THE MODIFIED SIMPLE EQUATION METHOD FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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In this study, the modified simple equation method is used to construct exact solutions of the space-time fractional modified Korteweg–de Vries equation, the space-time fractional modified regularized long-wave equation and the space-time fractional coupled Burgers’ equations in mathematical physics. The exact solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive. Also we can see that when the parameters are assigned special values, families of exact solitary wave solutions can be obtained by using this method.

Key words: Exact solution, modified simple equation method, modified Riemann-Liouville derivative, space-time fractional differential equation.

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1. INTRODUCTION

Nonlinear fractional partial differential equations (FPDEs) are generalizations of classical differential equations of integer order. A great number of crucial phenomena in physics, chemistry, biology, biomedical sciences, signal processing, systems identification, control theory, viscoelastic materials and polymers are well described by fractional ordinary differential equations and nonlinear FPDEs.

In the last two decades, a large amount of works have been provided to construct analytical and numerical solutions of fractional ordinary differential equations and fractional partial differential equations [1–3] of physical interest [4–6]. Recently, many powerful methods have been proposed to obtain numerical solutions and exact solutions of FPDEs [7–10], such as the Adomian decomposition method [11], the variational iterative method [12], the Jacobi collocation method [13–15], the exp-function method [16, 17], the \((G'/G)\)-expansion method [18–20], the first integral method [21–23], the sub-equation method [24–26], the functional variable method [27], the simplest equation method [28, 29], the modified trial equation method [30, 31] and so on [32].

The present paper is motivated by the desire to construct new exact solutions of nonlinear fractional differential equations by the modified simple equation (MSE) method. The rest of the paper is organized as follows: In Sec. 2, we present the modified Riemann-Liouville derivative, we give some preliminaries, and we describe the proposed modified simple equation method. In Sec. 3, we apply the proposed method to establish the exact solutions for the space-time fractional modified Korteweg-de Vries equation (mKdV), the space-time fractional modified regularized long-wave equation (mRLW) and the space-time fractional nonlinear coupled Burgers equations. In the last section some conclusions are given.

2. PRELIMINARIES AND THE MODIFIED SIMPLE EQUATION METHOD

Jumarie's modified Riemann-Liouville derivative of order $\alpha$ is defined as [33, 34]:

$$D_t^\alpha f(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, & 0 < \alpha \leq 1 \\
(f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, \ n \geq 1
\end{array} \right. \quad (2.1)$$

where $\Gamma(.)$ is the Gamma function.

Moreover, some significant properties for the proposed modified Riemann-Liouville derivative are given in [35, 36] as follows:

$$D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \gamma > 0 \quad (2.2)$$

$$D_t^\alpha (af(t) + bg(t)) = aD_t^\alpha f(t) + bD_t^\alpha g(t), \quad (2.3)$$

where $a$ and $b$ are constants, and

$$D_t^\alpha c = 0, \ c = \text{constant}, \quad (2.4)$$

which are direct consequences of the equality

$$d^\alpha x(t) = \Gamma(1+\alpha) dx(t), \quad (2.5)$$

which holds for nondifferentiable functions. We will use these properties in the following problems.

We consider the following general nonlinear fractional differential equation (FDE) of the type

$$P(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\beta u, D_t^\beta D_x^\alpha u, D_x^\beta D_x^\alpha u, \ldots) = 0, \ 0 < \alpha, \beta \leq 1, \quad (2.6)$$

where $u$ is an unknown function and $P$ is a polynomial of $u$ and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved.
We present the main steps of the modified simple equation method for fractional differential equations as follows [37]:

**Step 1.** Using the traveling wave transformation [20]

\[ u(x,t) = u(\xi), \quad \xi = \frac{x^\beta}{1+\beta} + \frac{wt^\alpha}{1+\alpha}, \]

(2.7)

where \( w \) is a non-zero arbitrary constant, we can rewrite Eq. (2.6) as nonlinear ordinary differential equation (ODE):

\[ Q(u, u', u'', u''', ... ) = 0, \]

(2.8)

where the prime denotes the derivation with respect to \( \xi \). If possible, we should integrate Eq. (2.8) term by term one or more times.

**Step 2.** We suppose that the exact solutions of Eq. (2.8) can be obtained in the following form:

\[ u(\xi) = \sum_{k=0}^{m} a_k \left[ \frac{\phi'(\xi)}{\phi(\xi)} \right]^k, \]

(2.9)

where \( a_k, (k = 0, 1, 2, 3, ..., m) \) are arbitrary constants to be determined such that \( a_m \neq 0 \), and \( \phi(\xi) \) is an unknown function to be determined later such that \( \phi'(\xi) \neq 0 \). In the tanh-function method, \( (G'/G) \)-expansion method, exp-function method, etc., the solution is represented in terms of some pre-defined functions, but in the modified simple equation method, \( \phi \) is not pre-defined or is not a solution of any pre-defined equation. Therefore, some new solutions may be found by this method. These are the main advantages of this method [38–40].

**Step 3.** We calculate the positive integer \( m \) that occur in Eq. (2.9) by looking at the homogeneous balance in Eq. (2.8) [41].

**Step 4.** Substituting (2.9) into (2.8), we can get a polynomial in \( \phi(\xi) \). Setting all the coefficients of \( \phi^j(\xi) \) \( (j = ..., -2, -1, 0) \) to zero yields a set of over determined nonlinear algebraic equations for \( a_k, (k = 0, 1, 2, 3, ..., m) \) and \( \phi(\xi) \).

**Step 5.** We solve the algebraic equations obtained in Step 4 using the Maple software, thus we can construct a variety of exact solutions for the nonlinear fractional differential equation (2.6).

### 3. APPLICATIONS

In the section, we present three typical examples to illustrate the applicability of the modified simple equation method to solve nonlinear fractional partial differential equations.
3.1. THE SPACE-TIME FRACTIONAL MODIFIED KORTEweg-de Vries EQUATION

In this sub-section, we apply the modified simple equation method to solve the space-time fractional modified Korteweg-de Vries equation (mKdV) of the form,

\[ D_\alpha^0 u + \mu u^2 D_\alpha^0 u + \tau D_\alpha^3 u = 0, \quad 0 < \alpha \leq 1, \] \tag{3.1.1} 

where \( \mu \) and \( \tau \) are non-zero constants. The function \( u(x,t) \) is assumed to be a causal function of time and space, i.e., vanishing for \( t < 0 \) and \( x < 0 \) [42]. Then by using the transformation (2.7) into Eq. (3.1.1), we obtain the following ODE:

\[ wu' + \mu u^2 u' + \tau u''' = 0. \] \tag{3.1.2} 

Integrating Eq. (3.1.2) once, we get:

\[ wu + \frac{\mu u^3}{3} + \tau u'' = 0. \] \tag{3.1.3} 

Balancing \( u'' \) with \( u^3 \) yields \( m = 1 \). Therefore, the solution (2.9) takes the form:

\[ u(\xi) = a_0 + a_1 \left( \frac{\phi'}{\phi} \right), \] \tag{3.1.4} 

where \( a_0 \) and \( a_1 \) are constants to be determined such that \( a_1 \neq 0 \), and \( \phi(\xi) \) is an unknown function such that \( \phi'(\xi) \neq 0 \). It is easy to see that:

\[ u''(\xi) = a_1 \left( \frac{\phi''}{\phi} - \frac{3\phi'\phi''}{\phi^2} + \frac{2\phi'^3}{\phi^3} \right), \] \tag{3.1.5} 

and

\[ u^3 = a_1^3 \left( \frac{\phi'}{\phi} \right)^3 + 3a_1^2 a_0 \left( \frac{\phi'}{\phi} \right)^2 + 3a_1 a_0^2 \left( \frac{\phi'}{\phi} \right) + a_0^3. \] \tag{3.1.6} 

Substituting Eqs. (3.1.4)-(3.1.6) into Eq. (3.1.3) and equating all the coefficients of \( \phi^0, \phi^{-1}, \phi^{-2}, \phi^{-3} \) to zero, we obtain:

\[ \phi^0 : \frac{1}{3} \mu a_0^3 + wa_0 = 0, \] \tag{3.1.7} 

\[ \phi^{-1} : wa_1 \phi' + \mu a_0^2 a_1 \phi' + \tau a_1 \phi''' = 0, \] \tag{3.1.8} 

\[ \phi^{-2} : \mu a_0 a_1^2 \left( \phi' \right)^2 - 3\tau a_1 \phi' \phi'' = 0, \] \tag{3.1.9} 

\[ \phi^{-3} : 2\tau a_1 + \frac{1}{3} \mu a_1^3 = 0. \] \tag{3.1.10} 

Since \( a_1 \neq 0 \), we can find from Eq. (3.1.7) and Eq. (3.1.10) that

\[ a_0 = \pm \sqrt{-\frac{3w}{\mu}}, a_1 = \pm \sqrt{-\frac{6\tau}{\mu}}. \] \tag{3.1.11}
Let us now discuss the following cases:

**Case 1:** When \( a_0 = \pm \sqrt{-\frac{3w}{\mu}} \) and \( a_1 = \pm \sqrt{-\frac{6\tau}{\mu}} \), we obtain a system of ODEs from Eqs. (3.1.8)-(3.1.9), which has the solution:

\[
\phi(\xi) = c_1 + c_2 e^{\sqrt{2w/\tau} \xi}.
\]  
(3.1.12)

Hence, the exact solution of Eq. (3.1.1) is

\[
u(\xi) = \mp \sqrt{-\frac{3w}{\mu}} \left(\frac{-c_1 + c_2 \cosh(\sqrt{2w/\tau} \xi) + c_2 \sinh(\sqrt{2w/\tau} \xi)}{c_1 + c_2 \cosh(\sqrt{2w/\tau} \xi) + c_2 \sinh(\sqrt{2w/\tau} \xi)}\right).
\]  
(3.1.13)

**Case 2:** When \( a_0 = \mp \sqrt{-\frac{3w}{\mu}}, a_1 = \pm \sqrt{-\frac{6\tau}{\mu}} \), we obtain

\[
\phi(\xi) = c_1 + c_2 e^{-\sqrt{2w/\tau} \xi}.
\]  
(3.1.14)

\[
u(\xi) = \pm \sqrt{-\frac{3w}{\mu}} \left(\frac{c_1 - c_2 \cosh(\sqrt{2w/\tau} \xi) + c_2 \sinh(\sqrt{2w/\tau} \xi)}{-c_1 - c_2 \cosh(\sqrt{2w/\tau} \xi) + c_2 \sinh(\sqrt{2w/\tau} \xi)}\right).
\]  
(3.1.15)

Note that these solutions are quite different from the travelling wave solutions found in [42, 43].

**3.2. THE SPACE-TIME FRACTIONAL MODIFIED REGULARIZED LONG-WAVE EQUATION**

We next consider the space-time fractional modified regularized long-wave equation (mRLW) [44]:

\[
D_\tau^\alpha u + v D_x^\alpha u + \mu u^2 D_x^\alpha u - \tau D_t^\alpha D_x^2 u = 0, 0 < \alpha \leq 1,
\]  
(3.2.1)

where \( \tau, \mu, v \) are arbitrary constants. For our purpose, we introduce the following transformations:

\[
u(x,t) = u(\xi), \quad \xi = \frac{x^\alpha}{\Gamma(1 + \alpha)} + \tau \frac{\mu^\alpha}{\Gamma(1 + \alpha)}.
\]  
(3.2.2)

Here \( w \) is a nonzero constant. Substituting Eq. (3.2.2) into Eq. (3.2.1), we get the following ODE:

\[
(w + v)u' + \mu u^2 u' - \tau uu'' = 0.
\]  
(3.2.3)

Further, integrating Eq. (3.2.3) with respect to \( \xi \), we get

\[
(w + v)u + \frac{1}{3} \mu u^3 - \tau uu'' = 0.
\]  
(3.2.4)

By balancing the highest-order derivative term and the nonlinear term in Eq. (3.2.4), the value of \( m \) can be determined, which is \( m = 1 \). Therefore \( u(\xi) \) takes same form
as former. Substituting the values of \(u, u', u''\) into Eq. (3.2.4) and equating the coefficients of \(\phi^0, \phi^{-1}, \phi^{-2}, \phi^{-3}\) to zero, we separately obtain:

\[
\phi^0: wa_0 + \frac{1}{3} \mu a_0^3 + va_0 = 0 \tag{3.2.5}
\]

\[
\phi^{-1}: -\tau wa_1 \phi''' + wa_1 \phi' + va_1 \phi' + \mu a_0^2 a_1 \phi' = 0 \tag{3.2.6}
\]

\[
\phi^{-2}: 3\tau wa_1 \phi'' + \mu a_0^2 (\phi')^2 = 0 \tag{3.2.7}
\]

\[
\phi^{-3}: -2\tau wa_1 + \frac{1}{3} \mu a_1^3 = 0. \tag{3.2.8}
\]

From Eq.(3.2.5), we get

\[
a_0 = \pm \sqrt{-\frac{3(w + v)}{\mu}}. \tag{3.2.9}
\]

And from Eq. (3.2.8), we obtain

\[
a_1 = \pm \sqrt{\frac{6\tau w}{\mu}}. \tag{3.2.10}
\]

Let us now discuss the following cases:

**Case 1:** When \(a_0 = \pm \sqrt{-\frac{3(w + v)}{\mu}}\) and \(a_1 = \pm \sqrt{\frac{6\tau w}{\mu}}\), we have the solution

\[
\phi(\xi) = c_1 + c_2 e^{-\sqrt{-\frac{2(w + v)}{\tau w}} \xi}. \tag{3.2.11}
\]

from Eqs. (3.2.6)-(3.2.7) and the exact solution of the space-time fractional mRLW equation can be written as

\[
u(\xi) = \mp \sqrt{-\frac{3(w + v)}{\mu}} \left( \frac{-c_1 + c_2 \cosh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right) - c_2 \sinh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right)}{c_1 + c_2 \cosh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right) - c_2 \sinh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right)} \right).
\]

**Case 2:** When \(a_0 = \pm \sqrt{-\frac{3(w + v)}{\mu}}\) and \(a_1 = \mp \sqrt{\frac{6\tau w}{\mu}}\), we have

\[
\phi(\xi) = c_1 + c_2 e^{-\sqrt{-\frac{2(w + v)}{\tau w}} \xi}, \tag{3.2.13}
\]

from Eqs. (3.2.6)-(3.2.7) and the exact solution of the space-time fractional mRLW equation can be written as

\[
u(\xi) = \mp \sqrt{-\frac{3(w + v)}{\mu}} \left( \frac{-c_1 + c_2 \cosh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right) + c_2 \sinh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right)}{c_1 + c_2 \cosh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right) + c_2 \sinh\left(\sqrt{-\frac{2(w + v)}{\tau w}} \xi\right)} \right).
\]
where $\xi = \frac{x^\alpha}{\Gamma(1+\alpha)} + w \frac{t^\alpha}{\Gamma(1+\alpha)}$.

Comparing our results with Abdel-Salam’s results [44] then it can be seen that our results are quite different.

3.3. THE SPACE-TIME FRACTIONAL COUPLED BURGERS’ EQUATIONS

The coupled Burgers’ equations are a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [45, 46]:

$$
D_t^\alpha u - D_x^{2\alpha} u + 2u D_x^\alpha u + pD_x^\alpha (uv) = 0,
$$

$$
D_t^\alpha v - D_x^{2\alpha} v + 2v D_x^\alpha v + qD_x^\alpha (uv) = 0,
$$

(3.3.1)

where $u = u(x,t)$, $v = v(x,t)$ are functions and $p, q$ are nonzero constants.

Applying the fractional complex transform (3.3.2), Eq. (3.3.1) can be reduced to the following nonlinear ODE system:

$$
w u' - u'' + 2uu' + p(uv)' = 0,
$$

$$
w v' - v'' + 2vv' + q(uv)' = 0.
$$

(3.3.3)

Further, integrating Eqs. (3.3.3) with respect to $\xi$, we get

$$
w u - u' + u^2 + pu v = 0,
$$

$$
w v - v' + v^2 + quv = 0.
$$

(3.3.4)

Balancing the highest order derivatives and highest nonlinear terms in Eqs. (3.3.4) we have the formal solutions:

$$
u(\xi) = a_0 + a_1 \left( \frac{\phi'}{\phi} \right),
$$

$$
u(\xi) = b_0 + b_1 \left( \frac{\phi'}{\phi} \right).
$$

(3.3.5)

Substituting Eqs. (3.3.5) into the first equation of Eqs. (3.3.4) and equating the coefficients of $\phi^0, \phi^{-1}, \phi^{-2}$ to zero, yields:

$$
\phi^0 : wa_0 + pa_0 b_0 + a_0^2 = 0,
$$

(3.3.6)

$$
\phi^{-1} : pa_0 b_1 \phi' + wa_1 \phi' + 2a_0 a_1 \phi' + pa_1 b_0 \phi' - a_1 \phi'' = 0,
$$

(3.3.7)

$$
\phi^{-2} : a_1^2 + a_1 + pa_1 b_1 = 0.
$$

(3.3.8)

And substituting Eqs. (3.3.5) into the second equation of Eqs. (3.3.4) and equating the coefficients of $\phi^0, \phi^{-1}, \phi^{-2}$ to zero, yields:

$$
\phi^0 : wb_0 + qa_0 b_0 + a_0^2 = 0,
$$

(3.3.9)
\[\phi^{-1} : qa_0b_1\phi' + wb_1\phi' + 2b_0b_1\phi' + qa_1b_0\phi' - b_1\phi'' = 0, \quad (3.3.10)\]

\[\phi^{-2} : b_1^2 + b_1 + qa_1b_1 = 0. \quad (3.3.11)\]

From Eq. (3.3.6) and Eq. (3.3.9), we get

\[
\begin{bmatrix}
a_0 = b_0 = 0, \\
a_0 = -w, b_0 = 0, \\
a_0 = -w \frac{(p-1)}{qp - 1}, b_0 = -w \frac{(q-1)}{qp - 1}
\end{bmatrix}
\quad \text{(3.3.12)}
\]

and from Eq. (3.3.8) and Eq. (3.3.11), we get

\[
\begin{bmatrix}
a_1 = b_1 = 0, \\
a_1 = -b_1, b_1 = 0, \\
a_1 = -b_1, b_1 = -1
\end{bmatrix}
\quad \text{(3.3.13)}
\]

Let us now discuss the following cases:

**Case 1:** When \(a_0 = b_0 = 0\) and \(\phi^{-1} \neq 0\), we have the solution

\[\phi(\xi) = c_1 + c_2 e^{w\xi}, \quad (3.3.14)\]

from Eq. (3.3.7) and Eq. (3.3.10). Then, the exact solution of the space-time fractional coupled Burgers’ equations can be written as

\[
u(\xi) = \frac{(1 - p) c_2 w(cosh(w\xi) + sinh(w\xi))}{(qp - 1) (c_1 + c_2 \cosh(w\xi) + c_2 \sinh(w\xi))}, \quad (3.3.15)\]

\[
u(\xi) = \frac{(1 - q) c_2 w(cosh(w\xi) + sinh(w\xi))}{(qp - 1) (c_1 + c_2 \cosh(w\xi) + c_2 \sinh(w\xi))}. \quad (3.3.16)\]

**Case 2:** When \(a_0 = -\frac{w(p-1)}{qp - 1}, b_0 = -\frac{w(q-1)}{qp - 1}\) and \(a_1 = -b_1, b_1 = -\frac{q-1}{qp - 1}\), we obtain the solution

\[\phi(\xi) = c_1 + c_2 e^{-w\xi}, \quad (3.3.17)\]

from Eq. (3.3.7) and Eq. (3.3.10). Then, the exact solution of the space-time fractional coupled Burgers’ equations can be written as

\[
u(\xi) = \frac{(1 - p) c_1 w}{(qp - 1) (c_1 + c_2 \cosh(w\xi) - c_2 \sinh(w\xi))}, \quad (3.3.18)\]

\[
u(\xi) = \frac{(1 - q) c_1 w}{(qp - 1) (c_1 + c_2 \cosh(w\xi) - c_2 \sinh(w\xi))}. \quad (3.3.19)\]

Here \(\xi = \frac{x^\alpha}{\Gamma(1+\alpha)} + w \frac{t^\alpha}{\Gamma(1+\alpha)}\) and \(qp \neq 1\).

Note that these solutions are quite different from the travelling wave solutions found in [47]. These solutions may be important and of significance for the explanation of some practical physical problems.
4. CONCLUSION

We have proposed the modified simple equation method for solving FPDEs. We have successfully obtained exact solutions of the space-time fractional modified Korteweg-de Vries equation, the space-time fractional modified regularized long-wave equation and the space-time fractional nonlinear coupled Burgers’ equations by aid of Maple software. The performance of this method is reliable, direct and effective, and also this method gives more general solutions. So, we dealt with a method that can be extended to solve many other systems of nonlinear FPDEs, which are arising in the theory of solitons and in other areas of mathematical physics and engineering.

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