CONSERVATION LAWS FOR COUPLED WAVE EQUATIONS

P. MASEMOLA1, A.H. KARA1, A.H. BHRAWY2, A. BISWAS3,4

1 School of Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa
2 Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
3 Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
4 Department of Mathematics, King Abdulaziz University, Jeddah-21589, Saudi Arabia
E-mail: biswas.anjan@gmail.com

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This paper reports conservation laws for coupled wave equations that are studied in several contexts. These include two-layered shallow water waves, long-short wave interactions, longitudinal and transverse wave interactions and others. The conserved densities are secured with the aid of Lie symmetry analysis, while the conserved quantities are obtained from the soliton solutions that were reported earlier.

Key words: Conservation laws, solitons, Lie symmetry.

1. INTRODUCTION

The dynamics of nonlinear waves is governed by various forms of nonlinear evolution equations (NLEEs) [1]-[32]. These include shallow water waves along lake shores and beaches, nonlinear pulse propagation through optical fibers and metamaterials, solitons and vortices in nonlinear optical media and Bose-Einstein condensates, wave propagation through elastic media, etc. For double-layered shallow water waves, the coupled nonlinear wave equations come into play. The study of nonlinear waves is always complete after all conservation laws are determined and the conserved quantities are listed. There are four NLEEs that study two-layered shallow water waves in various situations. Their soliton solutions were all reported earlier. This paper computes conservation laws for these models and the conserved quantities are all listed. Lie symmetry analysis is applied to extract the conserved densities as well as the flux. These densities lead to conserved quantities that are being reported. These are all listed in the subsequent sections.

2. LONG-SHORT WAVE EQUATION

The long-short wave equation arises in the study of fluid dynamics. The dimensionless form for this model is given by [1, 4]

\[ iq_t + aq_{xx} + b|q|^{2m}q + icqr_x + dqr^2 = 0, \]  

(1)

\[ r_t + k r_x + b \left( |q|^{2n} \right)_x = 0, \]  
(2)

where the parameters \( a, b, c, d, \) and \( k \) are constants. Here \( q(x,t) \) is the complex-valued dependent variable, while \( r(x,t) \) is the real-valued dependent variable. Also, \( x \) and \( t \) are spatial and temporal variables, respectively. The parameters \( a \) and \( b \) represent coefficients of dispersion and nonlinearity, otherwise, they are constants. Furthermore, exponents \( m \) and \( n \) govern the power law parameter. This model was studied earlier and soliton solutions were reported \([1]\).

The single-soliton solution to (1) and (2) is given by \([1, 5]\)

\[ q(x,t) = A_1 \operatorname{sech}^{\frac{1}{m}} \left[ B(x - vt) \right] e^{i \left( -\kappa x + \omega t + \theta \right)} \]  
(3)

and

\[ r(x,t) = A_2 \operatorname{sech}^{\frac{2n}{m}} \left[ B(x - vt) \right] \]  
(4)

with the speed of the soliton being given by

\[ v = -2ak. \]  
(5)

The amplitudes \( A_j \), for \( j = 1, 2 \) and the inverse width \( B \) of the solitons are given by

\[ A_1 = \left[ \frac{(m+1) \left( \omega + ak^2 \right) (v-k)^2}{b(v-k)^2 + db^2} \right]^{\frac{1}{m}} \]  
(6)

\[ A_2 = \frac{b}{v-k} \left[ \frac{(m+1) \left( \omega + ak^2 \right) (v-k)^2}{b(v-k)^2 + db^2} \right]^{\frac{m}{m}} \]  
(7)

and

\[ B = \sqrt{\frac{\omega + ak^2}{a}}. \]  
(8)

This soliton solution will exist provided,

\[ c = 0, \]  
(9)

\[ a \left( \omega + ak^2 \right) > 0, \]  
(10)

\[ \left\{ b(v-k)^2 + db^2 \right\} (\omega + ak^2) > 0, \]  
(11)

and

\[ v \neq k. \]  
(12)

To proceed with the extraction of conservation laws, we set \( q = u + iv \); this splits into the the following system of three equations:

\[ a v_{xx} = bv(u^2 + v^2)^m + c v r_x + d v r^2 + u_t \]  
(13)
\[ au_{xx} + bu(u^2 + v^2)^m + cu_x + duv^2 - v_t \]  \hspace{1cm} (14)

\[ b((u^2 + v^2)^n)_x + kr_x + r_t. \]  \hspace{1cm} (15)

By using the mathematical package Maple, we get the following multipliers and conservation laws where \( Q \) represents the multiplier and \( T^t \) represents the conservation law:

(a) For all values of \( n \):

\[ Q = (-u, v, 1), \quad T^t = \frac{1}{2}u^2 - \frac{1}{2}v^2 + w \]

(b) For \( n = 1 \):

(i) \( Q = u, v, t, \frac{x-kt}{2b} \quad T^t = -\frac{1}{2}(bvt^2 + btv^2 + (kt - x)r) \)

(ii) \( Q = (-u, v, 1), \quad T^t = \frac{1}{2}(-u^2 + v^2) \)

(c) For \( n = 0 \)

(i) \( Q = (-u, v, \mathcal{F}(-\frac{x+kt}{k}), r), \quad T^t = \int_0^1 (r \mathcal{F}(t - \frac{x}{k}) - \lambda u - \lambda v^2) d\lambda \)

(ii) \( Q = (v, v + ku, u - cux - cvr, dr(u^2 + v^2)), \quad T^t = \frac{1}{m+1} [3b(u^2 + v^2)^{m+1} + (m+1)(u(3du^2 + 2d^2) - 2cvx^2 + 3dvx - 3ku) + u(-2cvx - 3av + 3cvx + 3kux) + u^2(3dv^2 + 2dv)] \)

We now calculate the conserved quantities and consider the cases for all \( n \) and for \( n = 1 \), excluding the case for \( n = 0 \) as this results in trivial conserved quantities.

**Case-1: \( n = n \)**

\[ I_1 = -\frac{1}{2} \int_{-\infty}^{\infty} (u^2 + v^2 - 2r) dx = -\frac{1}{2} \int_{-\infty}^{\infty} (|q|^2 - 2r) dx \]

\[ = -\frac{\Gamma(\frac{1}{2})}{2B} \left( A_1^2 \frac{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)} - 2A_2 \frac{\Gamma\left(\frac{n}{m}\right)}{\Gamma\left(\frac{n}{m}\right)} \right). \]  \hspace{1cm} (16)

**Case-2: \( n = 1 \)**

\[ I_2 = -\frac{1}{2} \int_{-\infty}^{\infty} (t|q|^2) dx - \frac{1}{2b} \int_{-\infty}^{\infty} (kt - x) rd\notx \]

\[ = -\frac{t\Gamma\left(\frac{1}{2}\right)}{2B} \left( A_1^2 \frac{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)} + A_2k \frac{\Gamma\left(\frac{n}{m}\right)}{b} \frac{\Gamma\left(\frac{n}{m} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{m} + \frac{1}{2}\right)} \right). \]  \hspace{1cm} (17)

Now, for \( I_2 \) to be a conserved quantity, one needs to have \( dI_2/dt = 0 \). Thus, the amplitudes of the solitons must maintain the relation

\[ \frac{A_1^2}{A_2} = -\frac{k}{b} \frac{\Gamma\left(\frac{n}{m}\right)}{\Gamma\left(\frac{n}{m} + \frac{1}{2}\right)}. \]  \hspace{1cm} (18)
for this conservation law to exist.

\[
I_3 = -\frac{1}{2} \int_{-\infty}^{\infty} |q|^2 dx = -\frac{A_1^2 \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{m}\right)}{2B \Gamma\left(\frac{1}{2} + \frac{1}{m}\right)}.
\]  
(19)

3. LONG-WAVE SHORT-WAVE EQUATION

This wave equation describes the interaction between one longitudinal wave with two short transverse waves propagating in a generalized elastic medium [5]. The model will be considered with quadratic and power law nonlinearity. These are separately analyzed in the following subsections.

3.1. QUADRATIC NONLINEARITY

The governing equations are:

\[
i\phi_t + \alpha \phi_{xx} = \beta u\phi,
\]  
(20)

\[
i\psi_t + \alpha \psi_{xx} = \beta u\psi,
\]  
(21)

\[
u_t = \pm \beta (|\phi|^2 + |\psi|^2)_x,
\]  
(22)

where \( u(x,t) \) characterizes the longitudinal wave and \( \phi(x,t) \), \( \psi(x,t) \) are complex-valued functions, representing short transverse waves. Further, \( x \) and \( t \) represent the spatial and temporal variables and \( \alpha \) and \( \beta \) are real constants. For this coupled system of wave equations, 1-soliton solution takes the form:

\[
\phi(x,t) = A_1 \text{sech}\left[B(x - vt)\right]e^{i(-kx + \omega t + \theta)},
\]  
(23)

\[
\psi(x,t) = A_2 \text{sech}\left[B(x - vt)\right]e^{i(-kx + \omega t + \theta)},
\]  
(24)

\[
u(x,t) = A_3 \text{sech}^2\left[B(x - vt)\right],
\]  
(25)

Here \( A_j \) for \( j = 1, 2, 3 \) are the amplitudes of the solitons and \( B \) is their inverse width. From the phase component, \( k \) is the soliton frequency, \( \omega \) is the wave number, and \( \theta \) is the phase constant.

In this section we consider the conservation laws, multipliers, and conserved quantities. First, we let \( \phi = u + iv \) and \( \psi = w + iz \). Then we get the following system of equations:

\[
-\alpha v_{xx} - \beta uv + u_t = 0
\]  
(26)

\[
-\alpha u_{xx} - \beta u^2 - v_t = 0
\]  
(27)

\[
\alpha z_{xx} - \beta uz + w_t = 0
\]  
(28)
\[-\alpha w_{xx} - \beta uw - z_t = 0\]  
\[u_t + \beta(u^2 + v^2 + w^2 + z^2)_x = 0.\]

Further, the system of equations admits the following multipliers and conservation laws, calculated using the mathematical software Maple, where \(Q\) represents the multiplier and \(T^t\) represents the conservation law respectively:

(i) \(Q = (tu, \tau v, tw, \frac{-x}{2\beta})\) \(T^t = \frac{1}{2B}[\beta t(u^2 + v^2 + w^2 + z^2) - xu]\)
(ii) \(Q = (w, -z, u, -v, 0)\) \(T^t = \frac{1}{2}(2uw + (t + 1)vz)\)
(iii) \(Q = (-z, -w, v, 0, u)\) \(T^t = (\beta(w^2 + z^2 + u^2 + v^2)) + \frac{1}{2}(-\alpha(u_{xx}u + v_{xx} + w_{xx}w + z_{xx}z) - \beta u(u^2 + v^2 + w^2 + z^2))\)

Let us now calculate the conserved quantities for the first three conservation laws:

\[I_1 = \frac{1}{2} \int_{-\infty}^{\infty} \left\{ t|\phi|^2 + t|\psi|^2 - \frac{x}{2}(\phi + \phi^*) \right\} dx \]
\[= \frac{t}{2B} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{m}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)} (A_1^2 + A_2^2).\]

For \(I_1\) to be a conserved quantity, \(dI_1/dt\) must vanish, which means
\[A_1^2 + A_2^2 = 0,\]
which is possible when
\[A_1 = A_2 = 0.\]

This shows that the amplitudes of the short waves must be both zero.

\[I_2 = \int_{-\infty}^{\infty} \left\{ \left(\frac{1}{4} - \frac{1}{8}(t + 1)\right) (\phi \psi + \phi^* \psi^*) + \left(\frac{1}{4} + \frac{1}{8}(t + 1)\right) (\phi^* \psi + \phi \psi^*) \right\} dx \]
\[= \frac{A_1 A_2}{B} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{m}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)} \left\{ \frac{1}{4} + \frac{1}{8}(t + 1) \right\},\]

which shows that \(I_2\) will remain conserved for \(dI_2/dt = 0\). This means
\[A_1 A_2 = 0.\]

This implies that either one or both of the first two soliton amplitudes must vanish.

\[I_3 = \frac{1}{2i} \int_{-\infty}^{\infty} (\psi^* \phi - \phi^* \psi) dx = 0.\]
For the fourth conserved quantity, the flux is

\[
T_t = \frac{1}{2} \alpha (u u_{xx} - vv_{xx} - w w_{xx} - z z_{xx}) - \frac{1}{2} \beta (u + v + w + z)
\]

\[
= \frac{1}{2} \alpha (|\phi_x|^2 + |\psi_x|^2) - \frac{1}{2} \beta (|\phi|^2 + |\psi|^2)
\]

(37)

so that

\[
I_4 = \frac{\alpha}{2} \int_{-\infty}^{\infty} (|\phi_x|^2 + |\psi_x|^2 - \beta |\phi|^2 + |\psi|^2) \, dx
\]

\[
= \frac{\alpha}{2m(m+2)B} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{m})}{\Gamma(\frac{1}{2} + \frac{1}{m})} \left[ B^2 + m \left( (m+2)k^2 - A_3 \right) \right].
\]

(38)

### 3.2. Power Law Nonlinearity

Next, we consider the following nonlinear wave equation with power law nonlinearity [5]:

\[
i\phi _t + \alpha _1 (t) \phi _{xx} = \beta _1 (t) u \phi,
\]

(39)

\[
i\psi _t + \alpha _2 (t) \psi _{xx} = \beta _2 (t) u \psi,
\]

(40)

\[
u_t = \gamma _1 (t) \left( |\phi |^{2m} + |\psi |^{2m} \right) _x,
\]

(41)

where \( \alpha _i (t) \), \( \beta _i (t) \), and \( \gamma _i (t) \) with \( (i = 1, 2) \) are arbitrary time-dependent coefficients and \( m \) is the power law nonlinearity parameter. In this case the 1-soliton solution to the coupled wave equations is given by

\[
\phi (x,t) = A_1 \text{sech}^\frac{1}{m} [B(x - vt)] e^{i( - k_1 x + \omega _1 t + \theta _1)}
\]

(42)

\[
\psi (x,t) = A_2 \text{sech}^\frac{1}{m} [B(x - vt)] e^{i( - k_2 x + \omega _2 t + \theta _2)}
\]

(43)

\[
u (x,t) = A_3 \text{sech}^2 [B(x - vt)]
\]

(44)

where the soliton parameters have the same physical meaning as in the case of quadratic nonlinearity.

We consider the case when \( m = 1 \) and let \( \phi = u + iv \) and \( \psi = w + iz \). This gives the following system of equations:

\[
-\alpha _1 (t) v_{xx} - \beta _1 (t) uv + u_t = 0
\]

(45)

\[
-\alpha _1 (t) u_{xx} - \beta _1 (t) u^2 - v_t = 0
\]

(46)

\[
-\alpha _2 (t) z_{xx} - \beta _2 (t) uz + w_t = 0
\]

(47)
\[-\alpha_2(t)w_{xx} - \beta_2(t)uw - z_t = 0 \quad (48)\]

\[\gamma_1(t)(u^2 + v^2 + \gamma_2(t)(w^2 + z^2))_x + u_t = 0. \quad (49)\]

The multiplier and conservation law admitted by this system is:

\[Q = (u \int \gamma_1 dt, -v \int \gamma_2 dt, w \int \gamma_1 \gamma_2 dt, 0, \frac{1}{2} [xu + (u^2 + v^2) \int \gamma_1 dt + (w^2 + z^2) \int \gamma_1 \gamma_2 dt]). \]

We obtain the following conserved quantity:

\[I_1 = \frac{1}{2} \int_{-\infty}^{\infty} (xu + |\phi|^2 \gamma_1 t + |\psi|^2 \gamma_1 \gamma_2 t) \, dx \]
\[= \frac{\gamma_1 t \Gamma (\frac{1}{2}) \Gamma (\frac{1}{m})}{2B \Gamma (\frac{1}{2} + \frac{1}{m})} (A_1^2 + A_2^2 \gamma_2). \quad (50)\]

For \(I_1\) to be a conserved quantity, it is necessary to have

\[\gamma_1 = 0 \quad (51)\]

or

\[\gamma_2 = -\frac{A_1^2}{A_2^2}. \quad (52)\]

### 4. LONG-SHORT WAVE RESONANCE EQUATION

Our governing equation for long-short wave resonance equation is [2]

\[iS_t + \alpha S_{xx} = bLS \quad (53)\]

\[L_t + \beta (|S|^{2n})_x = 0, \quad (54)\]

where \(S = u + iv\). The soliton solution is given by

\[S(x,t) = A_1 \sech^{\frac{1}{2}} [B(x - vt)] e^{i(-\kappa x + \omega t + \sigma_0)}, \quad (55)\]

\[L(x,t) = A_2 \sech^2 [B(x - vt)], \quad (56)\]

where the amplitudes of the solitons are given by

\[A_1 = \left[ \frac{(n + 1)(\omega + \alpha \kappa^2)}{b \beta} \right]^{\frac{1}{2n}} \quad (57)\]
and

\[ A_2 = \frac{(n+1)\left(\omega + \alpha \kappa^2\right)}{bv} \]  

(58)

while the inverse width is

\[ B = n\sqrt{\frac{\omega + \alpha \kappa^2}{\alpha v}}. \]  

(59)

The speed of the soliton is

\[ v = -2\alpha \kappa. \]  

(60)

The following constraint conditions guarantee the existence of these soliton solitons:

\[ b\beta \left(\omega + \alpha \kappa^2\right) > 0, \]  

(61)

\[ av \left(\omega + \alpha \kappa^2\right) > 0, \]  

(62)

and

\[ bv \neq 0. \]  

(63)

The system admits the following multipliers and conserved densities:

For all \( n \)

(i) \( Q = (-u, v, 0), T^t = -\frac{1}{2}(u^2 + v^2) \)

(ii) \( Q = (u_t, u_t, -\frac{1}{2}b(u^2 + v^2)), T^t = \frac{1}{2}(-buL + \alpha uu_{xx} + v(-bvL - \alpha v_{xx}) \).

For \( n = 1 \)

(iii) \( Q = (-ut, vt, \frac{x}{2\beta}), T^t = -\frac{2}{m}[\beta t(u^2 + v^2) - xL] \)

(iv) \( Q = (u_x, u_x, \frac{bl}{2\beta}), T^t = \frac{1}{4\beta}[bL^2 - 2\beta vux - 2\beta wux] \).

The conserved quantities are listed as follows:

(i) \[
I_1 = -\frac{1}{2} \int_{-\infty}^{\infty} |S|^2 \, dx = -\frac{A_1^2}{2B} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{m}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)}. 
\]  

(64)

(ii) \[
I_2 = \frac{1}{2} \int_{-\infty}^{\infty} \left(bL|S|^2 - |S_x|^2\right) \, dx = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{m}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)} \frac{1}{2m(m+2)} \{2mbA_1^2 A_2 - A_1^2 B^2 + 2m(m+2)k^2 A_1^2 \}. 
\]  

(65)

(iii) For \( n = 1 \)

\[
I_3 = -\frac{t}{2} \int_{-\infty}^{\infty} \left(|S|^2 + \frac{xL}{2\beta}\right) \, dx = -\frac{tA_1^2}{2B} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{m}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{m}\right)}. 
\]  

(66)

Similarly, \( I_3 \) will be a conserved quantity, if \( A_1 = 0 \).
\[ I_4 = \int_{-\infty}^{\infty} \left\{ \frac{1}{4\beta} bL^2 - \frac{1}{4i} (S^*_x S - S^* S_x) \right\} \, dx \]
\[ = \frac{b}{3\beta B} A_2^2 \frac{kA_2 \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{m} \right)}{2B_0 \Gamma \left( \frac{1}{2} + \frac{1}{m} \right)}. \]  

5. COUPLED WAVE EQUATION

The coupled wave equation that will be studied in this section is [3]
\[ q_t + b q^2 q_x + c q_x + d q_{xxx} = 0 \]  
\[ r_t + \alpha (qr)_x + \beta r^3 r_x = 0. \]

The soliton solution of this coupled system is given by [3]
\[ q(x,t) = A \sech \left[ B(x - vt) \right], \]
\[ r(x,t) = \left\{ \frac{4v}{\alpha} - \frac{4\alpha A}{\beta} \sech \left[ B(x - vt) \right] \right\}^{\frac{1}{3}}, \]

where the amplitude is given by
\[ A = \lambda = \sqrt{\frac{6(v - c)}{b}} \]

and the inverse width of the soliton is
\[ B = \sqrt{\frac{v - c}{d}}. \]

The speed of the soliton is given by
\[ v = \alpha q + \frac{1}{4} \beta v^3. \]

The constraint conditions for these solitons to exist are:
\[ d(v - c) > 0 \]

and
\[ b(v - c) > 0. \]
The conserved vectors, \((T^x, T^t)\), the latter being the density, are
\[
T^x = cq + \frac{1}{3}q^3 + dq_{xx}, \quad T^t = q
\]
\[
T^x = \frac{1}{4}(2cq^2 + bq^4 - 2q_x^2 + 4dq_{xx}), \quad T^4 = q^2
\]
\[
T^x = \frac{1}{4}br^4 + \alpha qr, \quad T^4 = r.
\]
Therefore the conserved quantities in this case are
\[
I_1 = \int_{-\infty}^{\infty} qdx = \frac{\pi A}{B}
\]
\[
I_2 = \int_{-\infty}^{\infty} q^2 dx = \frac{2A}{B}
\]
and
\[
I_3 = \int_{-\infty}^{\infty} rdx = \frac{1}{B} \sqrt{\frac{4}{\alpha \beta}} \int_{-\infty}^{\infty} (\beta - \alpha A \sech \tau)^{\frac{1}{3}} d\tau.
\]
This third conserved quantity cannot be integrated in closed form and is therefore written in the form of quadratures.

6. CONCLUSIONS

This paper lists the conservation laws of coupled wave equations that are studied in the form of dimensionless coupled NLEEs. The conserved densities and fluxes are recovered by the aid of Lie symmetry analysis. The conserved quantities are obtained from these densities from the soliton solutions of these wave equations that were reported earlier. These laws will portray a complete picture for the wave dynamics for the respective models. Later, this paper will be extended to coupled wave equations with (2+1)-dimensions. Three-coupled wave equation will also be considered. These results will be reported elsewhere.

REFERENCES