THE SUCCESSIVE DIFFERENTIATION METHOD FOR SOLVING BRATU EQUATION AND BRATU-TYPE EQUATIONS

ABDUL-MAJID WAZWAZ
Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA
E-mail: wazwaz@sxu.edu
Received January 10, 2016

In this work, we apply the successive differentiation method for solving the nonlinear Bratu problem and a variety of Bratu-type equations. We use the successive differentiation of any linear or nonlinear ordinary differential equation to determine the values of the function’s derivatives at $x = 0$. The Taylor series of the solution can be established by using the derived coefficients. The algorithm handles the problem in a direct manner without any need to restrictive assumptions. We emphasize the power of the method by applying it to Bratu equation and a variety of Bratu-type equations.

Key words: Successive differentiation method; Taylor series; Bratu equation.

1. INTRODUCTION

The ordinary differential equations (ODEs) [1] are used in mathematical modeling to describe a variety of real-world problems in science and engineering, such as population growth models, the predator-prey equations, and chemical and biological models. The discovery of ODEs goes back to Leibniz, Newton, Bernoulli, and others. Many traditional methods, analytic and numerical, were used in the literature to solve ODEs and partial differential equations (PDEs), either linear or nonlinear [1]-[20].

The main concern of many researchers is to find a unified method that works for almost, but not all, ODEs and PDEs, linear or nonlinear. Researchers have been attempting to discover new methods for solving differential and integral equations, analytically and numerically. The Adomian decomposition method (ADM) [2]-[6], [16, 17], developed by Adomian is a powerful method that can be used to handle ODEs, homogeneous or nonhomogeneous, linear or nonlinear ones. However, the ADM requires the so-called Adomian polynomials to solve nonlinear equations, ODEs or PDEs. To facilitate the computational work of the ADM, Wazwaz [2]-[4] developed the so-called modified decomposition method, where a slight change was proposed in the first two components of the solution. The variational iteration method (VIM) developed by He [7], is another useful systematic technique for solving differential and integral equations. The VIM requires the determination of the Lagrange multipliers, which are developed optimally by using the variational theory.

The Lagrange multipliers are found to be either constants or functions. The homotopy perturbation method (HPM) is a series expansion method used to find the solution of linear and nonlinear equations as well. The HPM was developed by merging the standard homotopy and the perturbation techniques. Other methods were developed and used in the literature. The aforementioned methods were adapted for obtaining the series solution for the equation under discussion. The derived series solution was studied for exact solutions or just for numerical purposes.

In a parallel manner, we aim to use a powerful technique for solving linear and nonlinear, homogeneous and nonhomogeneous ODEs. The proposed scheme relies mainly from using the ODE and finding the successive differentiation of that equation without any need for transforming nonlinear terms or using Lagrange multipliers. In this case, we are able to compute the values of the derivatives at \( x = 0 \), and this will lead to the derivation of the Taylor series of the solution. The successive differentiation method (SDM) will be used to derive series solutions for the Bratu equation [21] and three Bratu-type equations. It will also be used for solving a variety of Bratu-type equations which were first introduced by Wazwaz [4].

The Bratu’s boundary value problem in 1-dimensional planar coordinates is given as

\[
\begin{align*}
  u'' + \lambda e^u &= 0, \\ 0 < x < 1, \\ u(0) &= u(1) = 0,
\end{align*}
\]

which was used to model a combustion problem in a numerical slab, the fuel ignition of the thermal combustion theory, and appeared in the Chandrasekhar model of the expansion of the universe. It stimulates a thermal reaction process in a rigid material where the process depends on a balance between chemically generated heat and heat transfer by conduction [1]-[7].

The exact solution to (1) reads [1]-[6]

\[
  u(x) = -2 \ln \left( \frac{\cosh((x - \frac{1}{2}) \theta)}{\cosh(\frac{\theta}{4})} \right),
\]

where \( \theta \) satisfies

\[
  \theta = \sqrt{2\lambda} \cosh \left( \frac{\theta}{4} \right).
\]

The Bratu problem has zero, one or two solutions when \( \lambda > \lambda_c, \lambda = \lambda_c \), and \( \lambda < \lambda_c \) respectively, where the critical value \( \lambda_c \) satisfies the equation

\[
  \lambda_c = 8 \text{csch}^2 \left( \frac{\theta_c}{4} \right).
\]

It was evaluated in [1]-[6] that the critical value \( \lambda_c \) is given by

\[
  \lambda_c = 3.513830719.
\]
In addition to the standard Bratu problem, there are other Bratu-type problems which were introduced and examined in the literature. In [5], a variety of Bratu-type equations was introduced as

$$u'' - \pi^2 e^u = 0, \quad 0 < x < 1,$$

$$u(0) = u(1) = 0.$$  \hspace{1cm} (6)

$$u'' + \pi^2 e^{-u} = 0, \quad 0 < x < 1$$

$$u(0) = u(1) = 0.$$  \hspace{1cm} (7)

and

$$u'' - e^u = 0, \quad 0 < x < 1,$$

$$u(0) = u(1) = 0.$$  \hspace{1cm} (8)

and will be referred to as type-I, type-II, and type-III, respectively. The last equation (8) is of great interest in magnetohydrodynamics.

2. THE SUCCESSIVE DIFFERENTIATION METHOD

In what follows, we highlight the main steps of the SDM, where the given differential equation will be differentiated successively for many times, and the values of the resulting functions at $x = 0$, or at other values such as $x = 1$, will be determined for each obtained derivative. Having determined these values, we construct the Taylor series of the solution of the equation under discussion. As usual, the obtained series converges to an exact solution, if this solution exists, otherwise, the series can be used for numerical computations. The SDM can be used directly, for both linear and nonlinear equations, and for homogeneous and nonhomogeneous cases, without any need for any restrictive assumptions.

We consider a generalized ordinary differential equation of $n$th-order as

$$u^{(n)}(x) = R(u(x), u'(x), \cdots, u^{(n-1)}(x)) + g(x),$$

$$u(0) = \alpha_0, u'(0) = \alpha_1, \cdots, u^{(n-1)}(0) = \alpha_{n-1}.$$  \hspace{1cm} (9)

where $R$ can be a linear or a nonlinear operator of $u(x)$ and its derivatives with constant coefficients or variable coefficients, and $\alpha_i, 0 \leq i \leq (n-1)$ are given initial conditions. By differentiating both sides of (9) many times, say $k$ times for example, we obtain

$$\left(u^{(n)}\right)^{(k)}(x) = R^{(k)}(u(x), u'(x), \cdots, u^{(n-1)}(x)) + g^{(k)}(x), \quad k \geq 1.$$  \hspace{1cm} (10)

By substituting $x = 0$ at each step of differentiating we compute the values of the functions $(u^{(n)})'(0), (u^{(n)})''(0), (u^{(n)})'''(0), \cdots$. Having determined these values, and by using the given initial conditions, the Taylor series of the solution $u(x)$ follows immediately.
For simplicity, we will introduce the dynamics of the successive differentiation method for first order, and the second-order ODEs, and hence it can be generalized to any linear or nonlinear differential equation of any order.

2.1. THE FIRST-ORDER ODEs

Consider the first-order ordinary differential equation
\[ u'(x) - f(x)u(x) = g(x), \quad u(0) = \alpha_0. \]  
(11)

Following the analysis presented earlier, we set
\[ u(x), \]  
\[ u'(x) = f(x)u(x) + g(x), \]  
\[ u''(x) = f(x)u'(x) + f'(x)u(x) + g'(x), \]  
\[ u'''(x) = f(x)u''(x) + 2f'(x)u'(x) + f''(x)u(x) + g''(x), \]  
\[ u^{(iv)}(x) = f(x)u'''(x) + 3f'(x)u''(x) + 3f''(x)u'(x) + f'''(x)u(x) + g'''(x), \]  
\[ \vdots \]  
(12)

Substituting \( x = 0 \) in each step of (12) gives the values of the functions as
\[ u(0) = \alpha_0 \]  
\[ u'(0) = f(0)u(0) + g(0) = \alpha_1, \]  
\[ u''(0) = f(0)u'(0) + f'(0)u(0) + g'(0) = \alpha_2, \]  
\[ u'''(0) = f(0)u''(0) + 2f'(0)u'(0) + f''(0)u(0) + g''(0) = \alpha_3, \]  
\[ u^{(iv)}(0) = f(0)u'''(0) + 3f'(0)u''(0) + 3f''(0)u'(0) + f'''(0)u(0) + g'''(0) = \alpha_4, \]  
\[ \vdots \]  
(13)

Having determined the values of the function at \( x = 0 \), we can easily set the Taylor series of the solution \( u(x) \) as
\[ u(x) = \alpha_0 + \alpha_1 x + \frac{\alpha_2}{2!} x^2 + \frac{\alpha_3}{3!} x^3 + \frac{\alpha_4}{4!} x^4 + \cdots. \]  
(14)
2.2. THE SECOND-ORDER ODEs

Consider the second-order ordinary differential equation

\[ u''(x) - f(x)u'(x) - h(x)u(x) = g(x), \quad u(0) = \alpha_0, \quad u'(0) = \alpha_1. \]  

(15)

Proceeding as before, we set

\[ u(x), \quad u'(x), \quad u''(x) = f(x)u'(x) + (f'(x) + h(x))u(x) + g'(x), \]
\[ u^{(iv)}(x) = f(x)u''(x) + (2f'(x) + h(x))u'(x) + (f''(x) + 2h'(x))u(x) + g''(x), \]
\[ \vdots \]

(16)

Substituting \( x = 0 \) in each step of (16) gives the values of the function as

\[ u(0) = \alpha_0, \quad u'(0) = \alpha_1, \]
\[ u''(0) = f(0)u'(0) + h(0)u(0) + g(0) = \alpha_2, \]
\[ u'''(0) = f(0)u''(0) + (f'(0) + h(0))u'(0) + h'(0)u(0) + g'(0) = \alpha_3, \]
\[ u^{(iv)}(0) = f(0)u'''(0) + (2f'(0) + h(0))u''(0) + (f''(0) + 2h'(0))u'(0) + h''(0)u(0) + g''(0) = \alpha_4, \]
\[ \vdots \]

(17)

Having determined the values of the function at \( x = 0 \), we can easily set the Taylor function of \( u(x) \) as

\[ u(x) = \alpha_0 + \alpha_1 x + \frac{\alpha_2}{2!} x^2 + \frac{\alpha_3}{3!} x^3 + \frac{\alpha_4}{4!} x^4 + \cdots. \]

(18)

In what follows, we will employ the SDM to the standard Bratu problem and to a variety of Bratu-type problems, which were handled by other methods in the literature.

3. THE BRATU EQUATION

The Bratu’s boundary value problem in 1-dimensional planar coordinates is given as

\[ u'' + \lambda e^u = 0, \quad 0 < x < 1, \]
\[ u(0) = u(1) = 0. \]

(19)
Using the SDM gives

\[ u(x), \quad u(0) = 0, \]
\[ u'(x) =, \quad u'(0) = \alpha, \]
\[ u''(x) = -\lambda e^{u(x)}, \quad u''(0) = -\lambda, \]
\[ u'''(x) = -\lambda u'(x)e^{u(x)}, \quad u'''(0) = -\alpha \lambda, \]
\[ u^{(iv)}(x) = -\left(\lambda(u'(x))^2 + \lambda u''(x)\right)e^{u(x)}, \quad u^{(iv)}(0) = -\alpha^2 \lambda + \lambda^2, \]
\[ u^{(v)}(x) = -\left(\lambda(u'(x))^3 + 3\lambda u'(x)u''(x) + \lambda u'''(x)\right)e^{u(x)}, \quad u^{(v)}(0) = -\alpha^3 \lambda + 4\lambda^2 \alpha, \]
\[ \vdots \]

In view of this, we obtain the series approximation

\[ u(x) = \alpha x - \frac{\lambda}{2!} x^2 - \frac{\alpha \lambda}{3!} x^3 - \frac{\alpha^2 \lambda - \lambda^2}{4!} x^4 - \frac{\alpha^3 \lambda - 4\lambda^2 \alpha}{5!} x^5 + \cdots . \]  \hspace{1cm} (21)

To determine \( \alpha \), we use the other boundary conditions \( u(1) = 0 \) for the case \( \lambda = 3.0 < \lambda_c = 3.513830719 \), and by solving the resulting equation we find

\[ \alpha = 1.675116658, \ 3.515010276. \]  \hspace{1cm} (22)

Substituting these two values of \( \alpha \), and for \( \lambda = 3 \), we obtain the two approximate solutions

\[ u(x) = 1.675116658 x - 1.5 x^2 - 0.837558329 x^3 + 0.2774167694 x^4 + 0.3850249013 x^5 + \cdots , \]  \hspace{1cm} (23)

and

\[ u(x) = 3.515010276 x - 1.5 x^2 - 1.757505138 x^3 - 0.226283017 x^4 - 0.03122183667 x^5 + \cdots . \]  \hspace{1cm} (24)

The obtained result is consistent with the fact that the Bratu equation has two solutions for \( \lambda < \lambda_c \).

4. THE BRATU-TYPE EQUATION I

The Bratu-type I equation reads

\[ \begin{align*}
  u'' - \pi^2 e^u &= 0, \quad 0 < x < 1, \\
  u(0) &= u(1) = 0.
\end{align*} \]  \hspace{1cm} (25)
Proceeding as before, the SDM gives

\[ u(x), \quad u(0) = 0, \]
\[ u'(x) =, \quad u'(0) = \alpha, \]
\[ u''(x) = \pi^2 e^{u(x)}, \quad u''(0) = \pi^2, \]
\[ u'''(x) = \pi^2 u'(x) e^{u(x)}, \quad u'''(0) = \pi^2 \alpha, \]
\[ u^{(iv)}(x) = (\pi^2 (u'(x))^2 + u''(x)) e^{u(x)}, \quad u^{(iv)}(0) = \pi^2 (\pi^2 + \alpha^2), \]
\[ u^{(v)}(x) = \pi^2 (u'(x))^3 + 3 u'(x) u''(x) + u'''(x)) e^{u(x)}, \quad u^{(v)}(0) = \pi^2 (4 \alpha \pi^2 + \alpha^3), \]
\[ \vdots \]
\[ (26) \]

In view of this, we obtain the series approximation

\[ u(x) = \alpha x + \frac{\pi^2}{2} x^2 + \frac{\pi^3}{3} \alpha x^3 + \frac{\pi^2(\pi^2 + \alpha^2)}{4} x^4 + \frac{\pi^2(4 \alpha \pi^2 + \alpha^3)}{5} x^5 + \cdots. \]  
\[ (27) \]

Notice that

\[ u(x) = -\ln[1 - \sin(\alpha x)], \]  
\[ (28) \]
gives the Taylor series

\[ u(x) = \alpha x + \frac{\alpha^2}{2} x^2 + \frac{\alpha^3}{6} x^3 + \frac{\alpha^4}{12} x^4 + \frac{\alpha^5}{24} x^5 + \frac{\alpha^6}{45} x^6 + \cdots. \]  
\[ (29) \]

However, note that

\[ u(x) = -\ln[1 - \sin(\pi x)], \]  
\[ (30) \]
gives the following Taylor series

\[ u(x) = \pi x + \frac{\pi^2}{2} x^2 + \frac{\pi^3}{6} x^3 + \frac{\pi^4}{12} x^4 + \frac{\pi^5}{24} x^5 + \frac{\pi^6}{45} x^6 + \cdots \]  
\[ (31) \]

It is obvious that (30) satisfies the boundary condition \( u(1) = 0 \), and by comparing the two Taylor series we find that \( \alpha = \pi \). This clearly means that (30) is the exact solution of the Bratu-type problem (25).

5. THE BRATU-TYPE EQUATION II

The Bratu-type II equation reads

\[ u'' + \pi^2 e^{-u} = 0, \quad 0 < x < 1, \]
\[ u(0) = u(1) = 0. \]  
\[ (32) \]

In a manner parallel to the analysis presented before, the successive differenti-
The successive differentiation method gives

\[ u(x), \quad u(0) = 0, \]
\[ u'(x) = , \quad u'(0) = \alpha, \]
\[ u''(x) = -\pi^2 e^{-u(x)}, \quad u''(0) = -\pi^2, \]
\[ u'''(x) = \pi^2 u'(x)e^{-u(x)}, \quad u'''(0) = \alpha\pi^2, \]
\[ u^{(iv)}(x) = -\pi^2 \left( (u'(x))^2 - u''(x) \right) e^{-u(x)}, \quad u^{(iv)}(0) = -\pi^2(\pi^2 + \alpha^2), \]
\[ u^{(v)}(x) = \pi^2 \left( (u'(x))^3 - 3u'(x)u''(x) + u'''(x) \right) e^{u(x)}, \quad u^{(v)}(0) = \pi^2(4\alpha\pi^2 + \alpha^3), \]
\[ \vdots. \] (33)

In view of this, we obtain the series approximation

\[ u(x) = \alpha x - \frac{\pi^2}{2!} x^2 + \frac{\pi^2\alpha}{3!} x^3 - \frac{\pi^2(\alpha^2 + \pi^2)}{4!} x^4 + \frac{\pi^2(4\alpha\pi^2 + \alpha^3)}{5!} x^5 + \cdots. \] (34)

To determine \( \alpha \), we use the boundary condition \( u(1) = 0 \) to find that \( \alpha = \pi \). This in turn gives the exact solution as

\[ u(x) = \ln[1 + \sin(\pi x)], \]
(35)

6. THE BRATU-TYPE EQUATION III

The Bratu-type equation III reads

\[ u'' - e^u = 0, \quad 0 < x < 1, \]
\[ u(0) = u(1) = 0, \] (36)

which is of great interest in magnetohydrodynamics [6]. Proceeding as before, we find

\[ u(x), \quad u(0) = 0, \]
\[ u'(x) = , \quad u'(0) = \alpha, \]
\[ u''(x) = e^{u(x)}, \quad u''(0) = 1, \]
\[ u'''(x) = u'(x)e^{u(x)}, \quad u'''(0) = \alpha, \]
\[ u^{(iv)}(x) = \left( (u'(x))^2 + u''(x) \right) e^{u(x)}, \quad u^{(iv)}(0) = 1 + \alpha^2, \]
\[ u^{(v)}(x) = \left( (u'(x))^3 + 3u'(x)u''(x) + u'''(x) \right) e^{u(x)}, \quad u^{(v)}(0) = 4\alpha + \alpha^3, \]
\[ \vdots. \] (37)
The series approximation is therefore given by

\[ u(x) = \alpha x + \frac{1}{2!} x^2 + \frac{\alpha}{3!} x^3 + \frac{1 + \alpha^2}{4!} x^4 + \frac{4\alpha + \alpha^3}{5!} x^5 + \cdots. \]  

(38)

Using the boundary condition \( u(1) = 0 \) gives

\[ \alpha = u'(0) = -0.463639988227675. \]  

(39)

This in turn gives the series approximation [6]

\[ u(x) = -0.463639988227675x + 0.5x^2 - 0.07727333137x^3 + 0.05062341828x^4 - 0.01628520791x^5 + 0.008903876534x^6 - 0.00364639988227675x^7 + 0.001868477052x^8 - 0.000851751709x^9 + 0.0002576095109x^{10} - 0.00005105422613x^{11} + 0.00002576095109x^{12} + \cdots. \]  

(40)

However, an exact solution [6] is given by

\[ u(x) = -\ln 2 + \ln[\lambda(x)], \]  

(41)

where

\[ \lambda(x) = \left\{ c \sec \left[ \frac{c(2x - 1)}{4} \right] \right\}^2, \]  

(42)

and \( c \) is the root of

\[ \left[ c \sec \left( \frac{c}{4} \right) \right]^2 = 2. \]  

(43)

The root \( c \) lies between 0 and \( \frac{\pi}{2} \), namely \( c = 1.336055695 \), to ten figures.

7. CONCLUSIONS

We applied the successive differentiation method for reliable treatment of the standard Bratu problem and for three kinds of Bratu-type equations. The method depends mainly on differentiating the given equation, and evaluating the obtained derivatives at \( x = 0 \). We then obtain the Taylor series for the solution of the equation. The successive differentiation method handles both linear and nonlinear ordinary differential equations, either homogeneous or nonhomogeneous ones, in a direct manner without any need to restrictive conditions. The method works effectively to the Volterra integral equations as will be discussed in a forthcoming work.

REFERENCES


