
<td>−2εΓ_3 + Γ_4</td>
<td>ε^2Γ_3 − εΓ_4 + Γ_5</td>
</tr>
<tr>
<td>Γ_3</td>
<td>Γ_1</td>
<td>Γ_2</td>
<td>Γ_3</td>
<td>Γ_4</td>
<td>Γ_5</td>
</tr>
<tr>
<td>Γ_4</td>
<td>Γ_1</td>
<td>Γ_2 + 2εΓ_3</td>
<td>Γ_3</td>
<td>Γ_4</td>
<td>Γ_5</td>
</tr>
<tr>
<td>Γ_5</td>
<td>Γ_1</td>
<td>Γ_2 + εΓ_4</td>
<td>Γ_3</td>
<td>Γ_4</td>
<td>Γ_5</td>
</tr>
</tbody>
</table>

2.2. SYMMETRY REDUCTIONS OF qZ SYSTEM (2)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions that transform Eqs. (2) into their corresponding systems of ODEs. In order to obtain symmetry reductions and exact solutions, one has to solve the associated Lagrange equations

\[
\frac{dx}{ξ_1} = \frac{dt}{ξ_2} = \frac{du}{η_1} = \frac{dv}{η_2} = \frac{dn}{η_3}. \tag{8}
\]

We consider the following cases.

**Case 1.** C_3 = C_4 = C_5 = 0

In this case by solving the corresponding Lagrange system for the subalgebra c_1Γ_1 + Γ_2 a combination translation symmetries of symmetry field Γ gives rise to the group-invariant solution of the form

\[
u = U(ξ), \quad v = V(ξ), \quad n = N(ξ), \tag{9}\]

where ξ = x − c_1t is an invariant of the symmetry c_1Γ_1 + Γ_2 and c_1 = C_1/C_2 is an arbitrary constant. The functions U, V, and N satisfy the following system of ODEs

\[
\begin{align*}
c_1 U' - V'' + NV + H^2 V^{(4)} &= 0, \\
c_1 V' + U'' - NU - H^2 U^{(4)} &= 0, \\
(c_1^2 - 1) N' - (U^2 + V^2)'' + H^2 N^{(4)} &= 0. \tag{10}\end{align*}
\]

**Case 2.** C_4 = C_5 = 0
The subalgebra \( c_1 \Gamma_1 + \Gamma_2 + c_3 \Gamma_3 \), which is a combination of the translation and rotation symmetries of the symmetry field \( \Gamma \) gives rise to the group-invariant solution of the form

\[
G^2(\xi) = u^2 + v^2, \\
u = G(\xi) \cos(x + K(\xi)), \\
v = G(\xi) \sin(x + K(\xi)),
\]

(11)

where \( \xi = x - c_1 t \) is an invariant of the symmetry \( c_1 \Gamma_1 + \Gamma_2 + c_3 \Gamma_3 \), \( c_1 \) is a constant wave speed, and the similarity functions \( G, K \), and \( N \) satisfy the following similarity reduction equations

\[
FG - cGK' + G(4)H^2 - 6H^2G''(K' + 1) - G'' - 12H^2G' (K' + 1) K'' - 3GH^2(K''^2 + GH^2 (K' + 1)^4 - 4GH^2K(3) (K' + 1) + G (K' + 1)^2 = 0, \\
G'(-c - 4H^2K(3) + 2K'(2H^2K'(K' + 3) + 6H^2 + 1) + 4H^2 + 2) + H^2(-4G(3)(K' + 1) - 6G''K'' + G(6(K' + 1)^2K'' - K'(4))) + GK'' = 0,
\]

(12)

\[
(c^2 - 1)F'' - (G^2)'' + H^2F(4) = 0.
\]

If \( K(\xi) = A\xi = (\lambda - 1)x + \mu t \), i.e., \( A = \lambda - 1, A c_1 = -\mu \), (11) becomes

\[
G^2(\xi) = u^2 + v^2, \\
u = G(\xi) \cos(\lambda x + \mu t), \\
v = G(\xi) \sin(\lambda x + \mu t), \\
n = F(\xi).
\]

(13)

Substitution of (13) into (2), and multiplying by \( \cos(\eta) \) and \( \sin(\eta) \), results in the following system of ODEs

\[
(4H^2\lambda^3 + 2\lambda - c)G' - 4H^2\lambda G(3) = 0, \\
(F + H^2\lambda^4 + \lambda^2 + \mu)G - (6H^2\lambda^2 + 1)G'' + H^2G(4) = 0, \\
(c^2 - 1)F''(\xi) - (G^2)'' + H^2F(4) = 0,
\]

(14)

with \( \eta = \lambda x + \mu t \).

Inserting the differential of the first equation of Eqs. (14) into the second equation of Eqs. (14) gives

\[
(F(\xi) + H^2\lambda^4 + \lambda^2 + \mu)G(\xi) - \left(\frac{c + 2\lambda}{4\lambda} + 5H^2\lambda^2\right)G''(\xi) = 0,
\]

(15)

then Eqs. (14) become

\[
(F(\xi) + H^2\lambda^4 + \lambda^2 + \mu)G(\xi) - \left(\frac{c + 2\lambda}{4\lambda} + 5H^2\lambda^2\right)G''(\xi) = 0,
\]

(16)

**Case 3.** \( C_3 = 0 \)
The subalgebra \( c_1 \Gamma_1 + \Gamma_2 + c_4 \Gamma_4 + c_5 \Gamma_5 \), which is a combination of the translation and rotation symmetries of the symmetry field \( \Gamma \) gives rise to the group-invariant solution of the form

\[
\begin{align*}
  u &= \cos(A_1(\xi) + \frac{1}{2} c_4 t^2 + c_5 t), \\
  v &= -\sin(A_1(\xi) + \frac{1}{2} c_4 t^2 + c_5 t), \\
  n &= c_4 t + A_2(\xi),
\end{align*}
\]

(17)

where \( \xi = x - c_1 t \) is an invariant of the symmetry \( c_1 \Gamma_1 + \Gamma_2 + c_4 \Gamma_4 + c_5 \Gamma_5 \), \( c_1 \) is a constant wave speed, and the similarity functions \( A_1 \) and \( A_2 \) satisfy the following similarity reduction equations

\[
\begin{align*}
  A_2 - 3H^2(A_1'')^2 + H^2(A_1')^4 + (A_1')^2 + A_1(c_1 - 4H^2 A_1^{(3)}) - c_5 &= 0, \\
  (6H^2(A_1')^2 + 1)A_1'' - H^2 A_1^{(4)} &= 0, \\
  (c_1^2 - 1)A_2'' + H^2 A_2^{(4)} &= 0.
\end{align*}
\]

(18)

**Case 4.** \( c_i, \ i = 1, 2, \ldots, 5 \) are arbitrary constants

For the most general generator \( \Gamma = c_1 \Gamma_1 + \Gamma_2 + c_3 \Gamma_3 + c_4 \Gamma_4 + c_5 \Gamma_5 \), \( c_i = \frac{c_i}{c_2}, \ i = 1, 2, 3, 4, 5 \), the corresponding invariant solutions of Eqs. (2) are given by

\[
\begin{align*}
  u &= B_1(\xi) \sin\left(\frac{1}{6} c_3 t^3 + \frac{1}{2} c_4 t^2 + c_5 t\right) + B_2(\xi) \cos\left(\frac{1}{6} c_3 t^3 + \frac{1}{2} c_4 t^2 + c_5 t\right), \\
  v &= B_1(\xi) \cos\left(\frac{1}{6} c_3 t^3 + \frac{1}{2} c_4 t^2 + c_5 t\right) - B_2(\xi) \sin\left(\frac{1}{6} c_3 t^3 + \frac{1}{2} c_4 t^2 + c_5 t\right), \\
  n &= \frac{1}{2} c_3 t^2 + c_4 t + F(\xi),
\end{align*}
\]

(19)

where \( B_1(\xi), B_2(\xi), \) and \( F(\xi) \) are functions of the similarity variable \( \xi = x - c_1 t \) that satisfy the nonlinear system of ODEs

\[
\begin{align*}
  (c_5 - F)B_1 + c_1 B_2' + B_1'' - H^2 B_1^{(4)} &= 0, \\
  (c_5 - F)B_2 - c_1 B_2' + B_2'' - H^2 B_2^{(4)} &= 0, \\
  c_3 - (B_1' + B_2')'' + (c_1 - 1)F'' + H^2 F^{(4)} &= 0.
\end{align*}
\]

(20)

In the special case with \( B_1 = B_2 = 0 \) (i.e., \( E = 0 \)), a pure general periodic similarity nonlinear ion-acoustic wave solution is obtained only in the presence of constant linear and time-dependent nonlinear shears and time-dependent background [23].

### 2.3. Exact Solutions Using the Sub-Equation Method

There are many recently published related papers [34-45] in the area of exactly solvable nonlinear evolution equations in a large variety of physical settings.
where both standard mathematical techniques in the theory of solitons and Lie symmetry analysis were used. In this subsection, we employ the sub-equation method \cite{24,25,46-48} to solve the system \eqref{16}. Amongst the sub-equations, the Riccati and Bernoulli equations are the ones that are often used. Here we use the Riccati equations as the corresponding sub-equation for obtaining the solitary wave solutions of the $qZ$ system \eqref{1}.

Our main goal is to solve Eqs. \eqref{16} using the sub-equation method. We suppose that the solution can be expressed by

\begin{align}
G &= \sum_{j_1=0}^{M} \alpha_{j_1} (\frac{\phi'}{\phi})^{j_1} + \sum_{j_2=0}^{M-1} \beta_{j_2} (\frac{\phi'}{\phi})^{j_2} \frac{1}{\phi}, \quad \alpha_M \beta_{M-1} \neq 0, \\
F &= \sum_{j_1=0}^{N} \lambda_{j_1} (\frac{\phi'}{\phi})^{j_1} + \sum_{j_2=0}^{N-1} \mu_{j_2} (\frac{\phi'}{\phi})^{j_2} \frac{1}{\phi}, \quad \lambda_N \mu_{N-1} \neq 0,
\end{align}

(21)

where $\phi = \phi(\xi)$ satisfies the Riccati equation

\begin{equation}
\phi' = K + \phi^2
\end{equation}

(22)

with its solutions

\begin{equation}
\phi = \begin{cases}
-\sqrt{-K} \tanh(\sqrt{-K} \xi), & K < 0, \\
-\sqrt{-K} \coth(\sqrt{-K} \xi), & K < 0, \\
-\frac{1}{\xi}, & K = 0, \\
\sqrt{K} \tan(\sqrt{K} \xi), & K > 0, \\
-\sqrt{K} \cot(\sqrt{K} \xi), & K > 0.
\end{cases}
\end{equation}

(23)

By the balancing procedure \cite{9}, we obtain two positive integers $M = N = 2$. Thus, the solutions of Eqs. \eqref{16} are of the form

\begin{align}
G(\xi) &= a_0 + a_1 \frac{\phi'}{\phi} + a_2 (\frac{\phi'}{\phi})^2 + a_3 \frac{1}{\phi} + a_4 \frac{\phi'}{\phi} \frac{1}{\phi}, \\
F(\xi) &= b_0 + b_1 \frac{\phi'}{\phi} + b_2 (\frac{\phi'}{\phi})^2 + b_3 \frac{1}{\phi} + b_4 (\frac{\phi'}{\phi}) \frac{1}{\phi},
\end{align}

(24)

from Eqs. \eqref{21}. Substituting \eqref{24} into \eqref{16} and making use of \eqref{22} and then equating the coefficients of the functions $\phi'$ to zero, we obtain an algebraic system of equations in terms of $a_i, b_i$ ($i = 0, 1, ..., 4$). Solving the resultant system of algebraic equations
with the aid of Mathematica, one possible set of values of \( a_i, b_i \) \((i = 0, 1, \ldots, 4)\) are

\[
a_0 = 0, a_1 = 0, a_2 = \pm \frac{3}{2K} \sqrt{c_1^2 - 1} \sqrt{c_1 K + 5c_1^2 \lambda^3 + 2K \lambda - 5\lambda^3},
\]

\[
a_3 = 0, a_4 = \pm \frac{3}{2} \sqrt{c_1^2 - 1} \sqrt{c_1 K + 5c_1^2 \lambda^3 + 2K \lambda - 5\lambda^3},
\]

\[
b_0 = \frac{-4c_1 K^2 - 20c_1^2 K^3 - c_1^2 \lambda^5 - 8K^2 \lambda + 16K \lambda^3 - 4K \lambda \mu + \lambda^5}{4K \lambda},
\]

\[
b_1 = 0, b_2 = \frac{3(c_1 K + 5c_1^2 \lambda^3 + 2K \lambda - 5\lambda^3)}{2K \lambda}, b_3 = 0,
\]

\[
b_4 = -\frac{3(c_1 K + 5c_1^2 \lambda^3 + 2K \lambda - 5\lambda^3)}{2K \lambda}, H = \frac{\sqrt{c_1^2 - 1}}{2\sqrt{K}}.
\]

By substituting (25) and the general solution (23) of (22) into the ansatz (24), meanwhile with the help of the transformation \( E = u + iv \) and (13), we obtain the following five solitary wave solutions of the qZ system (1)

\[
E_1(x, t) = M_1 \sec^2(\sqrt{K}(x - c_1 t)) e^{i(\mu t + \lambda x)}, K > 0,
\]

\[
n_1(x, t) = -\frac{1}{2\lambda} \sec^2(\sqrt{K}(x - c_1 t))[M_2 + M_3 \cos(2\sqrt{K}(x - c_1 t))];
\]

\[
E_2(x, t) = M_1 \csc^2(\sqrt{K}(x - c_1 t)) e^{i(\mu t + \lambda x)}, K > 0,
\]

\[
n_2(x, t) = -\frac{1}{2\lambda} \csc^2(\sqrt{K}(x - c_1 t))[M_2 - M_3 \cos(2\sqrt{K}(x - c_1 t))];
\]

\[
E_3(x, t) = M_1 \text{sech}^2(\sqrt{-K}(x - c_1 t)) e^{i(\mu t + \lambda x)}, K < 0,
\]

\[
n_3(x, t) = -\frac{1}{2\lambda} \text{sech}^2(\sqrt{-K}(x - c_1 t))[M_2 + M_3 \cosh(2\sqrt{-K}(x - c_1 t))];
\]

\[
E_4(x, t) = -M_1 \text{csch}^2(\sqrt{-K}(x - c_1 t)) e^{i(\mu t + \lambda x)}, K < 0,
\]

\[
n_4(x, t) = \frac{1}{2\lambda} \text{csch}^2(\sqrt{-K}(x - c_1 t))[M_2 - M_3 \cosh(2\sqrt{-K}(x - c_1 t))],
\]

where \( M_1 = \frac{3\sqrt{H^2K}\sqrt{K((20H^2\lambda^3 \pm 4H^2K + 1)) + 2\lambda}}{\sqrt{\lambda}} \), \( M_2 = H^2\lambda^5 - 4K \lambda + \lambda^3 + \lambda \mu - \left(40H^2K\lambda^3 \pm 2K\sqrt{4H^2K + 1}\right) \) and \( M_3 = 20H^2K\lambda^3 + \left(2K \lambda \pm K\sqrt{4H^2K + 1}\right) + H^2\lambda^5 + \lambda^3 + \lambda \mu \).

In the solutions (26)-(29), (26) and (27) are two plane periodic solutions; (28) is a bright envelope soliton; (29) is an envelope soliton. From (26)-(29), one can easily see that these solitary wave solutions are induced by the quantum parameter \( H \). When \( H \) is neglected (\( H \to 0 \)), the solitary wave solutions disappear (\( |E|^2 \to 0 \)). The results provide strong evidence that the terms proportional to \( H^2 \), in Eqs. (1) and (2), modified the dispersion-linearity equilibrium, which is ultimately responsible for the existence of solitons.
3. CONSERVATION LAWS

Let $x = (x_1, x_2, \ldots, x_n)$ be $n$ independent variables and $u = (u^1, u^2, \ldots, u^m)$ be $m$ dependent variables. Consider a system of $r$ PDEs of $k$th-order given by

$$P_\alpha[u] = P_\alpha(x, u, u^{(1)}, \ldots, u^{(k)}) = 0, \quad \alpha = 1, 2, \ldots, r,$$

where $u^{(1)} = \{u^{(1)}_i\}$, $u^{(2)} = \{u^{(2)}_{ij}\}$, and $u^{(k)}_i = \frac{\partial u^{(k)}_i}{\partial x_i}$, $u^{(k)}_{ij} = \frac{\partial^2 u^{(k)}_i}{\partial x_i \partial x_j}$, $\cdots$. We let $U = (U^1, U^2, \ldots, U^N)$ denote arbitrary functions of the independent variables $x$ and the partial derivatives $\partial/\partial x_i$ are denoted by subscripts $i$, i.e., $U_i^\sigma = \partial U^\sigma / \partial x_i$, $U_{ij}^\sigma = \partial^2 U^\sigma / \partial x_i \partial x_j$, etc.

1). The total derivative operators $D_i$ with respect to $x_i$ are

$$D_i = \frac{\partial}{\partial x_i} + u_\alpha^i \frac{\partial}{\partial u_\alpha^i} + u_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots$$

where $i, j, k, \ldots = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$.

2). Multipliers for the PDE system (30) are a set of functions $\{\Lambda^\alpha[U]\}$ satisfying

$$\Lambda^\alpha[U] P_\alpha[U] = D_i T^i[U]$$

for some functions $\{T^i[U]\}$.

If $U^\sigma = U^\sigma(x)$ is a solution of PDE system (30), then from (32) we obtain the conservation law

$$D_i T^i[u] = 0$$

of system (30), and for each $i$, $T^i[u]$ is a flux.

3). The standard Euler operators with respect to the differentiable function $U^j$ and its derivatives $U_i^j, U_{ij}^j, \ldots$ are the operators defined by

$$E_{U^j} = \frac{\partial}{\partial U^j} - D_i \frac{\partial}{\partial U_i^j} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial U_{i_1}^{i_2} \cdots i_s} + \cdots,$$

for each $j = 1, 2, \ldots, m$.

$\{\Lambda^\alpha[U]\}$ yields a set of multipliers for a conservation law of system (30) if and only if each Euler operator (34) annihilates the left-hand side of (32), i.e.,

$$E_{U^j} (\Lambda^\alpha[U] P_\alpha[U]) \equiv 0, \quad j = 1, \ldots, n,$$

for arbitrary $U, U_i, U_{ij}, \ldots$, etc.

From the determining equations (35) for multipliers, we obtain the zeroth-order multipliers (with the aid of GeM [49]) $\Lambda^1(x_1, x_2, U^1, U^2, U^3), \Lambda^2(x_1, x_2, U^1, U^2, U^3)$
and $\Lambda^3(x_1, x_2, U^1, U^2, U^3)$ for the qZ system (2), which are given by

$$\begin{align*}
\Lambda^1 &= \frac{1}{2} \gamma_1 t^2 + \gamma_2 t + \gamma_3 U, \\
\Lambda^2 &= -\frac{1}{2} \gamma_1 t^2 + \gamma_2 t + \gamma_3 V, \\
\Lambda^3 &= -\frac{1}{12} \gamma_1 t^3 - \frac{1}{4} \gamma_2 t^2 + \frac{1}{12} t(-3\gamma_1 x^2 + 12\gamma_6 x + 12\gamma_4) \\
&\quad - \frac{1}{4} \gamma_2 x^2 + \gamma_7 x + \gamma_5,
\end{align*}
$$

(36)

where $\gamma_i, i = 1, 2, \cdots, 7$ are arbitrary constants and $t = x_1, x = x_2, U = U^1, V = U^2$.

Then, from (32) and (36), we have the following seven conserved vectors of Eqs. (2) satisfying $\frac{\partial}{\partial t} T^1_i[u, v, n] + \frac{\partial}{\partial x} T^x_i[u, v, n] = 0, i = 1, 2, \cdots, 7$ with

$$(T^1_1, T^x_1) = \left(-\frac{1}{12} (t^3 + \frac{1}{4} tx^2) n_t + \frac{1}{4} (t^2 + x^2) n + \frac{1}{4} t^2 (u^2 + v^2),
- \frac{1}{2} t^2 (u_x v - u v_x) + \frac{1}{2} H^2 t^2 (u_{xxx} v - u v_{xxx} + u_x v_{xx} - u_{xx} v_x)
+ \frac{1}{6} t^3 (uu_x + vv_x) - \frac{1}{2} t x (u^2 + v^2) - \frac{1}{2} H^2 t x n_{xxx}
+ \frac{1}{12} t^3 + \frac{1}{4} t x^2 - \frac{1}{2} H^2 t n_x - \frac{1}{2} t x n - \frac{1}{12} H^2 t^3 + \frac{1}{4} H^2 t x^2) n_{xxx}\right);$$

$$(T^2_1, T^x_1) = \left[-\frac{1}{4} (t^2 + x^2) n_t + \frac{1}{2} t n + \frac{1}{2} t (u^2 + v^2),
\frac{1}{2} (t^2 + x^2) (uu_x + vv_x + \frac{1}{2} n_x - \frac{1}{2} H^2 n_{xxx}) + t (u_x v - u v_x) - \frac{1}{2} t x (u^2 + v^2)
+ H^2 t (u_{xxx} v - u v_{xxx} + u_x v_{xx} - u_{xx} v_x) + \frac{1}{2} H^2 (x n_{xxx} - n_x) - \frac{1}{2} x n\right];$$

$$(T^3_1, T^x_1) = \left[\frac{1}{2} (u^2 + v^2), uu_x - u_x v + H^2 (u_{xxx} v - u v_{xxx} + u_x v_{xx} - u_{xx} v_x)\right];$$

$$(T^1_2, T^x_2) = \left[t n_t - n, -2t (uu_x + vv_x) - t n_x + H^2 t n_{xxx}\right];$$

$$(T^2_2, T^x_2) = \left[n_t, -2(uu_x + vv_x) - n_x + H^2 n_{xxx}\right];$$

$$(T^3_2, T^x_2) = \left[x (tn_t - n), -2t (uu_x + vv_x) + t (u^2 + v^2) + tx (H^2 n_{xxx} - n_x)
+ t (n - H^2 n_{xxx})\right];$$

$$(T^4_1, T^x_1) = \left[x n_t - xn_x - 2x (uu_x + vv_x) + u^2 + v^2 + H^2 (x n_{xxx} - n_x)\right].$$

(37)

**Remark.** One can show that there are no higher-order multipliers for conservation laws of Eqs. (2) by applying the direct method. Since there are seven potential (nonlocal) variables resulting from the conservation laws (37) of Eqs. (2), one could obtain a tree of up to 127 nonlocally related PDE systems [13]. Nonlocal symmetries
of Eqs. (2) might arise from some of these nonlocally related PDE systems through applying Lie’s algorithm to each nonlocally related system in the tree [13].

4. CONCLUSIONS

We have studied the Lie symmetry analysis for the qZ system. Similarity reductions and some exact solutions with the aid of sub-equation method were obtained based on the optimal system of one-dimensional subalgebras for the qZ system. The exact solutions contained two plane periodic, a bright envelope soliton, and an envelope soliton waves. We have checked the correctness of the solutions obtained here by substituting them back into the qZ system. Finally, the conservation laws for the qZ system were derived by employing the direct (multipliers) approach. It is worth to mention that the current study of the qZ system can be further perused from the point of view of Hamiltonian structures and numerical simulations.

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