Bearing the thermodynamic arguments together with the two definitions of mass in mind, we try to find metrics with spherical symmetry. We consider the adiabatic condition along with the Gong-Wang mass, and evaluate the $g_{rr}$ element which points to a null hypersurface. In addition, we generalize the thermodynamics laws to this hypersurface to find its temperature and thus the corresponding surface gravity which enables us to get a relation for the $g_{tt}$ element. Moreover, we investigate the mathematical and physical properties of the discovered metric in the Einstein relativity framework which shows that the primary mentioned null hypersurface is an event horizon. The obtained energy-momentum tensor equals the energy-momentum tensor of a polytropic black hole embedded into an anti-de Sitter background. We also show that if one considers the Misner-Sharp mass in the calculations, the Schwarzschild metric will be got. The relationship between the two mass definitions in each metric is studied. The results of considering the geometrical surface gravity are also addressed. Our investigation shows that the geometrical surface gravity’s definition is not always compatible with the validity of the first law of thermodynamics on the horizons of spherically symmetric static metrics.

Key words: Thermodynamics; spherically symmetric static metrics; Einstein equations.

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1. INTRODUCTION

Spherically symmetric metrics have vast implications in describing the world around us. For instance, the FRW metric used to describe the universe expansion [1], moreover, these metrics are used to get the Tolman-Oppenheimer-Volkoff equations which help us to model some compact objects such as the Neutron stars and the white dwarfs [2]. Therefore, finding out such solutions is important from both of the cosmological and gravitational point of views [3]. It is also useful to note here that these metrics may include Black Holes (BHs) which respect certain rules [4]. Nowadays, these laws are known as the thermodynamics laws of BHs [4–8].

Thanks to the Jacobson work [9], it seems that the mutual relation between the BH laws and those of thermodynamics is more than a simple similarity. Indeed, Jacobson showed that one can recover the Einstein equations by applying the first law
of thermodynamics on the local Rindler horizon [9]. Its generalization to the \( f(R) \) gravity which leads to an entropy production, due to the \( R^{n\neq 1} \) terms, can also be found in Ref. [10]. In addition, it is shown that one can also recover the Einstein equations on the event horizon of spherically symmetric static metrics by applying the first law of thermodynamics on it [11]. Generalization to more gravitational theories are addressed in Ref. [12]. We should note here that in order to investigate the thermodynamics of systems, one should define a boundary for the system. Moreover, since the event horizon is a boundary between the timelike and spacelike phenomena and it includes the total mass, it can be considered as a boundary for the gravitational systems [11]. In fact, there are various horizons which can play the role of boundary in studying the gravitational systems [11]. The equivalence between the Friedmann equations and the first law of thermodynamics is also shown by some authors [13–21]. Additionally, the same deduction is valid in braneworld scenarios [22–27]. A suitable review on this topic can be found in Ref. [28]. Briefly, such attempts may help us provide a thermodynamic description for gravity.

As already mentioned, one may reach a thermodynamic origin for gravity by taking into account the Jacobson works [9, 10]. Another approach to get a thermodynamic motivation for the gravity is proposed by Padmanabhan [29]. Recently, Verlinde proposed a new origin for the gravity leading to emerge the spacetime [30]. In his theory, the tendency of system to increase its entropy together with the availability of the first law of thermodynamics on a holographic surface (as a causal boundary) lead to emerge the spacetime as well as the Einstein equations and therefore, the first and second laws of thermodynamics are automatically satisfied in this approach which attracts more attempts to itself [31–51]. Loosely speaking, he generalized the thermodynamic laws to the holographic surfaces to find out a motivation for the spacetime and gravity as the emerging phenomena in accordance with thermodynamic arguments. Finally, it should be reminded that although all of these approaches try to provide a thermodynamic interpretation for gravity, their definitions of energy differ from each other. Moreover, in another attempt, Padmanabhan used the Komar mass definition, the same as Ref. [30], as well as the difference between the surface and bulk degrees of freedom to get the Friedmann equations which is a thermodynamic motivation for emerging the FRW metric [52, 53]. Therefore, this approach may be considered as a thermodynamic motivation for the FRW metric.

Based on the mentioned attempts it is apparent that a deep mutual relation between the thermodynamics and gravity is unavoidable which may lead to a thermodynamic motivation for emerging the spacetime and thus its metric. Moreover, it is also useful to investigate the possibility of getting the static spherically symmetric metrics by considering the thermodynamical arguments. In order to get a thermodynamical motivation for the Schwarzschild metric, Zhang et al. have been considered
the general form of a static spherically symmetric metric [54]
\[ ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega_2^2, \]  \hspace{1cm} (1)
and used the Misner-Sharp mass definition [55]
\[ M_{ms} = r(1 - g^{ab}r_a r_b)/2, \] \hspace{1cm} (2)
together with the geometrical surface gravity definition [56, 57]. Thereinafter, since
an adiabatic system satisfies the \( dM_{ms} = 0 \) condition which means that the Misner-
Sharp mass is constant, they got
\[ h(r) = (1 - 2M_{ms}/r)^{-1}. \] \hspace{1cm} (3)
Finally, by equating the geometrical surface gravity [56, 57] with ordinary surface
gravity [4], they could get the Schwarzschild metric \( f(r) = 1 - 2M_{ms}/r \). A nice
historical review on the Schwarzschild metric can be found in ref. [58]. Inasmuch as
authors have not used the thermodynamic arguments to get a relation for the surface
gravity in ref. [54], their recipe is not a fully thermodynamic one. But, we should
note that their approach is powerful and can easily be generalized to the Reissner-
Nordstrom and Schwarzschild de-Sitter spacetimes, the Gauss-Bonnet and \( F(R) \) the-
ories [54, 59]. Accordingly, we confront two questions. First, is it possible to find
an expression for the surface gravity by using thermodynamic arguments? Second,
what are the results of considering another definitions of mass?

It has been shown by Gong and coworker that if one takes into account the
Misner-Sharp mass, as a definition for energy in spherically symmetric spacetimes,
then the first law of thermodynamics is not satisfied on the apparent horizon of FRW
metric in Brans-Dicke, nonlinear and scalar-tensor theories of gravity [60]. In addition,
they proposed a new definition for mass in spherically symmetric spacetimes
(Gong-Wang) which leads to the same value as that of Misner-Sharp if one evaluates
the mass confined to the apparent horizon of FRW metric. It is also shown that, for
non-static spherically symmetric metrics describing an object embedded in the FRW
background, the Gong-Wang definition of mass may lead to the same value as that of
the Misner-Sharp definition for energy [61].

Our aim in this paper is finding some spherically symmetric static metrics by
considering thermodynamic arguments. Here, we use the Gong-Wang as well as
the Misner-Sharp definitions of mass and follow the approach of authors introduced
in Refs. [54, 59] to get the \( g_{rr} \) element of metric. In continue, we try to find an
expression for the surface gravity, which leads to the \( g_{tt} \) element, by using the ther-
modynamics laws as well as the mutual relation between the surface gravity and
temperature. For the sake of simplicity we take \( G = c = \hbar = 1 \) and the \((-{}, +, +, +)\)
signature throughout this paper.

The paper is organized as follows. In the next section, at first, by taking into ac-
count the spherically symmetric static metrics along with the Gong-Wang definition of mass, we use the adiabatic condition to evaluate the $g_{rr}$ element of metric. Moreover, we try to find the surface gravity and thus the $g_{tt}$ element of metric by using thermodynamic arguments. Some of the mathematical properties of the discovered metric are studied. Bearing the Einstein equations in mind, we also investigate some relativistic properties of metric. Finally, we repeat our recipe by considering the Minser-Sharp mass which leads to the Schwarzschild metric. Moreover, the relation between the Gong-Wang and Misner-Sharp definitions of mass in the discovered metrics is also addressed. In addition, the geometrical surface gravity is studied and its results are compared with other definitions of surface gravity and thermodynamic considerations. The last section is devoted to a summary and concluding remarks.

2. SCHWARZSCHILD AND SCHWARZSCHILD-LIKE SOLUTIONS

Consider the

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega_2^2,$$

metric together with the Gong-Wang definition of Mass [60]

$$M_{gw} = r \frac{1 + g_{ab}r_ar_b}{2},$$

leading to

$$M_{gw} = \frac{r}{2}(1 + h(r)^{-1}),$$

for metric (4). For a spacetime with constant mass

$$dM_{gw} = 0,$$

which is nothing but the adiabatic condition [54, 59], and therefore

$$h(r) = \left(\frac{2C}{r} - 1\right)^{-1},$$

where $C$ is a constant of integration. Now, substituting (8) into (5) to get

$$C = M_{gw},$$

which means that, since energy and thus mass are positive, $C$ should meet the $C \geq 0$ condition. Finally, we obtain

$$h(r) = \left(\frac{2M_{gw}}{r} - 1\right)^{-1}.$$

Therefore, we get

$$ds^2 = -f(r)dt^2 + \frac{1}{\frac{2M_{gw}}{r} - 1}dr^2 + r^2d\Omega_2^2,$$
which is very similar to the Schwarzschild metric. Now, consider hypersurface \( \phi = r - c = 0 \) with normal vector \( n_\alpha = \partial_\alpha \phi \), simple calculations yield

\[
n_\alpha n^\alpha = \frac{2M_{gw}}{r} - 1, \tag{12}
\]

proving that \( r = 2M_{gw} \) is a null hypersurface, and thus it may be considered as a causal boundary for this spacetime and therefore, it may also be leaded to an event horizon located at the \( r = 2M_{gw} \) radii. In fact, in order to deal with an event horizon at this radii, metric should change its signature at this radii in a proper way [4], while curvature scalars should not also be diverged at this radii [4].

By using (2) one gets \( M_{ms}(r) = r - M_{gw} \) and thus \( M_{ms} = M_{gw} \) whiles \( r = 2M_{gw} \), telling us that both of these mass definitions lead to the same value if we evaluate the mass confined to the null hypersurface with radii \( r = 2M_{gw} \). This mutual consistency between the results of using either the Gong-Wang or Misner-Sharp masses in order to evaluate the energy confined to a null hypersurface is in line with the dynamic situations [60, 61]. It is useful to note here that the adiabatic condition together with the Gong-Wang definition of mass helped us to get an expression for \( g_{rr} \) leading to relations for the Gong-Wang and Misner-Sharp masses. Finally, since the null hypersurface located at \( r = 2M_{gw} \) includes the total energy \( E = M_{gw} = M_{ms} \), we consider it as the causal boundary for the system. Therefore, this hypersurface may carry an entropy in accordance with Bekenstein’s argument [4, 5]

\[
S_A = \frac{A}{4} = \frac{\int_{r=2M_{gw}} r^2 d\Omega}{4} = 4\pi M_{gw}^2. \tag{13}
\]

Now, using \( E = M_{gw} \) together with the Clausius-Clapeyron relation [62] to obtain the temperature of the null hypersurface as

\[
T = \frac{\partial E}{\partial S_A} = \frac{1}{8\pi M_{gw}}. \tag{14}
\]

Nowadays, \( T = \frac{|\kappa|}{2\pi} \) is attributed to various horizons, as the boundary for system, which are null hypersurfaces [4, 11]. Here, since we take into account the null hypersurface located at \( r = 2M_{gw} \) as the boundary of system, we generalize this mutual relation between the surface gravity and temperature to the mentioned null hypersurface.

Moreover, since Eq. (11) is a spherically symmetric static metric, by following the recipe of Refs. [4, 56], we get

\[
\kappa(r) = \frac{1}{2\sqrt{f(r)h(r)}}|f'(r)|, \tag{15}
\]

where \( \kappa \) is the surface gravity. In deriving this equation, \( |f'(r)| \) is used instead of \( f'(r) \) in order to avoid the negative values for the surface gravity, and thus its corres-
ponding temperature [4, 11]. Therefore, we get
\[ \kappa = \frac{1}{2 \sqrt{f(r)h(r)|r=2M_{gw}|}} |f'(r)|_{r=2M_{gw}} \]
for the hypersurface located at \( r = 2M_{gw} \). Finally, by combining Eq. (14) with equation (16) one gets
\[ \frac{1}{2 \sqrt{(f(r)h(r))|r=2M_{gw}|}} |f'(r)|_{r=2M_{gw}} = \frac{1}{4M_{gw}}. \]
It is easy to check that \( f(r) = h(r)^{-1} = \frac{2M_{gw}}{r} - 1 \) is a solution for this equation, leading to
\[ ds^2 = -\left(\frac{2M_{gw}}{r} - 1\right)dt^2 + \frac{1}{\frac{2M_{gw}}{r} - 1}dr^2 + r^2 d\Omega^2, \]
which is similar to the Schwarzschild metric. Therefore, by using the thermodynamic arguments we get an expression for the surface gravity which leads to a relation for the \( g_{tt} \) element of metric. In this metric, radii should meet the \( r < 2M_{gw} \) condition in order to preserve the \((- , + , + , +)\) signature of metric. A photon which was emitted at \( r_e \) with wavelength \( \lambda_e \) is observed at radii \( r_o \) with wave length \( \lambda_o \). For this observed photon the redshift is
\[ 1 + z = \frac{\lambda_o}{\lambda_e} = \sqrt{\frac{2M_{gw}}{r_o} - 1}, \]
which signals us that there is no observable photon emitted from \( r_e = 2M_{gw} \), because \( z \to \infty \). Moreover, the redshift of a photon observed at origin \( (r_0 = 0) \) diverges, meaning that one cannot communicate with an observer located at origin. Therefore, there are two singularities in this metric located at \( r = 0 \) and \( r = 2M_{gw} \). For the curvature scalars we get
\[ K = \frac{16(3m^2 + r^2 - 2rm)}{r^6}, \quad R = \frac{4}{r^2} \]
\[ W = \frac{16(3m - r)^2}{3r^6}, \quad R_S = \frac{8}{r^4}, \]
where \( K \) and \( R \) are the Kretschmann invariant and the Ricci scalar, respectively. Moreover, \( W \) and \( R_S \) also denote the Weyl and Ricci squares, respectively. It is obvious that none of them diverges at \( r = 2M_{gw} \). Finally, based on the above discussions, we conclude that \( r = 2M_{gw} \) points to an event horizon which confirms our primary conjectures about the corresponding hypersurface including that it may be a causal boundary which may carry an entropy (13) together with a temperature (14).
which satisfies Eq. (17). For the surface area of a sphere with radii $r$ we have

$$A = \int r^2 d\Omega = 4\pi r^2,$$  \hspace{1cm} (21)

leading to $A \to 0$ whiles $r \to 0$. It is also clear that the curvature scalars diverge at origin. Therefore, based on the above discussions, we think that $r = 0$ is a naked singularity [4]. It is shown that a naked singularity leads to time delay as well as gravitational lensing which yields the magnification of images [63, 64]. In fact, since the existence of naked singularities is not completely rejected in physics [65–68], the existence of the $r = 0$ naked singularity in this solution is not bad. Bearing the Einstein equations in mind, we are going to investigate the properties of metric (18) in the Einstein general relativity framework. Calculations lead to

$$\rho = -G_{t}^{t} = \frac{2}{r^2}, \quad P_{r} = G_{r}^{r} = -\frac{2}{r^2},$$  \hspace{1cm} (22)

where the other elements vanish. $\rho$ and $P_{r}$ denote the energy density and radial pressure of a fluid supporting this spacetime, respectively. If we define the radial state parameter of supporter fluid as $\omega_{r} = \frac{P_{r}}{\rho}$, we get $\omega_{r} = -1$ which is similar to the state parameter of cosmological constant used to explain the current accelerating phase of the universe [1]. It is apparent that all of the energy conditions are marginally satisfied. It is because the transverse pressures are zero ($G^{\theta}_{\theta} = G^{\phi}_{\phi} = 0$) whiles $\rho + P_{r} = 0$ and $\rho > P_{r}$. It is useful to note here that the energy-momentum tensor (22), needed for supporting the geometry (18), is equal to the energy-momentum tensor of a polytropic black hole embedded into the anti-de Sitter background [69, 70]. In fact, since the adiabatic approximation is the backbone of getting metric (18), and because an adiabatic process is indeed a polytropic process [71, 72], such resemblance between the obtained energy-momentum tensor (22) and that of Refs. [69, 70] is reasonable.

The geometrical surface gravity ($\kappa_{g}$) due to an energy-momentum source with energy $M$ and work density $w = -\frac{1}{2}(T_{0}^{0} + T_{1}^{1})$ is defined as [54, 56, 57, 59]

$$\kappa_{g} = \frac{M}{r^2} - 4\pi r w$$  \hspace{1cm} (23)

leading to

$$\kappa_{g} = \frac{M_{gw}}{r^2} - \frac{8\pi}{r},$$  \hspace{1cm} (24)

for metric (18). For the event horizon, this relation is reduced to

$$\kappa_{g} = \frac{1}{4M_{gw}} - 4\pi \frac{\pi}{M_{gw}}.$$  \hspace{1cm} (25)

It is clear that these results differ from those obtained from Eqs. (15), (14). Indeed, it is easy to check that the $\kappa_{g} = \kappa$ condition does not lead to $f(r) = \frac{1}{4\pi r}$ and thus (18). It is shown that if one either uses (15) or (14) as a definition for the surface gravity
or the temperature of horizon, then the first law of thermodynamics is available on
the event horizon of the spherically symmetric static metrics \cite{9, 11}. Hence, since
Eq. (24) does not yield the same value as those of Eqs. (15) and (14), the first law of
thermodynamics is not available on the event horizon of metric (18) if one uses (24)
as a suitable definition for the surface gravity and the corresponding temperature. The
latter means that this definition of surface gravity is not extendable to every metric
such as (18).

**SCHWARZSCHILD METRIC**

By using the Misner-Sharp mass (2) and following the recipe used to derive (10)
and (12), one gets

\[
h(r) = (1 - \frac{2M_{ms}}{r})^{-1},
\]

(26)

and

\[
n_{\alpha}n^{\alpha} = 1 - \frac{2M_{ms}}{r},
\]

(27)

respectively. The latter signals that there is a null hypersurface located at \(r = 2M_{ms}\).
Therefore, metric can be written as

\[
ds^{2} = -f(r)dt^{2} + \frac{1}{1 - \frac{2M_{ms}}{r}}dr^{2} + r^{2}d\Omega^{2}.
\]

(28)

Now, let us calculate the Gong-Wang mass for this metric which yields \(M_{gw}(r) =
\[r - M_{ms}\) indicating that both of the Misner-Sharp and Gong-Wang mass definitions
point to the same value if we evaluate the mass confined to the \(r = 2M_{ms}\) radii.
Once again, we see that both of these definitions estimate the same value for the mass
confined to the horizon which is in agreement with attempts in which the dynamic
black holes have been studied \cite{60, 61}. Therefore, as the previous case, \(r = 2M_{ms}\)
may be considered as a boundary for the system. Finally, in accordance with (27),
by assuming that the thermodynamics laws are available on the null hypersurface
located at \(r = 2M_{ms}\), and following the recipe used to derive (17), we get

\[
\frac{1}{2\sqrt{(f(r)h(r))_{r=2M_{ms}}}}|f'(r)_{r=2M_{ms}}| = \frac{1}{4M_{ms}}.
\]

(29)

It is apparent that \(f(r) = h(r)^{-1}\) is a solution for this equation leading to

\[
ds^{2} = -(1 - \frac{2M_{ms}}{r})dt^{2} + \frac{1}{1 - \frac{2M_{ms}}{r}}dr^{2} + r^{2}d\Omega^{2},
\]

(30)

which is nothing but the Schwarzschild metric. Previously, Zhang et al. \cite{54, 59}
derived this metric by focusing on the Misner-Sharp mass (2) together with the ge-
ometrical definition of surface gravity (23). Here, bearing the adiabatic condition in
mind, we used the Misner-Sharp mass to get the $g_{rr}$ element of metric which leads to discover a null hypersurface. Moreover, the mass analysis guided us to take into account this null hypersurface as a boundary for the system. Finally, by establishing the Clausius-Clapeyron relation on the mentioned null hypersurface and generalizing the mutual relation between the surface gravity and temperature to the assumed boundary, we reached the $g_{tt}$ element of metric leading to get the Schwarzschild metric. We should note here that if one uses the Misner-Sharp mass then all of the definitions of surface gravity, introduced in this paper, lead to the same value for the surface gravity in the Schwarzschild spacetime. It is due to this fact that the Schwarzschild metric is a vacuum solution which leads to $w = 0$.

3. SUMMARY AND CONCLUDING REMARKS

Here, we took into account a spherically symmetric static metric with unknown $g_{tt}$ and $g_{rr}$. Thereinafter, we used the Gong-Wang definition of mass and applied the adiabatic condition to the system ($dE = 0$) in order to find an expression for $g_{rr}$ which points to a null hypersurface. By this work, we could find a relation for the Gong-Wang and Misner-Sharp masses. We also saw that both of these mass definitions lead to the same value for the mass confined to the discovered null hypersurface located at $r = 2M_{gw}$. In addition, we considered this hypersurface as a boundary for the system and assigned an entropy and a temperature to it in accordance with Bekenstein’s arguments as well as the Clausius-Clapeyron relation. In continue, by generalizing the mutual relation between the temperature of the horizons and their surface gravity to the boundary of our system we could find an expression for the corresponding surface gravity. Moreover, we have estimated the surface gravity of the primary assumed metric and equated it with the thermodynamic outcomes. The latter leaded to a relation for the $g_{tt}$ element of metric. Therefore, we could find a new metric by relying on thermodynamic arguments. Additionally, by studying the redshift and the mathematical properties of the metric, we found that the primary null hypersurface, assumed as the causal boundary, is an event horizon whiles, there is a naked singularity located at origin ($r = 0$). Bearing the Einstein equations in mind, we investigated the physics behind this metric including the properties of a fluid needed for supporting this spacetime and the validity of the energy conditions. Our study shows that the energy conditions are marginally satisfied which are due to this fact that $\rho + P_r = 0$ whiles, $\rho = \frac{2}{r^2}$ and the other components vanish. As we have previously mentioned, the obtained energy-momentum tensor is equal to that of a polytropic black hole embedded into an anti-de Sitter background. This similarity is due to this fact that the adiabatic process (the primary assumption used to get the $g_{rr}$ element of metrics in this paper) is in fact a polytropic process. Thereinafter, we considered the Misner-Sharp mass, and followed our hypothetical recipe
to get the Schwarzschild metric. Although some authors gave a thermodynamical motivation for the Schwarzschild metric in various theories of gravity [54, 59], their approach differs from ours. Here, we focused on the thermodynamic considerations for getting the surface gravity whiles, they have used the geometrical surface gravity definition. Moreover, we have also shown that the geometrical surface gravity is not always compatible with other definitions of surface gravity and thus the thermodynamic considerations, such as the validity of the first law of thermodynamics on the event horizon of spherically symmetric static metrics. Moreover, our study shows that both of the Misner-Sharp and Gong-Wang definitions of mass lead to the same value for the energy confined to the horizon and thus the black hole mass which is in line with the dynamic situations [60, 61]. Finally, we think that our investigation may help us to provide the thermodynamic motivations for the spherically symmetric static metrics which leads to clarify the thermodynamic aspects of spacetime and gravity.

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