VARIOUS EXACT RATIONAL SOLUTIONS OF THE TWO-DIMENSIONAL
MACCARI’S SYSTEM

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A general explicit form of rational solutions for the Maccari’s system is given in terms of the Gram determinants by the bilinear method. It is shown that the fundamental rational solutions are made up of the lump and the simplest line rogue wave solutions. Under certain parametric conditions, the lumps can be classified into three patterns: bright, bimodal, and dark states. The fundamental rogue wave is called line rogue wave, because it arises from a constant background with a line profile and then disappears into the same background. The multi-rational solutions consist of either fundamental lumps or fundamental line rogue waves, or hybrid of lumps and line rogue waves. The multi-rational solutions also show that the multi-rogue waves describe the interaction of several fundamental rogue waves, which also arise from the constant background and then decay back to it. Moreover, higher-order rogue waves exhibit quite different patterns, such as bright, dark, and intermediate waveforms.

Key words: Maccari’s system, bilinear method, line rogue wave, higher-order rogue wave.

1. INTRODUCTION

Nonlinear phenomena are ubiquitous in engineering and natural sciences, and even in social sciences. Nonlinear evolution equations (NEEs) are extensively used in these vast research areas to describe various phenomena. The analytical, explicit solutions of NEEs can provide an effective description of the evolution of the associated phenomena. Thus, in recent years, various approaches have been established to construct the explicit solutions in closed form of the NEEs, and these methods include the inverse scattering method [1], the Hirota bilinear method [2], the Darboux transformation [3, 4] and so on. Resonance often occurs in nonlinear systems when special criteria among wavenumbers and frequencies are met; the Maccari’s system [5] is a classic example, where the phase velocity of the long wave and the group velocity of the short wave match with each other. Maccari’s system is a kind of NEE that is often presented to describe the motion of isolated waves, localized in small part of space, in many research fields such as hydrodynamics, plasma physics, and

Various exact rational solutions of the two-dimensional Maccari’s system. Over the last few years, several methods have been used to study the Maccari’s system. For example, Chow [6] obtained two-dromion solutions by the technique of coalescence of wavenumbers. By applying the variable separation approach [7–9], many coherent soliton structures such as dromions, breathers, foldons etc., were recently reported in literature [10, 11]. To the best of our knowledge, lump and rogue-wave solutions have never been analyzed up to now for the two-dimensional Maccari’s system, and thus we shall construct them by using Hirota’s bilinear method.

In recent years, the rogue wave phenomena [12] has been observed in different fields, such as Bose-Einstein condensates [13–15], optical systems [16–23], oceans science [24], superfluids [25], capillary waves [26], atmosphere science [27], and plasma physics [28, 29]. The research on the rogue wave phenomena has recently become one of the most active and important areas of focus in both theoretical analysis and experimental observations. The most widely used model is the rogue-wave solution of focusing nonlinear Schrödinger equation, for which plane waves are modulationally unstable. The simplest (fundamental) rogue-wave solution of the nonlinear Schrödinger equation was first discovered by Peregrine [30], which is localized in both space and time. At the point \((0, 0)\) on the \((x, t)\)-plane its amplitude reaches the maximum value that is three times that of the constant background, and decays algebraically to the background state eventually. Recently, different kinds of general rogue waves of the nonlinear Schrödinger equation have been reported in many works [31–35]. Besides, a hierarchy of general rogue wave solutions for other soliton equations has also been reported in Refs. [36–41], where their authors have demonstrated that the higher-order waves were also localized in both space and time, which could exhibit higher peak amplitudes and more interesting different patterns. As the ocean surface waves are always two-dimensional ones, the study about two-dimensional rogue waves would be much more challenging and important for understanding the dynamics of multi-dimensional physical systems. For example, the two-dimensional analogue of rogue waves has been reported in the Davey-Stewartson (DS) equation [42, 43], Kadomtsev-Petviashvili-I equation [44], and multicomponent Yajima-Oikawa (YO) equation [45]. For the YO equation and two kinds of DS systems, the fundamental rogue wave is a line rogue wave that arises from a constant background and then retreat back to the constant background again. Nonfundamental rogue waves can be classified into two patterns: multi-rogue and higher-order rogue waves, which have different behaviors. These distinct properties of the rogue waves in two-dimensional systems also inspire us to study the rational solutions of the Maccari’s system.

The organization of this paper is as follows. In Sec. 2, the exact and explicit rational solutions of the Maccari’s system are presented in the determinant form by using the Hirota bilinear method. The first-order rational solution and its features are
given in Sec. 3, and the dynamics of multi-rogue waves are discussed in Sec. 4. The higher-order rational solutions including fundamental lumps and line rogue waves, and more general rogue waves are discussed in detail in Sec. 5. The main results of the paper are summarized in Sec. 6.

2. DETERMINANT FORM OF THE RATIONAL SOLUTIONS

The two-dimensional Maccari’s system is of the form [5]:

\[
\begin{align*}
    iA_t + A_{xx} + LA &= 0, \\
    iB_t + B_{xx} + LB &= 0, \\
    L_y &= (AA^* + \alpha BB^*)_x, \\
\end{align*}
\]

(1)

where \( \alpha \neq \pm 1 \) is a real constant, \( A(x,y,t) \) and \( B(x,y,t) \) are two complex functions while \( L(x,y,t) \) is a real function, and the asterisk “*” denotes the complex conjugation. When \( A = B^* \), the Maccari’s system is identical to the so-called simplest (2+1)-dimensional extension of the nonlinear Schrödinger equation proposed by Fokas [46]. In particular, the reduction \( x = y \) leads to the nonlinear Schrödinger equation [47]. If we take \( y = t \) in Eq. (1), it reduces to the system of coupled long-wave resonance equations [48]. We next perform the dependent variable transformation:

\[
\begin{align*}
    A &= \rho_1 e^{i(K_1 x + \omega_1 t)} g/f, \\
    B &= \rho_2 e^{i(K_2 x + \omega_2 t)} h/f, \\
    L &= \epsilon + 2(\log f)_{xx}, \\
\end{align*}
\]

(2)

where \( f \) is a real function, \( g \) and \( h \) are complex functions with respect to variables \( x, y, \) and \( t; \rho_1, \rho_2, K_1, K_2, \omega_1, \) and \( \omega_2 \) are real constants, and \( \epsilon \) satisfies the constraint conditions

\[
\epsilon = K_1^2 + \omega_1 = K_2^2 + \omega_2. 
\]

(3)

Then Eq. (1) can be transformed into the following bilinear form

\[
\begin{align*}
    (D_x^2 + 2iK_1D_x + iD_t)g \cdot f &= 0, \\
    (D_x^2 + 2iK_2D_x + iD_t)h \cdot f &= 0, \\
    (D_{xy} + \lambda)f \cdot f &= \rho_1^2 gg^* + \alpha \rho_2^2 hh^*, \\
\end{align*}
\]

(4)

where \( D \) is the Hirota’s bilinear differential operator and \( \lambda \) is a real constant of integration.

**Theorem 1:** The Maccari’s system has rational solutions (2) with \( f, g, \) and \( h \) given by \( N \times N \) determinants in the form

\[
\begin{align*}
    f &= \tau(0,0), g = \tau(1,0), h = \tau(0,1), \\
\end{align*}
\]

(5)
Various exact rational solutions of the two-dimensional Maccari’s system

where \( \tau(k,l) = det_{1 \leq r,s \leq N}(m_{r,s}(k,l)) \), and the matrix elements are defined by

\[
m_{r,s}(k,l) = \left( \frac{p_r - iK_1}{p_s^* + iK_1} \right)^k \left( \frac{-p_r - iK_2}{p_s^* + iK_2} \right)^l \cdot \sum_{k_1=0}^{n_r} c_{r,k_1} (\partial_{p_r} + \xi'_r + \frac{k}{p_r - iK_1} + \frac{l}{p_r - iK_2})^{n_r - k_1} \cdot \sum_{l_1=0}^{n_s} d_{s,l_1} \left( \partial_{p_s^*} + \xi'^*_s + \frac{k}{-p_s^* - iK_1} - \frac{l}{-p_s^* - iK_2} \right)^{n_s - l_1} \cdot \frac{1}{p_r + p_s^*},
\]

\( \xi'_r = x + 2i p_r t + \frac{y}{2} \left( \frac{\rho_1^2}{(p_r - a)^2} + \frac{\alpha \rho_2^2}{(p_r - b)^2} \right), \)

where \( p_r \) and \( c_{r,k} \) are arbitrary complex constants, which are independent of each other, and \( a = iK_1 \) and \( b = iK_2 \). Here \( n_r \) and \( n_s \) are two arbitrary positive integers, which are independent of each other. The proof of this theorem is given in Appendix, and these rational solutions can also be expressed in terms of Schur polynomials as shown in [38, 42, 43]. In addition, the non-singularity of these rational solutions has been proved in [42, 43, 49] by assuming the real parts of wave numbers \( p_r \) (\( 1 \leq r \leq N \)) are all either positive or negative.

3. FUNDAMENTAL RATIONAL SOLUTION

The simplest rational solution, namely, the first-order rational solution, is given by Theorem 1 with \( N = 1 \), and \( n_r = n_s = n_1 = 1 \). For this case with \( c_{10} = 1 \), the functions \( f \), \( g \), and \( h \) are given by the following formulas

\[
f = \sum_{k_1=0}^{1} c_{1k_1} (\partial_{p_1} + \xi'_1)^{1-k_1} \cdot \sum_{l_1=0}^{1} c^{*}_{1l_1} (\partial_{p_1^*} + \xi'^*_1)^{1-l_1} \cdot \frac{1}{p_1 + p_1^*},
\]

\[
= (\partial_{p_1} + \xi'_1 + c_{11})(\partial_{p_1^*} + \xi'^*_1 + c^{*}_{11}) \frac{1}{p_1 + p_1^*},
\]

\[
= \frac{1}{p_1 + p_1^*} \left[ (\xi'_1 - \frac{1}{p_1 + p_1^*} + c_{11})(\xi'^*_1 - \frac{1}{p_1 + p_1^*} + c^{*}_{11}) + \frac{1}{(p_1 + p_1^*)^2} \right],
\]

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\[
g = \left( \frac{p_1 - iK_1}{p_1^* + iK_1} \right) \sum_{k_1=0}^{1} c_{1k_1} (\partial_{p_1} + \xi_1' + \frac{1}{p_1 - iK_1})^{1-k_1}.
\]
\[
\cdot \sum_{l_1=0}^{1} c_{1l_1} (\partial_{p_1^*} + \xi_1'^* - \frac{1}{p_1^* + iK_1})^{1-l_1} \frac{1}{p_1 + p_1^*} = 
\]
\[
= \left( \frac{p_1 - iK_1}{p_1^* + iK_1} \right) \frac{1}{p_1 + p_1^*} \left[ \left( \xi_1' - \frac{1}{p_1 + p_1^*} + \frac{1}{p_1 - iK_1} + c_{11} \right) \frac{1}{p_1 + p_1^*} 
\right]
\cdot \left( \xi_1'^* - \frac{1}{p_1 + p_1^*} - \frac{1}{p_1^* + iK_1} + c_{11}^* \right) + \frac{1}{(p_1 + p_1^*)^2}, \quad (9)
\]

\[
h = \left( \frac{p_1 - iK_2}{p_1^* + iK_2} \right) \sum_{k_1=0}^{1} c_{1k_1} (\partial_{p_1} + \xi_1' + \frac{1}{p_1 - iK_2})^{1-k_1}.
\]
\[
\cdot \sum_{l_1=0}^{1} c_{1l_1} (\partial_{p_1^*} + \xi_1'^* - \frac{1}{p_1^* + iK_2})^{1-l_1} \frac{1}{p_1 + p_1^*} = 
\]
\[
= \left( \frac{p_1 - iK_2}{p_1^* + iK_2} \right) \frac{1}{p_1 + p_1^*} \left[ \left( \xi_1' - \frac{1}{p_1 + p_1^*} + \frac{1}{p_1 - iK_2} + c_{11} \right) \frac{1}{p_1 + p_1^*} 
\right]
\cdot \left( \xi_1'^* - \frac{1}{p_1 + p_1^*} - \frac{1}{p_1^* + iK_2} + c_{11}^* \right) + \frac{1}{(p_1 + p_1^*)^2}, \quad (10)
\]

and
\[
\xi_1' = x + 2ip_1t + \frac{1}{2} \left( \frac{\rho_1^2}{(p_1 - iK_1)^2} + \frac{\alpha \rho_2^2}{(p_1 - iK_2)^2} \right)y. \quad (11)
\]

Here \(p_1\) and \(c_{11}\) are arbitrary complex constants. We further simplify them as
\[
f = \frac{1}{p_1 + p_1^*} (\theta_1 \theta_1^* + \Delta), \quad (12)
\]
\[
g = \left( \frac{p_1 - iK_1}{p_1^* + iK_1} \right) \frac{1}{p_1 + p_1^*} \left[ (\theta_1 + \frac{1}{p_1 - iK_1}) (\theta_1^* - \frac{1}{p_1^* + iK_1}) + \Delta \right], \quad (13)
\]
\[
h = \left( \frac{p_1 - iK_2}{p_1^* + iK_2} \right) \frac{1}{p_1 + p_1^*} \left[ (\theta_1 + \frac{1}{p_1 - iK_2}) (\theta_1^* - \frac{1}{p_1^* + iK_2}) + \Delta \right], \quad (14)
\]
where

\( \theta_1 = x + (b_1 + ib_2)y + (c_1 + ic_2)t + d_1 + id_2, = \frac{1}{(p_1 + p_1)^2}, \)

\( b_1 = \frac{1}{2} \left( \frac{\rho_1^2(p_{1R}^2 - (p_{1I} - K_1)^2)}{(p_{1I} + (p_{1I} - K_1)^2)^2} + \frac{\alpha \rho_1^2(p_{1R}^2 - (p_{1I} - K_2)^2)}{p_{1I}^2 + (p_{1I} - K_2)^2} \right), \)

\( b_2 = \frac{\rho_1^2(p_1I - K_1)}{(p_{1I}^2 + (p_{1I} - K_1)^2)^2} - \frac{\alpha \rho_1^2(p_1I - K_2)}{(p_{1I}^2 + (p_{1I} - K_2)^2)^2}, \)

\( c_1 = -2p_{1I}, c_2 = 2p_{1R}, d_1 = c_{11R} - \frac{1}{2p_{1R}}, d_2 = c_{11I}, \)

\( a_{11} = \frac{p_{1R}}{p_{1R}^2 + (p_{1I} - K_1)^2}, a_{21} = \frac{2p_{1I}^2 + (p_{1I} - K_1)^2}{2}, \)

\( a_{12} = \frac{p_{1R}}{p_{1R}^2 + (p_{1I} - K_2)^2}, a_{22} = \frac{2p_{1I}^2 + (p_{1I} - K_2)^2}{2}, \)

with \( p_{1R} = Re(p_r) \) and \( p_{1I} = Im(p_r) \).

From the above manipulations, it is clear that we not only separate the real and imaginary parts of \( \theta_1 \) but we also get the fundamental rational solutions

\[
A = \rho_1 e^{i(K_1x + \omega_1t)} \left( \frac{p_1 - iK_1}{p_1^2 + iK_1} \right)(1 - \frac{2i(a_{11}l_2 - a_{21}l_1) + (a_{11}^2 + a_{21}^2)}{l_1^2 + l_2^2 + \theta_0}), \tag{15}
\]

\[
B = \rho_2 e^{i(K_2x + \omega_2t)} \left( \frac{p_1 - iK_2}{p_1^2 + iK_2} \right)(1 - \frac{2i(a_{12}l_2 - a_{22}l_1) + (a_{12}^2 + a_{22}^2)}{l_1^2 + l_2^2 + \theta_0}), \tag{16}
\]

\[
L = \epsilon - \frac{4(l_1^2 - l_2^2 - \theta_0)}{(l_1^2 + l_2^2 + \theta_0)^2}. \tag{17}
\]

where \( l_1 = x + b_1y + c_1t + d_1, l_2 = b_2y + c_2t + d_2, \) and \( \theta_0 = \frac{1}{4p_{1I}} \).

Then we can see that there are two different kinds of fundamental rational solutions, namely, the lump solutions and the fundamental line rogue waves [42]. The fundamental lumps and the prototype rogue waves [50] behave very similarly, such that both of them are space localized, but there is a great difference that the rogue wave is time localized and the lump-type solution is not. The dynamical behaviors of the two kinds of solutions are shown below.

**I. Lump solution.** If \( b_2 \neq 0 \), it is easy to see that \( p_{1I} \neq K_1 \). In this case, \((A, B, L)\) are constants along the \((x(t), y(t))\) trajectory where \( x + b_1y + c_1t = 0, \)

\( b_2y + c_2t = 0. \) Besides, at any given time, \(|A|, |B|, L| \to (|\rho_1|, |\rho_2|, \epsilon) \) when \((x, y)\) goes to infinity. Thus these rational solutions are lumps moving on the constant background and do not disappear at a certain time. Without loss of generality, we just discuss the lump solutions for patterns at \( t = 0 \) as follows.
1. In this situation, $|A|^2$ in (15) has the following possible five critical points:

$$(x_1, y_1) = (-d_1 + \frac{d_2 b_1}{b_2}, -\frac{d_2}{b_2}),$$

$$(x_2, y_2) = \left(\frac{b_1}{b_2}(d_2 + \frac{\sqrt{a_{11}^2 \delta_1}}{2p_{1R}}) + \frac{a_{11}}{a_{21}} \cdot \frac{\sqrt{a_{11}^2 \delta_2}}{2p_{1R}} - d_1, -\frac{d_2}{b_2} + \frac{\sqrt{a_{11}^2 \delta_2}}{2p_{1R} b_2}\right),$$

$$(x_3, y_3) = \left(\frac{b_1}{b_2}(d_2 - \frac{\sqrt{a_{11}^2 \delta_2}}{2p_{1R}}) - \frac{a_{11}}{a_{21}} \cdot \frac{\sqrt{a_{11}^2 \delta_2}}{2p_{1R}} - d_1, -\frac{d_2}{b_2} - \frac{\sqrt{a_{11}^2 \delta_2}}{2p_{1R} b_2}\right),$$

$$(x_4, y_4) = \left(\frac{b_1}{b_2}(d_2 - \frac{\sqrt{a_{21}^2 \delta_1}}{2p_{1R}}) + \frac{a_{11}}{a_{21}} \cdot \frac{\sqrt{a_{21}^2 \delta_1}}{2p_{1R}} - d_1, -\frac{d_2}{b_2} + \frac{\sqrt{a_{21}^2 \delta_1}}{2p_{1R} b_2}\right),$$

$$(x_5, y_5) = \left(\frac{b_1}{b_2}(d_2 + \frac{\sqrt{a_{21}^2 \delta_1}}{2p_{1R}}) - \frac{a_{11}}{a_{21}} \cdot \frac{\sqrt{a_{21}^2 \delta_1}}{2p_{1R}} - d_1, -\frac{d_2}{b_2} - \frac{\sqrt{a_{21}^2 \delta_1}}{2p_{1R} b_2}\right),$$

where $\delta_1 = 3(p_{1I} - K_1)^2 - p_{1R}^2$, and $\delta_2 = 3p_{1R}^2 - (p_{1I} - K_1)^2$. If $\delta_1 > 0$ and $\delta_2 > 0$, it has five critical points, but in other cases it doesn’t have so many critical points.

At this point, we can also get the following second-order derivatives: $H_1 = \frac{\partial^2 |A|^2}{\partial x^2}$, $H_2 = \frac{\partial^2 |A|^2}{\partial y^2}$, $H_3 = \frac{\partial^2 |A|^2}{\partial x \partial y}$, and $H = H_1 H_2 - H_3^2$. So we can get the explicit
form of the above derivatives:

\[ H_1(x_1, y_1) = \frac{192p_1^4R_1^2\delta_4}{\delta_3^2}, \]

\[ H(x_1, y_1) = -\frac{4096\delta_1\delta_2^2\rho_1^8\rho_1^4}{\delta_3^4}, \]

\[ H_1(x_2, y_2) = H_1(x_3, y_3) = -\frac{6p_1^4\rho_1^2\delta_3}{(K_1 - p_{11})^4}, \]

\[ H(x_2, y_2) = H(x_3, y_3) = \frac{16\delta_3^2\delta_1\delta_2^2\rho_1^4}{(K_1 - p_{11})^{10}}, \]

\[ H_1(x_4, y_4) = H_1(x_5, y_5) = 6\rho_1^2\delta_3, \]

\[ H(x_4, y_4) = H(x_5, y_5) = \frac{16\delta_3^2\delta_2^2\rho_1^4b_2^4}{p_1^4R_1}. \]

with \( \delta_3 = p_{11}^2 + (p_{11} - K_1)^2 \), and \( \delta_4 = (p_{11} - K_1)^2 - p_{11}^2 \).

Thus the fundamental lump solutions can be classified into the following three patterns:

(a1) Bright lump. When \( 0 \leq (p_{11} - K_1)^2 \leq \frac{1}{3}p_{11}^2 \), it has two minimum points and one maximum point;

(b1) Bimodal lump. When \( \frac{1}{3}p_{11}^2 < (p_{11} - K_1)^2 < 3p_{11}^2 \), it has two minimum points and two maximum points;

(c1) Dark lump. When \( (p_{11} - K_1)^2 \geq 3p_{11}^2 \), it has one minimum point and two maximum points.

The profiles of the three kinds of fundamental lumps are given in Fig. 1, which confirm the above analysis in an intuitive way.

2. \(|B|^2\) in (16) has the following possible critical points:

\((x_1, y_1) = (-d_1 + \frac{d_2b_1}{b_2}, -\frac{d_2}{b_2})\),

\((x_2, y_2) = (\frac{b_1}{b_2}(d_2 + \frac{\sqrt{a_{12}^2b_2^2}}{2p_{11}^2}) + \frac{a_{12}}{a_{22}}, \frac{\sqrt{a_{12}^2b_2^2}}{2p_{11}^2} - d_1, -\frac{d_2}{b_2} + \frac{\sqrt{a_{22}^2b_2^2}}{2p_{11}^2}),\)

\((x_3, y_3) = (\frac{b_1}{b_2}(d_2 - \frac{\sqrt{a_{12}^2b_2^2}}{2p_{11}^2}) - \frac{a_{12}}{a_{22}}, \frac{\sqrt{a_{12}^2b_2^2}}{2p_{11}^2} - d_1, -\frac{d_2}{b_2} - \frac{\sqrt{a_{22}^2b_2^2}}{2p_{11}^2}),\)

\((x_4, y_4) = (\frac{b_1}{b_2}(d_2 + \frac{\sqrt{a_{22}^2b_2^2}}{2p_{11}^2}) + \frac{a_{12}}{a_{22}}, \frac{\sqrt{a_{22}^2b_2^2}}{2p_{11}^2} - d_1, -\frac{d_2}{b_2} + \frac{\sqrt{a_{12}^2b_2^2}}{2p_{11}^2}),\)

\((x_5, y_5) = (\frac{b_1}{b_2}(d_2 + \frac{\sqrt{a_{22}^2b_2^2}}{2p_{11}^2}) - \frac{a_{12}}{a_{22}}, \frac{\sqrt{a_{22}^2b_2^2}}{2p_{11}^2} - d_1, -\frac{d_2}{b_2} - \frac{\sqrt{a_{12}^2b_2^2}}{2p_{11}^2}),\)

where \( \delta_5 = 3(p_{11} - K_2)^2 - p_{11}^2 \), and \( \delta_6 = 3p_{11}^2 - (p_{11} - K_2)^2 \). If \( \delta_5 > 0 \) and \( \delta_6 > 0 \), it has five critical points, but in other cases it doesn’t have so many critical points.
At this point, we can also get the following second-order derivatives: $G_1 = \frac{\partial^2 |B|^2}{\partial x^2}$, $G_2 = \frac{\partial^2 |B|^2}{\partial y^2}$, $G_3 = \frac{\partial^2 |B|^2}{\partial x \partial y}$, and $G = G_1 G_2 - G_3^2$. Thus we get the explicit form of the above derivatives:

$$G_1(x_1, y_1) = \frac{192 p_{1R}^4 \rho_2^2 \delta_8}{\delta_7^2},$$

$$G(x_1, y_1) = \frac{-4096 \delta_5 \delta_6 \delta_2^2 p_{1R}^4}{\delta_7^4},$$

$$G_1(x_2, y_2) = G_1(x_3, y_3) = \frac{-6 p_{1R}^2 \rho_2^2 \delta_7}{(K_2 - p_{1I})^2},$$

$$G(x_2, y_2) = G(x_3, y_3) = \frac{16 \delta_2^2 \delta_5 \delta_6 \delta_2^2 p_{1R}^4}{(K_2 - p_{1I})^4},$$

$$G_1(x_4, y_4) = G_1(x_5, y_5) = \frac{6 \rho_2^2 \delta_7}{(K_2 - p_{1I})^2},$$

$$G(x_4, y_4) = G(x_5, y_5) = \frac{16 \delta_2^2 \delta_5 \delta_6 \delta_2^2}{p_{1I}^2},$$

with $\delta_7 = p_{1R}^2 + (p_{1I} - K_2)^2$, and $\delta_8 = (p_{1I} - K_2)^2 - p_{1R}^2$.

The fundamental lump solutions of $|B|^2$ can be classified into the following three patterns:
(a) Bright lump. When $0 \leq (p_{1f} - K_2)^2 \leq \frac{1}{3}p_{1, R}^2$, it has one minimum point and two maximum points;

(b) Bimodal lump. When $\frac{1}{3}p_{1, R}^2 < (p_{1f} - K_2)^2 < 3p_{1, R}^2$, it has two minimum points and two maximum points;

(c) Dark lump. When $(p_{1f} - K_2)^2 \geq 3p_{1, R}^2$, it has one minimum point and two maximum points.

The three kinds of fundamental lumps of $|B|$ are plotted in Fig. 2.

3. If $p_{1f} \neq K_r$ or $K_1 \neq K_2$, the resulting solution $L$ in (17) is a normal lump. It has the following three critical points:

\[
(x_1, y_1) = (-d_1 + \frac{d_2 b_1}{b_2}, \frac{d_2}{b_2}),
\]

\[
(x_2, y_2) = (-d_1 + \frac{d_2 b_1}{b_2} + \frac{\sqrt{3}}{2p_{1, R}}, -\frac{d_2}{b_2}),
\]

\[
(x_3, y_3) = (-d_1 + \frac{d_2 b_1}{b_2} - \frac{\sqrt{3}}{2p_{1, R}}, -\frac{d_2}{b_2}).
\]

And then, the associated critical values are

\[
L(x_1, y_1) = 16p_{1, R}^2 + \epsilon, \quad L(x_2, y_2) = L(x_3, y_3) = -2p_{1, R}^2 + \epsilon.
\]

II. Line rogue-wave solutions. When $p_{1f} = K_1 = K_2$, we get $b_2 = 0$, and the rational solution (15) becomes line rogue wave solution [42], which arises from a constant background with a line profile and then disappears into the background again. After a shift of time and space coordinates, $c_{11}$ can be eliminated and the fundamental rogue waves can be rewritten as

\[
A(x, y, t) = -\rho_1 e^{i(K_1 x + \omega_1 t)} (1 - \frac{4i\Omega t + 4}{(\lambda_1 x + \lambda_2 y - 2\lambda_1 K_1 t - 1)^2 + \Omega^2 t^2 + 1}),
\]

\[
B(x, y, t) = -\rho_2 e^{i(K_2 x + \omega_2 t)} (1 - \frac{4i\Omega t + 4}{(\lambda_1 x + \lambda_2 y - 2\lambda_1 K_2 t - 1)^2 + \Omega^2 t^2 + 1}),
\]

\[
L = \epsilon - 4\Omega((\lambda_1 x + \lambda_2 y - 2\lambda_1 K_1 t - 1)^2 + \Omega^2 t^2 - 1)
\]

\[
((\lambda_1 x + \lambda_2 y - 2\lambda_1 K_2 t - 1)^2 + \Omega^2 t^2 + 1)^2,
\]

from (15)-(17), where $\epsilon = K_1^2 + \omega_1$, $\lambda_1 = 2p_{1, R}$, $\lambda_2 = \frac{\rho_1^2 + \rho_2^2}{p_{1, R}}$, $\Omega = 4p_{1, R}^2$.

This kind of solution describes a rogue wave with the line profile oriented along the $(\lambda_2, -\lambda_1)$ direction in the $(x, y)$ plane, thus this kind of fundamental rogue waves are line rogue waves. The orientation angle $\beta$ of this line rogue wave is $\beta = -\arctan\left(\frac{\lambda_2}{\lambda_1}\right)$, and its width is $\frac{2\sqrt{\Omega}}{\sqrt{\lambda_1^2 + \lambda_2^2}}$. Along the line direction $\lambda_1 x + \lambda_2 y - 1 = 0$, the solution is a constant. As $t \to \pm \infty$, both solutions $|A|$ and $|B|$ uniformly approach the constant background ($|\rho_1|$ for $|A|$ and $|\rho_2|$ for $|B|$) in the $(x, y)$ plane; but in the intermediate times, $|A|$ reaches the maximum amplitude $3|\rho_1|$ ($3|\rho_2|$ for
|B|), which is three times of the background amplitude at time \(t = 0\). If we choose \(p_{1I} = K_1 = K_2\), we obtain bright line rogue waves \(|A|\) and \(|B|\). Figure 3 shows the bright line rogue wave \(|A|\). \(L\) is always bright, which is plotted in Fig. 4.

Fig. 3 – (Color online) Fundamental line rogue wave \(|A|\) in (24) with \(\omega_1 = 1, \alpha = 2, K_1 = 1, K_2 = 1, \rho_1 = 1, \rho_2 = 1, p_{1I} = 1, p_{1R} = 1\). (a) \(t = -3\), (b) \(t = -1\), (c) \(t = 0\), (d) \(t = 3\).

Fig. 4 – (Color online) Fundamental solution \(L\) in (24) with \(t = 0, \alpha = 2, K_1 = 1, K_2 = 1, \rho_1 = 1, \rho_2 = 1\). (a) Lump solution with \(\omega_1 = 1, p_{1I} = 1, p_{1R} = 0.5\). (b) Line rogue wave solution with \(\omega_1 = 2, p_{1I} = 0.5, p_{1R} = 1\).

From the above discussions on the fundamental rational solutions, it is noted that the choice of the parameters \(K_r\) and \(p_{1I} (Im(p_1))\) determines these local wave patterns. When \(p_{1I} = K_1 = K_2\), the fundamental rational solutions describe line rogue waves, the other choices admit different patterns of lumps. This fact also holds for \(n_r = 1\) as well as higher \(N\) integers or higher \(n_r\) integers in the solution (5). Non-fundamental rational solutions can also be obtained from solution (5), when either \(N > 1\) or \(n_r > 1\), or both \(N > 1\) and \(n_r > 1\). In the next section, we will consider two subclasses of these non-fundamental rogue waves.

4. MULTI-ROGUE WAVES

The multi-rational solutions can be obtained when we take \(N > 1, n_1 = n_2 = n_3 = \cdots = n_N = 1\) in rational solutions (5). These solutions describe the interac-
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...tion of \( N \) individual fundamental rational solutions, including normal rogue waves, and line rogue waves, which depend on whether or not the parameters satisfy the condition:

\[
Im(p_r) = K_1 = K_2, r = 1, 2, \cdots, N. \tag{25}
\]

If the above conditions are satisfied, the solutions will describe the interaction of \( N \) fundamental line rogue waves. If not, we get a lump solution or a hybrid of lump solution and line rogue wave. We will discuss these rational solutions in more detail in this section.

Fig. 5 – (Color online) A two-rogue-wave solution \(|A|\) generated by functions in (27) and parameters in (28) under (a),(b) condition. (a) \( t = -5 \), (b) \( t = -\frac{1}{2} \), (c) \( t = 0 \), (d) \( t = 5 \).

We set \( c_{r0} = 1 \) \((r = 1, 2, \cdots, N)\), and then we get

\[
m_{r,s}(0,0) = \frac{1}{p_r + p_s^*}[(\nu'_r + \nu'_s)^* + \frac{1}{(p_r + p_s^*)^2}],
\]

\[
m_{r,s}(1,0) = \left( \frac{p_r - iK_1}{p_s^* + iK_1} \right) \frac{1}{p_r + p_s^*}[(\nu'_r + \frac{1}{p_r - iK_1})(\nu'_s)^* + \frac{1}{(p_r - iK_1)^2} + \frac{1}{(p_r + p_s^*)^2}],
\]

\[
m_{r,s}(0,1) = \left( \frac{p_r - iK_2}{p_s^* + iK_2} \right) \frac{1}{p_r + p_s^*}[(\nu'_r + \frac{1}{p_r - iK_2})(\nu'_s)^* + \frac{1}{(p_r - iK_2)^2} + \frac{1}{(p_r + p_s^*)^2}],
\]

from (6). Here \( \nu_r = \xi_r' - \frac{1}{p_r + p_s^*} + c_{r1}, \nu_s = \xi_s^* - \frac{1}{p_r + p_s^*} + c_{s1}, \xi_r' \) is given by (7), and \( p_r, c_{r1} \((r = 1, 2, \cdots, N)\) are arbitrary complex parameters.

Fig. 6 – (Color online) A hybrid of a lump and a line rogue wave solution \(|A|\) generated by functions in (27) under (b3) condition with \( c_{11} = 0, c_{21} = 0, \omega_1 = 1, \alpha = 2, \rho_1 = 1, \rho_2 = 2, K_1 = K_2 = 2, p_{1R} = 1, p_{1I} = 2, p_{2R} = 4, p_{2I} = 1 \). (a) \( t = -1 \), (b) \( t = -\frac{1}{4} \), (c) \( t = 0 \), (d) \( t = \frac{1}{2} \).
To demonstrate these multi-rational solutions, we first consider the case of $N = 2$. In this case, three functions $f$, $g$, and $h$ can be obtained from (5) and (26) as

$$
\begin{align*}
    f &= \begin{vmatrix} m_{11}(0,0) & m_{12}(0,0) \\ m_{21}(0,0) & m_{22}(0,0) \end{vmatrix}, \\
    g &= \begin{vmatrix} m_{11}(1,0) & m_{12}(1,0) \\ m_{21}(1,0) & m_{22}(1,0) \end{vmatrix}, \\
    h &= \begin{vmatrix} m_{11}(0,1) & m_{12}(0,1) \\ m_{21}(0,1) & m_{22}(0,1) \end{vmatrix}.
\end{align*}
$$

The complex parameter $c_{11}$ can be removed by a shift of the $(x,y,t)$ axes. In this case, the solution admits two-usual rogue waves, two-line rogue waves and mixed solution that consists of one usual rogue wave and one line rogue wave. Thus these multi-rogue waves $|A|$, $|B|$, and $L$ can be classified into the following three patterns:

**$(a_3)$** When $p_{1I} = p_{2I} = K_1 = K_2$, the multi-rational solution is a multi-rogue wave that is made up of two fundamental line rogue waves.

**$(b_3)$** When $K_1 = K_2 = p_{1I} \neq p_{2I}$, the multi-rational solution is a combination of a lump and a line rogue wave.

**$(c_3)$** When $K_1 = K_2 \neq p_{rI}$ (r = 1, 2), or $K_1 \neq K_2$, the multi-rational solution admits two fundamental lumps.

![Fig. 7 – (Color online) Six different patterns of multi-lump solutions of $|A|$ generated by functions in (27) under condition $(c_1)$ when $c_{11} = 0$, $c_{21} = 0$, $\omega_1 = 1$, $\rho_1 = 1$, $\rho_2 = 1$, $\alpha = 2$, $K_1 = 1$, $K_2 = 1$.](image)

Now, we discuss these multi-rational solution patterns in more detail. To obtain
the patterns \((a_3)\), we can set the parameters:

\[
K_1 = K_2 = p_{11} = p_{21} = 1, c_{11} = 0, c_{21} = 0, \\
\alpha = 2, \rho_1 = 1, \rho_2 = 1, p_{1R} = 1, p_{2R} = 2
\]  

(28)

for the solution shown in Fig. 5. It is seen that these two line rogue waves arise from a constant background in the entire \((x, y)\) plane, and start to interact with each other, then lead to that the region of their intersection possesses a higher amplitude (see the \(t = -\frac{1}{2}\) panel). Then these line rogue waves start to separate and the higher amplitudes in the interaction region gradually disappear. When the two line rogue waves move towards far field, they can also rise to higher amplitude (see the \(t = 0\) panel).

The most interesting observation here is that the interaction of the two fundamental (line) rogue waves still exist, and this interaction has created two separated curvy waves at \(t = 0\). When \(t > 0\), the interaction disappears completely, and the solutions gradually disappears to the constant background again without a trace (see the \(t = 5\) panel). For this multi-line rogue waves, the maximum value of the solution \(|A|\) or \(|B|\) stays below 4 (four times of the background) all the time, which is quite different from the one-dimensional prototype rogue waves \([50]\), where the maximum value is five times of the background.

For the patterns \((b_3)\), as seen in Fig. 6, a lump stays on the constant background in the entire \((x, y)\) plane (see the \(t = -1\) panel). During the intermediate times, a line rogue wave is created from the constant background and the lump keep on moving (see the \(t = -1/3\) panel). As the time goes on, the lump moves towards the line rogue wave and interact with each other at a latter time (see the \(t = 0\) panel), the regions of their interaction acquire higher amplitudes. When time \(t > 0\), they start to separate from each other and the fundamental line rogue wave goes back to the constant background. Finally, the line rogue wave disappears just leaving the lump to continue moving on the constant background (see the \(t = 3/4\) panel).

For the patterns \((c_3)\), the non-fundamental rational solutions \(|A|\) and \(|B|\) actually consist of two lumps, and can also be classified into six distinct patterns. We consider the case \(K_1 = K_2\). Thus if \((p_{11} - K_1)^2 > 3p_{1R}^2\) and \((p_{21} - K_1)^2 > 3p_{2R}^2\), two fundamental lumps are dark states, and we can call them dark-dark states; if \((p_{11} - K_1)^2 > 3p_{2R}^2\) and \(\frac{1}{3}p_{2R}^2 < (p_{21} - K_1)^2 \leq 3p_{2R}^2\), we get dark-bimodal states; if \((p_{11} - K_1)^2 > 3p_{2R}^2\) and \((p_{21} - K_1)^2 \leq \frac{1}{3}p_{2R}^2\), we obtain dark-bright states; if \(\frac{1}{3}p_{2R}^2 < (p_{11} - K_1)^2 \leq 3p_{1R}^2\) and \(\frac{1}{3}p_{2R}^2 < (p_{21} - K_1)^2 \leq 3p_{2R}^2\), we get bimodal-bimodal states; if \(\frac{1}{3}p_{2R}^2 < (p_{11} - K_1)^2 \leq 3p_{1R}^2\) and \((p_{21} - K_1)^2 \leq \frac{1}{3}p_{2R}^2\), we obtain bimodal-bright states; if \((p_{11} - K_1)^2 \leq \frac{1}{3}p_{1R}^2\) and \((p_{21} - K_1)^2 \leq \frac{1}{3}p_{2R}^2\), we get bright-bright states. In Fig. 7 we have shown all the above patterns for \(|A|\). If \(N > 2\), we get much more states. \(L\) is always a bright state, and here we do not plot it.

For larger values of \(N\), these multi-rational solutions show qualitatively similar behavior, but more line rogue waves will be created from the constant background.
and more lumps are moving on the constant background, then interact with each other, and more and more complicated waveforms will emerge in the interaction region, which leads to more transient solution patterns. For example, for $N = 3$, the solution can be obtained from (5) and (26). And if the parameter choices are

$$c_{11} = 0, c_{21} = 0, c_{31} = 0, \rho_1 = 1, \rho_2 = 1, K_1 = 2, K_2 = 2,$$

$$\alpha = 2, p_{1R} = 1, p_{1I} = 2, p_{2R} = 2, p_{2I} = 2, p_{3R} = \frac{1}{2}, p_{3I} = 2,$$

(29)
a three-rogue-wave solution is shown in Fig. 8. As clearly seen, the wave patterns become much more different and highly complicated in their shapes.

![Fig. 8](image)

Fig. 8 – (Color online) A three-rogue-wave solution $|A|$ generated from (5) and (26) for parameters (29) and $N = 3$. (a) $t = -5$, (b) $t = -1$, (c) $t = 0$, (d) $t = \frac{1}{2}$, (e) $t = 5$.

For another set of parameter choices:

$$c_{11} = 0, c_{21} = 0, c_{31} = 0, \alpha = 2, \rho_1 = 1, \rho_2 = 1, \omega_1 = 1, K_1 = 1, K_2 = 1,$$

(30)
the corresponding rational solutions generated from (5) and (26) are shown in Fig. 9. To the best of our knowledge, these types of patterns of multi-rational solutions have not been reported in the literature.

### 5. HIGHER-ORDER RATIONAL SOLUTIONS

A different subclass of non-fundamental rational solution is the higher-order rational solution, which is obtained by taking $N = 1$ and $n_1 > 1$ in (6). Notice that if the parameters satisfy the following relations:

$$\text{Im}(p_1) = K_1 = K_2,$$

(31)
the resulting solution is a higher-order line rogue wave. Otherwise, they are higher-order lump solutions in certain conditions, which are quite different from the multi-rational solutions. The higher-order rational solutions do not show any sign of mixing of lumps and line rogue solutions, which is the first main difference compared with the multi-rational solutions.

For example, supposing \( n_1 = 2 \), we can get the second-order rational solution from (6) \( (c_{10} = 1, c_{11} = 0) \) through the following functions,

\[
f = [(\partial^2_{p_1} + \xi')^2 + c_{12}][((\partial^2_{p_1} + \xi_{1}^*)^2 + c_{12}^*)^2 + \frac{1}{p_1 + p_1'}],
\]

\[
g = (\frac{-p_1 - iK_1}{p_1^* + iK_1})[(\partial^2_{p_1} + \xi')^2 + \frac{1}{p_1 - iK_1} + \frac{1}{p_1^* - iK_1}]^2 + c_{12}
\]

\[
\cdot [(\partial^2_{p_1} + \xi_{1}^*)^2 + c_{12}^* + \frac{1}{-p_1 - iK_1}][\frac{1}{p_1 + p_1'}],
\]

\[
\text{Fig. 9 – (Color online) Different patterns of multi-rational solutions } |A| \text{ generated from (5) and (26) with parameters (30) and } N = 3. \text{ (a) Two bright lumps and a line rogue wave for } p_{1R} = -2.5, p_{1I} = 2, p_{2R} = 1, p_{2I} = 1, p_{3R} = 3, p_{3I} = 2, t = 0. \text{ (b) A bright lump, a bimodal lump, and a line rogue wave for } p_{1R} = 1, p_{1I} = 1, p_{2R} = 3, p_{2I} = 2, p_{3R} = -2, p_{3I} = 3, t = -0.75. \text{ (c) A bright lump, a dark lump, and a line rogue wave for } p_{1R} = 1, p_{1I} = 1, p_{2R} = -2.5, p_{2I} = 2, p_{3R} = -1, p_{3I} = 3, t = -1.5. \text{ (d) A line rogue wave and two bimodal lumps for } p_{1R} = 1, p_{1I} = 1, p_{2R} = 1.5, p_{2I} = -2, p_{3R} = -1.5, p_{3I} = 3, t = -1. \text{ (e) A line rogue wave with two dark lumps for } p_{1R} = 1, p_{1I} = 3, p_{2R} = 1, p_{2I} = 1, p_{3R} = 1.5, p_{3I} = -2, t = 2. \text{ (f) A bright lump with two line rogue waves for } p_{1R} = 1, p_{1I} = 1, p_{2R} = 3, p_{2I} = 2, p_{3R} = 1.5, p_{3I} = 1, t = -0.7.
\]
Fig. 10 – (Color online) Three different patterns of higher-order rogue-wave solutions $|A|$ generated by functions in (32) and (33) with $c_{12} = 0$, $\alpha = 2$, $\rho_1 = 1$, $\rho_2 = 1$, $\omega_1 = 1$, $K_1 = 1$, $K_2 = 1$. (a) A second-order bright lump for $p_{1R} = 2$, $p_{1I} = 1.5$, $t = 1$. (b) A second order bimodal lump for $p_{1R} = 2$, $p_{1I} = 3$, $t = 1$. (c) A second-order dark lump at intermediate state for $p_{1R} = 1.5$, $p_{1I} = 3.5$, $t = 0.5$. The second row provides the corresponding density plots of the first row.

$$h = \left( \frac{p_1 - iK_2}{p_1^2 + iK_2^2} \right) \left[ (\partial^2_{p_1} + \xi'_1 + \frac{1}{p_1 - iK_2})^2 + c_{12} \right]$$

$$\cdot \left[ (\partial^2_{p_1} + \xi_{1*}^2 + c_{12}^* + \frac{1}{-p_1^* - iK_2}) \frac{1}{p_1 + p_1^*} \right], \quad (34)$$

where $\xi_1'$ is defined by (7), and $p_1$ and $c_{12}$ are arbitrary complex parameters. As mentioned above, if we choose the parameters as per the constraint conditions (31), the rational solution reduces to the second-order line rogue wave solution. If not, it will be the second-order lump solution. When it is a second-order lump, $|A|$ and $|B|$ can also be classified into three patterns. We consider the case $K_1 = K_2 \neq p_{1I}$. If $(p_{1I} - K_1)^2 < \frac{1}{3}p_{1R}^2$, the rational solution is the bright higher-order lump wave; if $(p_{1I} - K_1)^2 > 3p_{1R}^2$, the rational solution is the dark higher-order lump; and if $\frac{1}{3}p_{1R}^2 \leq (p_{1I} - K_1)^2 \leq 3p_{1R}^2$, the rational solution turns out to be the bimodal lump. Figure 10 shows the three patterns of the higher-order rogue wave. Higher-order rogue wave with $n_1 > 2$ can be obtained in a similar way.

For the parameter values

$$\rho_1 = 1, \rho_2 = 1, \alpha = 2, K_1 = 1, K_2 = 1, \omega_1 = 1, p_{1R} = 1, p_{1I} = 1, c_{12} = 0, \quad (35)$$

the corresponding solution is a second-order line rogue wave as shown in Fig. 11. As is clearly seen, this kind of higher-order rogue wave illustrates a very funny phe-
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Fig. 11 – (Color online) A second-order line rogue wave $|A|$ generated by functions in (32) and (33) with parameters (35). (a) $t = -4$, (b) $t = -\frac{1}{2}$, (c) $t = 0$, (d) $t = 3$.

nomenon: the higher-order rogue waves do not uniformly approach the constant background but feature lumps as $t \to \pm\infty$, which is much different from the multi-rogue wave illustrated above. As in Fig. 11, these solutions are localized lumps staying on the constant background when $|t|$ is large enough (see the $t = -4$ and $t = 3$ panels), and as $t$ becomes smaller, the lump gradually disappears. When $t \to 0$ these lumps disappear. But at the same time, a parabola-shaped rogue wave is created from the constant background (see the $t = 0$ panel), which is the main difference from that of the multi-rogue waves discussed earlier.

Fig. 12 – (Color online) Four patterns of the third-order rational solutions $|A|$ from (5) with $N = 1, n_1 = 3, K_1 = 1, K_2 = 1, \alpha = 2, \rho_1 = 1, \rho_2 = 1, \omega_1 = 1$. (a) A line rogue wave for $p_{1R} = 1, p_{1I} = 1, t = 0$. (b) A bright lump for $p_{1R} = 2, p_{1I} = 1.5, t = 1$. (c) A bimodal lump for $p_{1R} = 2, p_{1I} = 3, t = 0.5$. (d) A dark lump for $p_{1R} = 2, p_{1I} = 5, t = 1$. The second row provides the corresponding density plots of the first row.

The higher-order rational solutions with $n_1 > 2$ can be similarly obtained as the second-order rational solutions, however they do not behave as multi-rogue waves, and there is no mixing of lump and line rogue waves in this case. In addition, the lumps or line rogue waves are in a same state. For example, for $n_1 = 3$, if the third-
order rational solutions consist of lumps, the resulting lumps are all either dark states or bright states, and there is no mixing of bright and dark states. Figure 12 shows all these different patterns of third-order rational solutions for $c_{10} = 1$ and $c_{1s} = 0$ ($s = 1, 2, 3$).

6. SUMMARY

In this paper, exact explicit rational solutions of the two-dimensional Maccari’s system were derived by using the Hirota bilinear method. These solutions are expressed in terms of Gram determinants. The fundamental rational solutions contain lumps and rogue waves. The lump solutions exhibit different patterns: dark, bright, bimodal, and intermediate lumps. The fundamental rogue waves are line rogue waves that arise from a constant background with a line profile and then disappear into the background again. We have also obtained two subclasses of non-fundamental rational solutions: multi-rational solutions and higher-order rational solutions. The multi-rational solutions are made of either lumps or rogue waves, and hybrid of lump and rogue waves. The multi-rogue waves describe the interaction of several fundamental rogue waves and funny curvy wave patterns occur because of the interaction. The hybrid rational solutions that feature both lump and rogue wave patterns also possess complex curvy patterns due to the interaction of lumps and line rogue waves. The higher-order rogue waves, however, show quite different features, a part of the wave structure comes from a far distance as a lump moving on the constant background, while the other part of the wave starts from the constant background and retreat to it. This transient wave displays various patterns such as parabola-shaped waves. Moreover, we reported that the higher-order rogue waves also have different shapes such as, dark, bright, and intermediate waveforms.

7. APPENDIX

In this Appendix, we will prove Theorem 1 given in Sec. 2 by using the bilinear Hirota method. First we present the following lemma.

Lemma 1. The bilinear equations in the KP hierarchy:

\[
\begin{align*}
(D^2_{x_1} + 2aD_{x_1} - D_{x_2})\tau(k + 1, l) \cdot \tau(k, l) &= 0, \\
(D^2_{x_1} + 2bD_{x_1} - D_{x_2})\tau(k, l + 1) \cdot \tau(k, l) &= 0, \\
\frac{1}{2} D_{x_1} D_u - 1)\tau(k, l) \cdot \tau(k, l) &= -\tau(k + 1, l)\tau(k - 1, l), \\
\frac{1}{2} D_{x_1} D_v - 1)\tau(k, l) \cdot \tau(k, l) &= -\tau(k, l + 1)\tau(k, l - 1),
\end{align*}
\]
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admit the Gram determinant solutions

$$\tau_{k,l} = \det_{1 \leq r,s \leq N}(m_{r,s}(k,l)),$$  

(37)

with the matrix element

$$m_{r,s}(k,l) = \frac{1}{p_r + q_s} \left( \frac{-p_r - a}{q_s + a} \right)^k \left( \frac{-p_r - b}{q_s + b} \right)^l e^{\xi_r + \eta_s},$$  

(38)

$$\xi_r = p_r x_1 + p_r^2 x_2 + \frac{1}{p_r - a} u + \frac{1}{p_r - b} v + \xi_{r0},$$  

(39)

$$\eta_s = q_s x_1 - q_s^2 x_2 + \frac{1}{q_s + a} u + \frac{1}{q_s + b} v + \eta_{s0},$$  

(40)

where $p_r, q_s, \xi_{r0},$ and $\eta_{s0}$ are complex constants. But these functions do not lead to rational solutions.

This Lemma can be proved by the same method as for Lemma 3.1 in [33], thus its proof is omitted here. We note that by the variable transformation

$$x_1 = x, x_2 = it, u = -\frac{1}{2} \rho_1^2 y, v = -\frac{1}{2} \alpha \rho_2^2 y,$$  

(41)

$$a = i K_1, b = i K_2, \lambda = \rho_1^2 + \alpha \rho_2^2,$$  

and imposing the complex conjugate condition

$$\tau(k,l) = \tau(-k,-l),$$  

(43)

the above bilinear equation (36) is reduced to the bilinear equation (4) of the Maccari’s system, which means equation (4) has also the Gram determinant solutions of the form (37). Moreover, taking parameter constraints

$$p_r = q_s, \quad c_{r_k l_1} = d_{r_k l_1}^s, \quad \xi_{r0} = \eta_{r0}^s$$  

(44)

and using the variable transformations (32), (33), and (34), we can obtain

$$\eta_s = \xi_s^r, \quad m_{r,s}^*(k,l) = m_{r,s}(-k,-l), \quad \tau^*(k,l) = \tau(-k,-l),$$  

(45)

thus the conjugate condition (43) is satisfied. Finally using the gauge freedom of $\tau_n$, we obtain the rational solutions to the two-dimensional Maccari’s system as given in (6) of Theorem 1.

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