ASYMMETRIC SOLITONS IN PARITY-TIME-SYMMETRIC DOUBLE-HUMP SCARFF-II POTENTIALS

PENGFEI LI$^1$, DUMITRU MIHALACHE$^2$, LU LI$^1$,*

$^1$Institute of Theoretical Physics, Shanxi University, Taiyuan 030006, China
E-mail*: llz@sxu.edu.cn

$^2$Horia Hulubei National Institute of Physics and Nuclear Engineering, Magurele-Bucharest, 077125, Romania

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Symmetric and asymmetric solitons that form in self-focusing optical waveguides with parity-time ($PT$)-symmetric double-hump Scarff-II potentials are investigated. It is shown that the branch corresponding to asymmetric solitons bifurcates out from the base branch of $PT$-symmetric solitons with the increasing of the input power. The stability of symmetric and asymmetric stationary solitons is investigated by employing both linear stability analysis and direct numerical simulations. The effects of the soliton power, the separation between the two humps of the potential, the width and the modulation strength of the potential, on the structure of the linear stability eigenvalue spectrum is also studied. The different instability scenarios of $PT$-symmetric solitons have also been revealed by using direct numerical simulations.

Key words: Bifurcation of eigenvalue spectrum, asymmetric soliton, parity-time-symmetry breaking.

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1. INTRODUCTION

Much attention has been paid in recent years to the study of localized optical waveforms in balanced gain and loss waveguides. These optical waveguides can be used to investigate subtle quantum concepts by using optical waves for the reason of similarity between paraxial optics wave equation and quantum mechanical Schrödinger equation. One of the key properties is that there exist real eigenvalue spectra in non-Hermitian parity-time ($PT$)-symmetric Hamiltonian systems [1–3]. It is well-known that a necessary condition for a Hamiltonian to be $PT$-symmetric is $U(\xi) = U^*(-\xi)$, where asterisk denotes the complex conjugation. Thus the $PT$-symmetry requires that the real part of the complex-valued potential $U(\xi)$ must be an even function of the position $\xi$, whereas the imaginary part must be an odd function.

In optical settings, the real and imaginary components of the potential stand for the distribution of refractive index and gain/loss, respectively [4–6]. Thus such specially realized optical devices provide a fertile ground for demonstrating experimentally the key characteristics of $PT$-symmetric Hamiltonians [7, 8]. Furthermore,
\( \mathcal{P}\mathcal{T} \)-symmetric soliton solutions and the corresponding beam dynamics have been also investigated in nonlinear regimes with complex-valued \( \mathcal{P}\mathcal{T} \)-symmetric potentials. Various kinds of optical solitons have been studied, including bright solitons, gap solitons, Bragg solitons, and gray or dark solitons and vortices [9–19]. The observation of such \( \mathcal{P}\mathcal{T} \)-symmetric optical solitons is a challenging issue in spite of a series of theoretical predictions in many optical settings, due to the difficulty of realizing gain/loss and optical nonlinearity in perfect synergy. Recently, stable optical discrete solitons in \( \mathcal{P}\mathcal{T} \)-symmetric mesh lattices have been experimentally demonstrated [20]. Unlike other non-conservative nonlinear systems where dissipative solitons appear as fixed points in the parameter space of the governing equations, the discrete \( \mathcal{P}\mathcal{T} \)-symmetric solitons in optical lattices form a continuous parametric family of solutions [20]. Also, the observation of Bloch oscillations in complex \( \mathcal{P}\mathcal{T} \)-symmetric photonic lattices has been recently reported [21].

Also, optical solitons in mixed linear-nonlinear lattices, optical lattice solitons in media described by the complex Ginzburg-Landau model with \( \mathcal{P}\mathcal{T} \)-symmetric periodic potentials, vector solitons in \( \mathcal{P}\mathcal{T} \)-symmetric coupled waveguides, defect solitons in \( \mathcal{P}\mathcal{T} \)-symmetric lattices, solitons in chains of \( \mathcal{P}\mathcal{T} \)-invariant dimers, solitons in nonlocal media, solitons and breathers in \( \mathcal{P}\mathcal{T} \)-symmetric nonlinear couplers, unidirectional optical transport induced by the balanced gain-loss profiles, the nonlinearly induced \( \mathcal{P}\mathcal{T} \) transition in photonic systems, and asymmetric optical amplifiers based on parity-time symmetry have been reported [22–43].

Recently, a family of stable \( \mathcal{P}\mathcal{T} \)-symmetry-breaking solitons with real eigenvalue spectra have been found for a special class of \( \mathcal{P}\mathcal{T} \)-symmetric potentials \( U(\xi) = g^2(\xi) + \alpha g(\xi) + idg(\xi)/d\xi \), where \( g(\xi) \) is a real and even function and \( \alpha \) is a real constant [44]. For this type of potentials, a precondition of the existence of non-\( \mathcal{P}\mathcal{T} \)-symmetric (asymmetric) solitons is miraculously satisfied [45]. These asymmetric solitons bifurcate out from the base branch of \( \mathcal{P}\mathcal{T} \)-symmetric solitons when the soliton power exceeds a certain threshold.

In this paper, we will study the key features of both symmetric and asymmetric solitons in \( \mathcal{P}\mathcal{T} \)-symmetric double-hump Scarff-II potentials. The paper is organized as follows. In Sec. 2, the governing model is introduced. The dependence of the nonlinear propagation constant of both asymmetric and symmetric solitons, on the input power, the width and the modulation strength of the complex-valued potential, and the separation between its two humps, together with the position of bifurcation points of asymmetric solitons from the symmetric ones are presented in Sec. 3. In Sec. 4, we analyze systematically the stability of both asymmetric and symmetric stationary solutions and their nonlinear evolution dynamics. Section 5 concludes the paper.
2. THE GOVERNING MODEL

In the context of the paraxial theory, the optical wave propagation in a Kerr nonlinear planar graded-index waveguide with a balanced gain/loss is governed by the following $(1+1)$-dimensional wave equation

\[
i \frac{\partial \psi}{\partial z} + \frac{1}{2k_0} \frac{\partial^2 \psi}{\partial x^2} + k_0 \frac{[F(x) - n_0]}{n_0} \psi + \frac{k_0 n_2}{n_0} |\psi|^2 \psi = 0,
\]

where $\psi(z, x)$ is the complex envelope function of the optical field, $k_0 = 2\pi n_0/\lambda$ is the wavenumber with $\lambda$ and $n_0$ being the wavelength and the background refractive index, respectively. Here $F(x) = F_R(x) + iF_I(x)$ is a complex function, in which the real part is the refractive index distribution and the imaginary part stands for the gain/loss, and $n_2$ is the Kerr nonlinear parameter.

Equation (1) can be rewritten in a dimensionless form

\[
i \frac{\partial \Psi}{\partial \zeta} + \frac{\partial^2 \Psi}{\partial \xi^2} + U(\xi) \Psi + \sigma |\Psi|^2 \Psi = 0,
\]

by introducing the normalized transformations $\psi(z, x) = \left(\frac{k_0 n_2}{L_D} w_0\right)^{-1/2} \psi(\zeta, \xi)$, $\xi = x/w_0$, and $\zeta = z/L_D$. Here $L_D = 2k_0 w_0^2$ is the diffraction length and $\sigma = n_2/|n_2| = \pm 1$ corresponds to self-focusing $(+1)$ or self-defocusing $(-1)$ Kerr-type nonlinearities, respectively. The complex-valued potential is $U(\xi) \equiv V(\xi) + iW(\xi)$ with $V(\xi) = 2k_0^2 w_0^2 [F_R(x) - n_0]/n_0$ and $W(\xi) = 2k_0^2 w_0^2 F_I(x)/n_0$.

For $\mathcal{PT}$-symmetric systems, the real and imaginary components of the complex potential are required to be even and odd functions, respectively. In general, such potentials cannot support continuous families of non-$\mathcal{PT}$-symmetric solutions [45]. However, for a special type of $\mathcal{PT}$-symmetric potentials $V(\xi) = g^2(\xi) + \alpha g(\xi)$ and $W(\xi) = dg(\xi)/d\xi$ with $g(\xi)$ being an arbitrary real and even function and $\alpha$ being an arbitrary real constant, it has been shown that stable $\mathcal{PT}$-symmetry-breaking solitons can occur [44], where as typical examples, the real functions $g(\xi)$ were taken as localized double-hump exponential functions or as periodic functions. Subsequently, these results were also extended to two-dimensional potentials [46, 47].

In this paper, we aim to search for the asymmetric soliton solutions corresponding to a $\mathcal{PT}$-symmetric double-hump Scarff-II potential, i.e.,

\[
g(\xi) = W_0 \left[ \text{sech} \left( \frac{\xi + \xi_0}{\chi_0} \right) + \text{sech} \left( \frac{\xi - \xi_0}{\chi_0} \right) \right].
\]

Here, $\chi_0$ and $\xi_0$ are related to the width and the separation between the two humps of the potential, respectively, and $W_0$ represents the modulation strength of the complex potential. Note that, in the $\mathcal{PT}$-symmetric Scarff-II potential with a single-hump, the symmetric solutions in both nonlinear and linear regimes have been studied in Refs. [9, 48].
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Fig. 1 – (Color online) Symmetric and asymmetric solutions in the $\mathcal{PT}$-symmetric double-hump Scarff-II potential. (a) Real (blue solid curve) and imaginary (red dotted curve) components of the potential with $\xi_0 = 2$, $\chi_0 = 1$, and $W_0 = 1$. (b) The propagation constant $\beta$ versus the power $P$ for symmetric (blue solid curve) and asymmetric (red dashed curve) solutions. (c) and (d) Distributions of symmetric and asymmetric solutions at the power $P = 1.5$ [see the marked square and circle in (b)], where the blue solid and red dotted curves represent the real and imaginary components, respectively. (e) and (f) Symmetric and asymmetric solutions for different soliton powers $P$.

3. SYMMETRIC AND ASYMMETRIC SOLITONS AND SYMMETRY-BREAKING BIFURCATIONS

In this Section, we will investigate the main characteristics of the stationary soliton solutions having either symmetric and asymmetric waveforms. We assume that the solutions of Eq. (2) are in the form $\Psi(\xi, \zeta) = \psi(\xi)e^{i\beta\zeta}$, where $\psi(\xi)$ is a complex-valued function and $\beta$ is the real propagation constant. Substituting it into Eq. (2), we can obtain the stationary equation in the form

$$\frac{d^2 \psi(\xi)}{d\xi^2} + U(\xi) \psi(\xi) + \sigma |\psi|^2 \psi(\xi) = \beta \psi(\xi),$$

where $U(\xi) = g^2(\xi) + idg(\xi)/d\xi$, and $g(\xi)$ is given by Eq. (3), as shown in Fig. 1(a).

Here, we consider the self-focusing nonlinearity ($\sigma = 1$) in Eq. (4) and we perform extensive numerical calculations in order to put forward the key features of both symmetric and asymmetric solitons by varying the separation between the two humps of the potential, the potential’s width, the modulation strength of the complex potential, and the soliton power $P = \int_{-\infty}^{+\infty} |\Psi|^2 d\xi$. The shape of the potential $U(\xi)$
Fig. 2 – (Color online) The nonlinear propagation constant $\beta$ as a function of the power $P$ for asymmetric (red circles) and symmetric (blue solid spheres) solutions for different (a) separation between the two humps of the potential for the fixed parameters $\chi_0 = 1$ and $W_0 = 1$; (b) potential's width for the fixed parameters $\xi_0 = 2$ and $W_0 = 1$, and (c) modulation strength of the potential for the fixed parameters $\chi_0 = 1$ and $\xi_0 = 2$, respectively. Here, the power $P$ ranges from 0.1 to 3 and $\sigma = 1$. 
is shown in Fig. 1 (a). The dependence of the propagation constant $\beta$ on the soliton power $P$ [the nonlinear dispersion curve $\beta = \beta(P)$] of both symmetric and asymmetric soliton solutions is shown in Fig. 1(b), where the blue solid and red dotted curves correspond to the symmetric solutions (SS) and the asymmetric solutions (AS), respectively. One can see that the stationary solutions in the $P^T$-symmetric potential are symmetric when $P < P_{th} = 1.1$ for our choice of the parameters. However, once the power exceeds the threshold value $P_{th}$, the asymmetric soliton solution begins to bifurcate out from the symmetric one. It should be emphasized that the asymmetric stationary solutions are non-$P^T$-symmetric but their eigenvalues still keep real. Thus, two different kinds of stationary solutions coexist in the region of $P \geq P_{th}$. We also find that all of the propagation constants $\beta$ belonging to the asymmetric solutions are larger than those of the symmetric ones at equal powers. Similar phenomena of bifurcation of asymmetric solitons from symmetric ones have been found in different physical settings involving Hermitian systems (i.e., without gain and loss); see, for example Refs. [49–54].

Because Eq. (4) is $P^T$-symmetric, if $\phi(\xi)$ is a solution, so is $\phi^*(\xi)$. Thus, the symmetric solutions of Eq. (4) satisfy naturally $\phi(\xi) = \phi^*(-\xi)$ due to they possess symmetric real profiles and antisymmetric imaginary ones, as shown in Fig. 1(c), which presents the profiles of the real and imaginary parts of a symmetric solution at the power $P = 1.5$. However, for the asymmetric solutions, the broken symmetry leads to $\phi(\xi) \neq \phi^*(-\xi)$. Thus, for each of the non-$P^T$-symmetric solution $\phi(\xi)$, there is a companion solution $\phi^*(-\xi)$ [44], which has the same propagation constant as $\phi(\xi)$, so they constitute a pair of degenerate nonlinear modes. In Fig. 1(d) we show the typical profiles of the real and imaginary parts of an asymmetric solution at the power $P = 1.5$, which is centered on a certain negative value of the coordinate $\xi$. However, its companion asymmetric solution is not shown here.

Furthermore, the symmetric and asymmetric solutions for the different powers $P$ are presented in Fig. 1(e) and (f), respectively. From them, one can see that the soliton amplitudes are increasing with the increasing of the power. It should be emphasized that for the asymmetric solutions, with the increasing of the power, their intensities are gradually concentrated on the left-hand side of the $\xi$-axis, as shown in Fig. 1(f). Similarly, there exists the corresponding companion solution for a fixed value of the propagation constant $\beta$ and its field distribution is focused on the right-hand side of the $\xi$-axis (this solution is not shown here).

Next, we discuss the influence of the separation between the two humps of the potential, the width, and the modulation strength of the complex potential on the nonlinear propagation constant $\beta$ of the symmetric and asymmetric solutions for Eq. (4). The results are summarized in Fig. 2, where the range of the power $P$ is chosen to vary from 0.1 to 3. We find that for given $\chi_0$ and $W_0$, the value of the power at the bifurcation point for the asymmetric solution is decreasing with the increasing of
the separation parameter $\xi_0$; see Fig. 2(a). On the contrary, for fixed $\xi_0$ and $W_0$, the value of the power at the bifurcation point is an increasing function of $\chi_0$, as shown in Fig. 2(b). Also, from Fig. 2(c), we find that the bifurcation point is more easily arising in the strong modulation regime. Thus by increasing the separation between the two humps of the potential and the modulation strength we can easily trigger the bifurcation of the asymmetric solutions, but we get the opposite situation when we increase the potential width.

4. LINEAR STABILITY ANALYSIS

In this Section, we will study the stability of the symmetric and asymmetric solutions by employing the linear stability analysis. The corresponding evolution dynamics is investigated by direct numerical simulations.

The linear stability analysis can be performed by adding a small perturbation to a known solution $\phi(\xi)$

$$\Psi(\xi, \zeta) = e^{i\beta \zeta} \left[ \phi(\xi) + u(\xi) e^{i\delta \zeta} + v^*(\xi) e^{i\delta \zeta} \right],$$

(5)

where $\phi(\xi)$ is the stationary solution with real propagation constant $\beta$, $u(\xi)$ and $v(\xi)$ are small perturbations with $|u|, |v| \ll |\phi|$. Substituting Eq. (5) into Eq. (2) and keeping only the linear terms, we obtain the following linear eigenvalue problem

$$i \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \delta \begin{pmatrix} u \\ v \end{pmatrix},$$

(6)

where $L_{11} = d^2/d\xi^2 + U - \beta + 2\sigma |\phi|^2$, $L_{12} = \sigma \phi^2$, $L_{21} = -L_{12}$, $L_{22} = -L_{11}$, and $\delta$ is the corresponding eigenvalue of the linear problem (6). If $\delta$ has a positive real part, the solution $\phi(\xi)$ is linearly unstable, otherwise, $\phi(\xi)$ is linearly stable. In the following, the linear stability of the stationary solution is characterized by the largest real part of $\delta$. Thus, if it is zero, the solution is linearly stable, otherwise, it is linearly unstable. Here, the linear eigenvalue problem (6) can be solved by making use of the Fourier collocation method [55].

As a typical example, Fig. 3 presents the dependence of the largest real part $\max(\delta_R)$ of the two eigenvalues corresponding to symmetric and asymmetric solutions of Eq. (6) on the power $P$ for $W_0 = 1$, $\chi_0 = 1$, and $\xi_0 = 2$. From it, one can see that the symmetric solutions are stable for $P < 1.1$, as shown by the blue solid curve in Fig. 3, while in the region of $P \geq 1.1$, the asymmetric solutions are stable, as shown by the red dotted curve in Fig. 3. Furthermore, compared with the results shown in Fig. 1(b), one finds that the point of occurrence of unstable symmetric soliton is just the point of bifurcation of the asymmetric branch of the curve $\beta = \beta(P)$ from the symmetric one, which is shown in Fig. 1(b).
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Fig. 3 – (Color online) The largest real part of eigenvalues $\delta$ corresponding to symmetric (blue solid curve) and asymmetric (red dotted curve) solutions for Eq. (6) versus the power $P$. Here, the parameters are $W_0 = 1$, $\chi_0 = 1$, and $\xi_0 = 2$.

Fig. 4 – (Color online) The modes’ profiles and the corresponding evolution plots. (a1), (b1), and (c1) The profiles of the symmetric solutions for the input powers $P = 0.8$, $1.1$, and $1.5$ corresponding to the open circles “a”, “b”, and “c” shown in Fig. 3; (d1) The profile of the asymmetric solution with $P = 2.8$ at the open circle “d” shown in Fig. 3. (a2), (b2), (c2), and (d2). The corresponding evolution plots of the field intensity. Here the other parameters are the same as in Fig. 3.
Fig. 5 – (Color online) The largest real part of the eigenvalue for Eq. (6) as a function of power for different (a) separation between the two humps of the potential, (b) width, and (c) modulation strength, respectively. The blue solid curves and the red open circle curves are for the symmetric and asymmetric solitons, respectively.
To confirm the results of linear stability analysis, we have studied the evolution with propagation distance $\zeta$ of four nonlinear modes by employing direct numerical simulations of the nonlinear partial differential equation 2. The results are summarized in Fig. 4, in which the three symmetric modes for three choices of the input power $P$ are shown in panels $(a_1)$, $(b_1)$, and $(c_1)$, whereas the asymmetric mode is shown in panel $(d_1)$. The blue solid curves show the real parts whereas the red dotted curves display the imaginary parts of the field profiles. The largest real parts of the corresponding eigenvalues for these solutions are marked in Fig. 3 with the open circles “a”, “b”, “c”, and “d”, respectively.

The linear stability analysis show that the solutions at the marked points “a” and “d” should be stable, while the solutions at the points “b” and “c” should be unstable. Therefore, we perturbed the linearly stable symmetric solution shown in Fig. $4(a_1)$ and the linearly stable asymmetric solution plotted in Fig. $4(d_1)$ by 5% random-noise perturbations. The numerical simulations clearly show that they propagate robustly, as displayed in Figs. $4(a_2)$ and Figs. $4(d_2)$. While for the linearly unstable symmetric solutions shown in Figs. $4(b_1)$ and $4(c_1)$, we have performed direct numerical simulations without adding any initial perturbations, see Figs. $4(b_2)$ and $4(c_2)$. Comparing Fig. $4(b_2)$ with Fig. $4(c_2)$, we find that the dynamical evolutions of the unstable symmetric solutions are quite different. For the symmetric solution with smaller nonzero real part of the eigenvalue $\delta$, the peak values of the optical field intensity can be switched periodically between the positions of the right and the left humps of the potential, as shown in Fig. $4(b_2)$. In this case, the symmetric soliton exhibits a weak instability. However, for the solution with larger nonzero real part of the eigenvalue $\delta$, the peak values of the optical field intensity reside mainly on the right side of the $\zeta$-axis and the optical field intensity oscillate periodically with propagation distance, as shown in Fig. $4(c_2)$. In this case, the symmetric soliton exhibits a much stronger instability than that shown in Fig. $4(b_2)$.

To present the influence of the separation between the two humps of the potential, the width, and the modulation strength of complex potential on the stability of either symmetric or asymmetric solitons, we investigate the dependence of the largest real part of the eigenvalues for Eq. (6) on the input power. The results of these numerical simulations are summarized in Fig. 5; see the blue solid curves and the red circle curves corresponding to symmetric and asymmetric solitons, respectively. An interesting consequence of the results plotted in Fig. 5 is that the locations of the points of bifurcations of asymmetric branches from the symmetric ones are in accordance with the onset of unstable symmetric solitons [see a typical example of such bifurcation in Fig. 1 (b)]. It indicates that instability of symmetric solutions gives rise to a bifurcation where stable asymmetric solutions begin to appear.
5. CONCLUSIONS

In summary, we have studied in detail both the symmetric and asymmetric soliton solutions that form in $\mathcal{PT}$-symmetric double-hump Scarff-II potentials. The results have shown that non-$\mathcal{PT}$-symmetric solitons, i.e., asymmetric solitons, bifurcate out from the base branch of $\mathcal{PT}$-symmetric solitons when the input soliton power exceeds a certain threshold. Also, the effects of the input power, the separation between the two humps of the potential, the width, and the modulation strength of the complex-valued double-hump Scarff-II potential on the solitons’ nonlinear propagation constant have been investigated. The stability and the robustness to small perturbations of these soliton solutions have been investigated and two generic instability scenarios of symmetric solitons have been put forward.

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