Higher-Order Rogue Wave Dynamics for a Derivative Nonlinear Schrödinger Equation

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Received September 30, 2016

Abstract. The mixed Chen-Lee-Liu derivative nonlinear Schrödinger (CLL-NLS) equation can be considered as the simplest model to approximate the dynamics of weakly nonlinear and dispersive waves, taking into account the self-steepening effect (SSE). The latter effect arises as a higher-order correction of the nonlinear Schrödinger (NLS) equation, which is known to describe the dynamics of pulses in nonlinear fiber optics, and constitutes a fundamental part of the generalized NLS equation. Similar effects are described within the framework of the modified NLS equation, also referred to as the Dysthe equation, in hydrodynamics. In this work, we derive fundamental and higher-order solutions of the CLL-NLS equation by applying the Darboux transformation. Exact expressions of non-vanishing solitons at boundaries, breathers, and a hierarchy of rogue wave solutions are presented. In addition, we discuss the localization properties of such rogue waves, by characterizing their length and width. In particular, we describe how the localization properties of first-order NLS rogue waves can be modified by taking into account the SSE, presented in the CLL-NLS equation. This is illustrated by use of an analytical and a graphical method. The results may motivate similar analytical studies, extending the family of the reported rogue wave solutions as well as possible experiments in several nonlinear dispersive media, confirming these theoretical results.

Key words: Chen-Lee-Liu derivative nonlinear Schrödinger equation, Darboux transformation, Rogue waves, Self-steepening effects.

1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most relevant equations in physics. This integrable equation can be rigorously derived as an approximation to governing equations of several nonlinear and dispersive media [1–4]. Recently, a wide class of solutions, such as the Peregrine soliton [5] and multi-Peregrine soliton, also referred to as Akhmediev-Peregrine breathers [6], of the NLS are intensively discussed in physical and mathematical communities [7]. The doubly-localized Peregrine soliton, which approaches a non-zero constant background in the
infinite limit of the spatial and temporal periodicity, amplifies the amplitude of the carrier by a factor of three at the origin of coordinates. Multi-Peregrine solitons [8] have similar dynamics, with the particular property to generate much higher maximal peak amplitudes, compared to background [9–16]. Due to these properties, Peregrine-type waves are suggested to model “rogue waves” (RWs), known to appear in the ocean [17] and in other media [18]. Mathematically speaking, modulationally unstable extreme waves admit high-intensity peaks, appearing from nowhere and disappearing without a trace, while evolving in time and space [19]. Recently, exact solutions of the NLS equation, describing a new form of modulation instability dynamics, have been derived [20, 21]. The concept of the RWs was first discussed in the studies of ocean waves [22–25], and gradually extended to other fields of research, such as capillary water waves [26], optical fibers [27–29], Bose-Einstein condensates [30] and other applications, which have been summarized in very recent review papers [7, 18].

Only recently, experimental validation of such RW model has been successfully conducted in nonlinear fibers [31], in water wave tanks [32–35], and in plasmas [36, 37]. The latter experimental studies have been performed based on the NLS modeling evolution equation.

In addition to the NLS equation, there are several other integrable evolution equations admitting Peregrine-type RW solutions such as the Hirota, the modified Korteweg-de Vries, the Sasa-Satsuma, the Fokas-Lenells, the NLS Maxwell-Bloch, the Hirota Maxwell-Bloch, the generalized NLS, the vector NLS, the derivative NLS, the variable coefficient NLS, the Davey-Stewartson, and the KP-I equations [38–71]. Lately, fundamental rogue wave modes of the mixed Chen-Lee-Liu derivative nonlinear Schrödinger (CLL-NLS) equation [72]

\[ i r_t + r_{xx} + |r|^2 r - i|r|^2 r_x = 0 \]

have been reported [73] by use of the Hirota bilinear method. Clearly, the latter solution is physically more complex and more accurate in describing the propagation of optical pulses compared to the NLS or simplified CLL equations [74]

\[ i r_t + r_{xx} + i|r|^2 r_x = 0, \]

since the CLL-NLS equation takes into dispersion, nonlinearity as well as self-steepening effect (SSE), described by the term $|r|^2 r_x$, however, while ignoring self-phase-modulation (SPM) [75]. The SSE of light pulses, originating from their propagation in a medium with an intensity dependent index of refraction, was first introduced in [76] and was observed in optical pulses with possible shock formation [77]. It receives a significant attention for the propagation of electromagnetic waves in nonlinear fibers, using a femtosecond laser, since it plays a crucial role in the generation of supercontinuum [78, 79]. In mathematical terms, its source is the first nonli-
near correction to the NLS equation in the description of very focused light pulses or significant sharp water wave packets for which the validity of the NLS equation is known to be violated, due to the related significant broadening of the spectrum \([80, 81]\). In hydrodynamics, the CLL-NLS equation can be obtained from the modified NLS equation, also known as the Dysthe equation \([82]\) by ignoring the mean flow term, whose contribution is small if the nonlinearity of the wave train is kept small. Therefore, exact CLL-NLS models may motivate experiments in nonlinear optical fibers as well as in water wave flumes \([73]\). Especially, taking into account the fact that exact RW solutions are closely related to the modulation instability of weakly nonlinear dispersive waves.

In this paper, we report exact solutions of the integrable CLL-NLS equation. To the authors’ best knowledge, this is so far the first derivation of such doubly-localized solutions using the Darboux transformation (DT). The construction of the DT for the CLL-NLS equation is highly non-trivial. For the CLL-NLS equation \((1)\), although it was introduced thirty years ago as an integrable system \([72]\), its DT is not given explicitly during the past three decades. Very recently, it has been shown that applying the DT technique to the CLL and CLL-NLS equations would engender major difficulties, due to the asymmetry of the Lax pair, see details in the appendix of \([73]\). Moreover, in \([73]\), the authors derived the first-order RW solution of the CLL-NLS equation by applying the bilinear method, and they pointed out that it is a challenging task to determine the higher-order RW modes for the CLL equation. The CLL-NLS equation is a higher-order equation, compared to CLL and NLS equations. Therefore, deriving higher-order RW modes of the CLL-NLS is even more demanding and worth to be investigated in detail. In Sec. 2 and Sec. 3 the integration scheme will be introduced and we will address the significant challenges using the DT, solving CLL-type equations. Exact solutions with particular focus on higher-order RWs is reported in Sec. 4, extending therefore the family of exact first-order solutions. Furthermore, we discuss the influence of the SSE on the localization properties of NLS RWs in Sec. 5. Due to obvious physical relevance of the CLL-NLS equation, we emphasize further analytical, numerical, and experimental studies, related to the presented exact solutions of this integrable evolution equation.

### 2. THE DT FOR THE COUPLED CLL-NLS EQUATION

In this section, we consider the \(n\)-fold DT for the coupled CLL-NLS equation

\[
\begin{align*}
    r_t - ir_{xx} + ir^2 q + rqr_x &= 0, \\
    q_t + iq_{xx} - iq^2 r + qrq_x &= 0,
\end{align*}
\]

which reduces to the CLL-NLS equation while \(q = -\overline{r}\) and the over-bar denotes complex conjugation. These two equations in \((3)\) are the compatibility conditions of
the following Lax pair [83, 84]:

\[
\begin{align*}
\Phi_x &= U \Phi = (i\sigma_3 \lambda^2 + Q \lambda - \frac{1}{2}i\sigma_3 + \frac{1}{4}iQ^2\sigma_3)\Phi, \\
\Phi_t &= V \Phi = [-2i\sigma_3 \lambda^4 - 2Q\lambda^3 + (2i\sigma_3 - iQ^2\sigma_3) \lambda^2 + (Q + i\sigma_3 Q_x - \frac{1}{2}Q^2)\lambda - \frac{1}{2}i\sigma_3 - \frac{1}{8}iQ^4\sigma_3 + \frac{1}{4}(QQ_x - Q_x Q)]\Phi,
\end{align*}
\]

with

\[
\Phi(x, t, \lambda) = \begin{pmatrix} f(x, t, \lambda) \\ g(x, t, \lambda) \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}.
\]

It is trivial to see that \( \Phi_k = \begin{pmatrix} f_k \\ g_k \end{pmatrix} = \Phi(x, t, \lambda)|_{\lambda = \lambda_k} = \begin{pmatrix} f(x, t, \lambda) \\ g(x, t, \lambda) \end{pmatrix}|_{\lambda = \lambda_k} \) gives the eigenfunction of the Lax pair equations corresponding to \( \lambda_k \). Indeed, we seek \( n \) eigenfunctions to get the determinant representation of the \( n \)-fold DT.

**Theorem 2.1** The \( n \)-fold DT for the coupled CLL-NLS is

\[
T_n = T_n(\lambda; \lambda_1, \ldots, \lambda_n) = \begin{cases} 
\frac{1}{\sqrt{|\Delta_n^1||\Delta_n^2|}} \begin{pmatrix} (T_n)_{11} & (T_n)_{12} \\ (T_n)_{21} & (T_n)_{22} \end{pmatrix} & \text{if } n \text{ even}, \\
\frac{1}{\sqrt{|\Delta_n^1||\Delta_n^2|}} \begin{pmatrix} \sqrt{H} & 1 \\ 1 & \sqrt{H} \end{pmatrix} \begin{pmatrix} (T_n)_{11} & (T_n)_{12} \\ (T_n)_{21} & (T_n)_{22} \end{pmatrix} & \text{if } n \text{ odd},
\end{cases}
\]

the elements \((T_n)_{ij}\) \((i, j = 1, 2)\) are defined by

\[
(T_n)_{11} = \begin{pmatrix} \lambda_n \xi_1^n \\ \eta_1^n \Delta_2^n \end{pmatrix}, \quad (T_n)_{12} = \begin{pmatrix} 0 \\ \eta_1^n \Delta_2^n \end{pmatrix}, \quad (T_n)_{21} = \begin{pmatrix} 0 \\ \eta_2^n \Delta_2^n \end{pmatrix}, \quad (T_n)_{22} = \begin{pmatrix} \lambda_n \xi_2^n \\ \eta_2^n \Delta_1^n \end{pmatrix},
\]

\( \eta_1^n, \xi_1^n \) and \( \Delta_2^n \) \((i = 1, 2)\) are defined by

\[
\eta_1^n = (\lambda_1^n f_1, \lambda_2^n f_2, \lambda_3^n f_3, \ldots, \lambda_n^n f_n)^T, \quad \eta_2^n = (\lambda_1^n g_1, \lambda_2^n g_2, \lambda_3^n g_3, \ldots, \lambda_n^n g_n)^T,
\]

- if \( n \) is even,
  \[
  \xi_1^n = (0, \lambda^{n-2}, 0, \lambda^{n-4}, \ldots, 0, 1), \quad \xi_2^n = (\lambda^{n-1}, 0, \lambda^{n-3}, 0, \ldots, \lambda, 0),
  \]
- if \( n \) is odd,
  \[
  \xi_1^n = (0, \lambda^{n-2}, 0, \lambda^{n-4}, \ldots, \lambda, 0), \quad \xi_2^n = (\lambda^{n-1}, 0, \lambda^{n-3}, 0, \ldots, 0, 1),
  \]

and

\[
\Delta_1^n = (A_1^n, A_2^n, A_3^n, \ldots, A_n^n)^T, \quad \Delta_2^n = (B_1^n, B_2^n, B_3^n, \ldots, B_n^n)^T,
\]

with \( A_k^n, B_k^n \) \((k = 1, 2, 3, \ldots, n)\) defined by
Theorem 2.2 The $n$-th order solutions $r_n$ and $q_n$ are

\[
\begin{align*}
\text{if } n & \text{ is even,} \\
A_n^k &= (\lambda_k^{n-1} f_k \lambda_k^{n-2} g_k \lambda_k^{n-3} f_k \lambda_k^{n-4} g_k \ldots \lambda_k^3 f_k \lambda_k^2 g_k \lambda_k^1 f_k \ f_k), \\
B_n^k &= (\lambda_k^{n-1} g_k \lambda_k^{n-2} f_k \lambda_k^{n-3} g_k \lambda_k^{n-4} f_k \ldots \lambda_k^3 g_k \lambda_k^2 f_k \lambda_k^1 g_k \ f_k), \\
\text{if } n & \text{ is odd,} \\
A_n^k &= (\lambda_k^{n-1} f_k \lambda_k^{n-2} g_k \lambda_k^{n-3} f_k \lambda_k^{n-4} g_k \ldots \lambda_k^3 g_k \lambda_k^2 f_k \lambda_k^1 g_k \ f_k), \\
B_n^k &= (\lambda_k^{n-1} g_k \lambda_k^{n-2} f_k \lambda_k^{n-3} g_k \lambda_k^{n-4} f_k \ldots \lambda_k^3 g_k \lambda_k^2 f_k \lambda_k^1 g_k \ f_k).
\end{align*}
\]

The solutions $(q_n, r_n)$ generated by the above $n$-fold DT have the following determinant representations.

\[
\text{Theorem 2.2 The } n\text{-th order solutions } r_n \text{ and } q_n \text{ are}
\begin{align*}
r_n &= \begin{cases} 
|\Delta_1^n| r - 2i |\Delta_2^n| & \text{if } n \text{ even,} \\
H \left(\frac{|\Delta_1^n|}{|\Delta_2^n|} r - 2i \frac{|\Delta_2^n|}{|\Delta_1^n|} \right) & \text{if } n \text{ odd,} 
\end{cases} \\
q_n &= \begin{cases} 
|\Delta_1^n| q - 2i |\Delta_2^n| & \text{if } n \text{ even,} \\
\frac{1}{H} \left(\frac{|\Delta_1^n|}{|\Delta_2^n|} q - 2i \frac{|\Delta_2^n|}{|\Delta_1^n|} \right) & \text{if } n \text{ odd,} 
\end{cases}
\end{align*}
\]

where $\Delta_i^n$ $(i = 3, 4)$ are defined by

\[
\begin{align*}
\Delta_3^n &= (C_1^n C_2^n C_3^n \ldots C_n^n)^T, \\
\Delta_4^n &= (D_1^n D_2^n D_3^n \ldots D_n^n)^T,
\end{align*}
\]

with $C_k^n, D_k^n$ $(k = 1, 2, 3, \ldots, n)$, given by

\[
\begin{align*}
\text{if } n & \text{ is even,} \\
C_n^k &= (\lambda_k^n f_k \lambda_k^{n-2} f_k \lambda_k^{n-3} g_k \lambda_k^{n-4} f_k \ldots \lambda_k^3 g_k \lambda_k^2 f_k \lambda_k^1 g_k \ f_k), \\
D_n^k &= (\lambda_k^n g_k \lambda_k^{n-2} g_k \lambda_k^{n-3} f_k \lambda_k^{n-4} g_k \ldots \lambda_k^3 f_k \lambda_k^2 g_k \lambda_k^1 f_k \ g_k),
\end{align*}
\]

\[
\begin{align*}
\text{if } n & \text{ is odd,} \\
C_n^k &= (\lambda_k^n f_k \lambda_k^{n-2} f_k \lambda_k^{n-3} g_k \lambda_k^{n-4} f_k \ldots \lambda_k^3 g_k \lambda_k^2 f_k \lambda_k^1 g_k), \\
D_n^k &= (\lambda_k^n g_k \lambda_k^{n-2} g_k \lambda_k^{n-3} f_k \lambda_k^{n-4} g_k \ldots \lambda_k^3 f_k \lambda_k^2 g_k \lambda_k^1 f_k).
\end{align*}
\]

In Theorem 2.1 and Theorem 2.2, $(q, r)$ is a “seed” solution of the coupled CLL-NLS equation, $H$ is an overall factor in the formula of the DT involved with an integral function depending on $q$ and $r$, which satisfies the following conditions

\[
\frac{\partial H}{\partial x} = \frac{1}{2} i(q r - 2) H, \quad \frac{\partial H}{\partial t} = -\frac{1}{4} (4i + 4q^2 r^2 - 2q r_x + 2q r_x) H.
\]

A general analytical expression of $H$ is

\[
H = \exp \left( \int_{(x_0, t_0)}^{(x, t)} \frac{1}{2} i(q r - 2) dx - \frac{1}{4} (4i + 4q^2 r^2 - 2q r_x + 2q r_x) dt \right).
\]
Let \( a \) and \( c \) be two real constants, \( b = a^2 + (a - 1)c^2 \), and then \( q = -\tau = c \exp[i(ax + bt)] \) is a “seed” solution of the CLL-NLS equation. For this case, \( H = \exp(-\frac{1}{2}i(2 + c^2)x - \frac{1}{4}i(4 + c^4 + 4c^2a)t) \), which will be used to generate the breather solution of the CLL-NLS equation by DT later.

3. DERIVATION OF THE \( n \)-FOLD DT

In this Section, we derive the \( n \)-fold DT and the \( n \)-th order solutions for the coupled CLL-NLS equation in order to prove Theorem 2.1 and Theorem 2.2. To obtain the \( n \)-fold DT we consider the one- and two-fold DT at first, and then the \( n \)-fold DT can be obtained by iteration.

3.1. THE ONE-FOLD DT

Without loss of generality, assuming the one-fold DT as

\[
T_1(\lambda) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \lambda,
\]

(9)

where \( a_k, b_k, c_k \) and \( d_k \) \((k = 0, 1)\) are complex functions of \( x \) and \( t \). Then, there exists \( \Phi_1 \Phi \) satisfying the following conditions \( \Phi_1^{[1]} = U_1^{[1]} \Phi_1^{[1]} \) and \( \Phi_1^{[1]} = V_1^{[1]} \Phi_1^{[1]} \), where \( U_1^{[1]} \) and \( V_1^{[1]} \) have the same form as \( U \) and \( V \) except that \( q \) and \( r \) are replaced by \( q_1 \) and \( r_1 \). If so, we have

\[
T_x + TU - U^{[1]}T = 0, \quad \text{and} \quad T_t + TV - V^{[1]}T = 0.
\]

(10)

Lemma 3.1 Let one-fold DT of the coupled CLL-NLS equation be the form of (9), then it is given by

\[
T_1(\lambda) = T_1(\lambda, \lambda_1) = \frac{1}{\sqrt{f_1g_1}} \left( \sqrt{H} \right) \left( \begin{array}{cc} \lambda g_1 & -\lambda_1 f_1 \\ -\lambda_1 g_1 & \lambda f_1 \end{array} \right),
\]

(11)

and the new solution \((q_1, r_1)\), generated by above \( T_1 \) from “seed” \((q, r)\) is

\[
r_1 = H \left( \frac{g_1}{f_1} r + 2i\lambda_1 \right), \quad q_1 = \frac{1}{H} \left( \frac{f_1}{g_1} q - 2i\lambda_1 \right).
\]

(12)

Here, the overall factor \( H \) is given by (7).
Let \( F(\lambda) = (F_{ij}) = T_x + TU - U^{[1]}T = 0 \) (\( i, j = 1, 2 \)) and substitute \( T_1 \) (9) into \( F \),
then
\[
F_{11} = (q b_1 - r_1 c_1) \lambda^2 + \left( q b_0 - r_1 c_0 + a_{1x} + \frac{1}{4} i a_1 (q r - q_1 r_1) \right) \lambda + a_{0x} + \frac{1}{4} i a_0 (q r - q_1 r_1),
\]
\[
F_{12} = -2 i \lambda^3 b_1 + (r a_1 - r_1 d_1 - 2 i b_0) \lambda^2 + \left( r a_0 - r_1 d_0 + i b_1 + b_{1x} - \frac{1}{4} i b_1 (q r + q_1 r_1) \right) \lambda
\]
\[
+ i b_0 + b_{0x} - \frac{1}{4} i b_0 (q r + q_1 r_1),
\]
\[
F_{21} = 2 i \lambda^3 c_1 + (q d_1 - q_1 a_1 + 2 i c_0) \lambda^2 + \left( q d_0 - q_1 a_0 - i c_1 + c_{1x} + \frac{1}{4} i c_1 (q r + q_1 r_1) \right) \lambda
\]
\[
- i c_0 + c_{0x} + \frac{1}{4} i c_0 (q r + q_1 r_1),
\]
\[
F_{22} = (r c_1 - q_1 b_1) \lambda^2 + \left( r c_0 - q_1 b_0 + d_{1x} - \frac{1}{4} i d_1 (q r - q_1 r_1) \right) \lambda + d_{0x} - \frac{1}{4} i d_0 (q r - q_1 r_1).
\]

Note that \( b_1 \) and \( c_1 \) are equal to zero from coefficient of \( \lambda^3 \), and then the remaining coefficients of \( \lambda^i (i = 0, 1, 2) \) imply
\[
r_1 = \frac{a_1}{d_1} r - \frac{2 i b_0}{d_1}, \quad q_1 = \frac{d_1}{a_1} q + \frac{2 i c_0}{a_1}, \quad (13)
\]
and
\[
a_{1x} = \frac{a_1 c_0}{2 d_1} r - \frac{i b_0 c_0}{2 d_1} q + \frac{b_0}{d_1}, \quad d_{1x} = \frac{d_1 b_0}{2 a_1} q + \frac{i b_0 c_0}{a_1} - \frac{c_0}{2}, \quad (14)
\]
\[
b_{0x} = \frac{b_0^2}{2 a_1} r - \frac{b_0 c_0}{2 d_1} q + \frac{b_0^2 c_0}{a_1 d_1} + \frac{1}{2} i b_0 q r - i b_0,
\]
\[
c_{0x} = \frac{c_0^2}{2 a_1} r - \frac{b_0 c_0}{2 a_1} q - \frac{i c_0^2 b_0}{a_1 d_1} - \frac{1}{2} i c_0 q r + i c_0.
\]

Let \( a_0 = d_0 = 0 \) according to the coefficients of \( \lambda \) in order to obtain the non-trivial solution. After simple calculations, we obtain \( (a_1 d_1)_x = 0, (b_0 c_0)_x = 0 \), and \( (a_1 b_0)_x = \frac{1}{2} i a_1 b_0 (q r - 2) \). Based on the above results and taking the similar procedure to the second formula of (10), we have \( (a_1 d_1)_t = 0, (b_0 c_0)_t = 0 \), and \( (a_1 b_0)_t = -\frac{1}{2} a_1 b_0 (4 i + i q r^2 - 2 r q_x + 2 q r_x) \). Now, let \( a_1 d_1 = 1 \) and \( b_0 c_0 = \lambda_1^2 \) without loss of generality. Moreover, according to \( (a_1 b_0)_{xt} = (a_1 b_0)_{tx} \), it is reasonable to let \( a_1 b_0 = \mu G \), where \( G \) is the primitive integral function and \( \mu \) is an integral constant. That is, \( G \) satisfies
\[
\frac{\partial G}{\partial x} = \frac{1}{2} i (q r - 2) G, \quad \frac{\partial G}{\partial t} = -\frac{1}{4} (4 i + i q r^2 - 2 r q_x + 2 q r_x) G. \quad (15)
\]

Thus, \( G = H \), if we disregard the integral constant.
The explicit form of \( T_1 \) can be determined by \( T_1 \Phi_1 |_{\lambda = \lambda_1} = 0 \), \( i.e. \)
\[
a_1 \lambda_1^1 f_1 + b_0 g_1 = 0, \quad c_0 f_1 + d_1 \lambda_1 g_1 = 0.
\]
For convenience, let \( \mu = -\lambda_1 \), then the unknown elements \( a_1, d_1, b_0, \) and \( c_0 \) are solved by

\[
\begin{align*}
a_1 &= \sqrt{H_1} \sqrt{\frac{g_1}{f_1}}, \\
d_1 &= \frac{1}{\sqrt{H_1}} \sqrt{\frac{f_1}{g_1}}, \\
b_0 &= -\lambda_1 \sqrt{H_1} \sqrt{\frac{f_1}{g_1}}, \\
c_0 &= -\lambda_1 \frac{1}{\sqrt{H_1}} \sqrt{\frac{g_1}{f_1}}.
\end{align*}
\]

That is, the form of one-fold DT is

\[
T_1(\lambda) = T_1(\lambda; \lambda_1) = \begin{pmatrix}
\lambda \sqrt{H_1} \sqrt{\frac{g_1}{f_1}} & -\lambda_1 \sqrt{H_1} \sqrt{\frac{f_1}{g_1}} \\
-\lambda_1 \frac{1}{\sqrt{H_1}} \sqrt{\frac{g_1}{f_1}} & \lambda \frac{1}{\sqrt{H_1}} \sqrt{\frac{f_1}{g_1}}
\end{pmatrix},
\]

and the new solution \((q_1, r_1)\) can be expressed as

\[
\begin{align*}
r_1 &= H \left( \frac{g_1}{f_1} r + 2i \lambda_1 \right), \\
q_1 &= \frac{1}{H} \left( \frac{f_1}{g_1} q - 2i \lambda_1 \right).
\end{align*}
\]

Q.E.D.

Note that transformed eigenfunctions associated with new solution \((q_1, r_1)\) are

\[
\Phi_j^{[1]} = \begin{pmatrix}
\frac{f_j}{g_j}
\end{pmatrix} = T(\lambda, \lambda_1)|_{\lambda = \lambda_j} \Phi_j.
\]

(16)

It is trivial to see that \( \Phi_1^{[1]} = 0 \). In other words, \( T_1 \) annihilates its generating function, which is a general property of the DT. Therefore, we have to use a transformed eigenfunction \( \Phi_2^{[1]} \) associated with \( \lambda_2 (\neq \lambda_1) \) in order to generate the next step DT.

### 3.2. THE TWO-FOLD DT

By iteration, the two-fold DT for the coupled CLL-NLS equation is calculated as

\[
T_2(\lambda) = T_2(\lambda; \lambda_1, \lambda_2) = T_1^{[1]}(\lambda, \lambda_2) T_1(\lambda, \lambda_1),
\]

where

\[
T_1^{[1]}(\lambda, \lambda_2) = \frac{1}{\sqrt{f_1^{[1]} g_2^{[1]}}} \begin{pmatrix}
\sqrt{H_1} & -\lambda_1 f_2^{[1]} \\
\lambda_2 g_2^{[1]} & \lambda_2 f_2^{[1]}
\end{pmatrix},
\]

\( H_1 \) possesses the same form as \( H \) in (7), except \( q \) and \( r \) are replaced by \( q_1 \) and \( r_1 \). The definitions of \( H_1 \) and \( \Phi_2^{[1]} \) are valid for \( H_k \) and \( \Phi_k^{[j]} \) (If \( k < j \), \( \Phi_k^{[j]} = 0 \)).

According to the specific matrix forms of \( T_1 \) and \( T_1^{[1]}(\lambda, \lambda_2) \), then \( T_2 \) is expressed by

\[
T_2(\lambda; \lambda_1, \lambda_2) = \begin{pmatrix}
a_2^{[1]} \\
d_2^{[1]}
\end{pmatrix} \lambda^2 + \begin{pmatrix}
b_1^{[1]} \\
c_1^{[1]}
\end{pmatrix} \lambda + \begin{pmatrix}
a_0^{[1]} \\
d_0^{[1]}
\end{pmatrix},
\]

(17)
and \[ a_0^{[1]} = \lambda_1 \lambda_2 \sqrt{\frac{H_1 f_1^{[1]} g_1}{H g_1^{[1]} f_1}}, \quad d_0^{[1]} = \lambda_1 \lambda_2 \sqrt{\frac{H g_2^{[1]} f_1}{H_1 f_2^{[1]} g_1}}. \]

Note that \( T_2(\lambda) \Phi_k|_{\lambda=\lambda_k} = 0 \) \((k = 1, 2)\), then the four unknown elements \( a_2^{[1]}, d_2^{[1]}, b_1^{[1]}, \) and \( c_1^{[1]} \) can be solved as follows according to Cramer’s rule,
\[
a_2^{[1]} = \frac{\delta_3}{\delta_1}, \quad b_1^{[1]} = \frac{\delta_4}{\delta_1}, \quad d_2^{[1]} = \frac{\delta_5}{\delta_2}, \quad c_1^{[1]} = \frac{\delta_6}{\delta_2},
\]
where \( \delta_k \) \((k = 1, 2, \ldots, 6)\) are defined by
\[
\delta_1 = \begin{vmatrix} \lambda_2^2 f_1 & \lambda_1 g_1 \\ \lambda_2^2 f_2 & \lambda_2 g_2 \end{vmatrix}, \quad \delta_2 = \begin{vmatrix} \lambda_2^2 g_1 & \lambda_1 f_1 \\ \lambda_2^2 g_2 & \lambda_2 f_2 \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} -a_0^{[1]} f_1 & \lambda_1 g_1 \\ -a_0^{[1]} f_2 & \lambda_2 g_2 \end{vmatrix},
\delta_4 = \begin{vmatrix} -a_0^{[1]} g_1 & \lambda_1 f_1 \\ -a_0^{[1]} g_2 & \lambda_2 f_2 \end{vmatrix}, \quad \delta_5 = \begin{vmatrix} \lambda_2^2 f_1 & -a_0^{[1]} f_1 \\ \lambda_2^2 f_2 & -a_0^{[1]} f_2 \end{vmatrix}, \quad \delta_6 = \begin{vmatrix} \lambda_2^2 g_1 & -a_0^{[1]} g_1 \\ \lambda_2^2 g_2 & -a_0^{[1]} g_2 \end{vmatrix}.
\]
Substituting the above elements in the matrix form of \( T_2(\lambda) \), then it becomes
\[
T_2(\lambda; \lambda_1, \lambda_2) = \frac{1}{\sqrt{\lambda_1 f_1 \ g_1 \ \lambda_1 f_1 \ g_1}} \left( \begin{array}{c} \sqrt{\frac{g_1 f_1}{H_1}} \\ \sqrt{\frac{g_2 f_2}{H_2}} \end{array} \right) \begin{pmatrix} (T_2)_{11} & (T_2)_{12} \\ (T_2)_{21} & (T_2)_{22} \end{pmatrix}, \quad (18)
\]
and the elements \((T_2)_{ij} \ (i, j = 1, 2)\) are given by the following determinants
\[
(T_2)_{11} = \begin{vmatrix} \lambda_2^2 f_1 & \lambda_1 g_1 & f_1 \\ \lambda_2^2 f_2 & \lambda_2 g_2 & f_2 \end{vmatrix}, \quad (T_2)_{12} = \begin{vmatrix} 0 & \lambda & 0 \\ \lambda_2^2 f_1 & \lambda_1 g_1 & f_1 \end{vmatrix},
(T_2)_{21} = \begin{vmatrix} 0 & \lambda & 0 \\ \lambda_2^2 g_1 & \lambda_1 f_1 & g_1 \end{vmatrix}, \quad (T_2)_{22} = \begin{vmatrix} \lambda_2^2 g_1 & \lambda_1 f_1 & g_1 \\ \lambda_2^2 g_2 & \lambda_2 f_2 & g_2 \end{vmatrix}.
\]
Note that the overall factor \( H_1 \) has an integral function depending on \( q_1 \) and \( r_1 \). It implies that we need to apply the one-fold DT in order to obtain the two-fold DT. Thus, \( T_2 \) is not an explicit formula of the two-fold DT. Especially, as one iterates the above method, more integrals in overall factors \( H_k \) \((k > 1)\) will be involved. This depends on \( q_k \) and \( r_k \). However, \( q_k \) and \( r_k \) are too cumbersome to be expressed in terms of explicit integrals in overall factors \( H_k \). That is, it is not possible to get the explicit expressions of \( T_k \) if \( H_k \) can not be eliminated. Thus, eliminating the integrals in the overall factors \( H_k \) is an unavoidable challenge. The next Lemma provides a crucial step to deal with this obstacle. In the following lemma, \( \frac{g_i^{[0]}}{f_i^{[0]}} \equiv g_i \frac{f_i^{[0]}}{f_i} \).
Lemma 3.2 Let $i \geq k + 1 \geq 1$, then $\frac{g^{[k]}_i}{f^{[k]}_i} H_{k+1}$ is a constant.

On one hand, according to the $x$-part of the Lax pair for $\Phi^{[k]}_i$ and the $k$-th step of DT, a straightforward calculation implies

$$f^{[k]}_{i,x} = (i\lambda^2 - \frac{1}{2}i + \frac{1}{4} iqkr_k) f^{[k]}_i + \lambda_i r_k g^{[k]}_i, \quad g^{[k]}_{i,x} = \lambda_i q_k f^{[k]}_i - (i\lambda^2 - \frac{1}{2}i + \frac{1}{4} iqkr_k) g^{[k]}_i,$$

$$r_{k+1} = H_k \left( \frac{g^{[k]}_i}{f^{[k]}_i} r_k + 2i \lambda_i \right), \quad q_{k+1} = \frac{1}{H_k} \left( \frac{f^{[k]}_i}{g^{[k]}_i} q_k - 2i \lambda_i \right).$$

According to the definition of $H_{k+1}$,

$$\frac{H_{k+1} + q_{k+1}}{H_{k+1}} = \frac{1}{2}i(q_{k+1} r_{k+1} - 2) = \frac{1}{2}iqr_k - i + 2i\lambda^2 - \lambda_i(\frac{f^{[k]}_i}{g^{[k]}_i} q_k - \frac{g^{[k]}_i}{f^{[k]}_i} r_k).$$

Thus,

$$\left( \frac{g^{[k]}_i}{f^{[k]}_i} H_{k+1} \right)_x = \frac{g^{[k]}_i}{g^{[k]}_i} \frac{f^{[k]}_i}{f^{[k]}_i} + \frac{H_{k+1,x}}{H_{k+1}} = 0. \quad (19)$$

On the other hand, according to the $t$-part of Lax pair for $\Phi^{[k]}_i$, and the definition of $H_{k+1,t}$, a straightforward calculation implies

$$\frac{f^{[k]}_i}{f^{[k]}_i} = -2i\lambda^4 + (2i - iqkr_k) \lambda^2 - \frac{1}{8} iqkr^2_k + \frac{1}{4} r_k q_k - \frac{1}{4} qkr_k x - \frac{1}{2}i$$

$$- (2r_k \lambda^3 - (r_k - \frac{1}{2}k q_k + ir_k x) \lambda_i) \frac{g^{[k]}_i}{f^{[k]}_i},$$

$$\frac{g^{[k]}_i}{g^{[k]}_i} = -(-2i\lambda^4 + (2i - iqkr_k) \lambda^2 - \frac{1}{8} iqkr^2_k + \frac{1}{4} r_k q_k - \frac{1}{4} qkr_k x - \frac{1}{2}i)$$

$$- (2q_k \lambda^3 - (q_k - \frac{1}{2}qkr_k - iqk x) \lambda_i) \frac{f^{[k]}_i}{g^{[k]}_i},$$

and

$$\frac{H_{k+1,t}}{H_{k+1}} = -\frac{1}{4}(4i + iq^2 k r_{k+1} - 2 r_{k+1} q_{k+1, x} + 2 q_{k+1} r_{k+1, x})$$

$$= -4i\lambda^4 + (4i - 2qkr_k) \lambda^2 - \frac{1}{4} iq_k r^2_k - \frac{1}{2} (q_k r_k x - q_k q_k x) - i$$

$$+ (2q_k \lambda^3 - (q_k - \frac{1}{2}qkr_k - iqk x) \lambda_i) \frac{f^{[k]}_i}{g^{[k]}_i}$$

$$- (2r_k \lambda^3 - (r_k - \frac{1}{2}r_k q_k + ir_k x) \lambda_i) \frac{g^{[k]}_i}{f^{[k]}_i}.$$
The above three expressions give
\[
\left( \frac{g_i^{[k]} f_i^{[k]}}{f_i^{[k]}} H_{k+1} \right)_t = \frac{g_i^{[k]} f_i^{[k]}}{f_i^{[k]}} - \frac{f_i^{[k]} g_i^{[k]}}{f_i^{[k]}} + \frac{H_{k+1}}{H_{k+1}} = 0, \tag{20}
\]
Q.E.D.

Based on lemma 3.2, let \( \frac{g_1^{[k]}}{f_1^{[k]}} H_{k+1} = 1 \) without loss of generality. In this case, \( \frac{g_1}{f_1} H_1 = 1 \), and then two-fold DT in (18) is simplified as
\[
T_2(\lambda) = T_2(\lambda, \lambda_1, \lambda_2) = \frac{1}{\sqrt{\lambda_1 f_1 g_1 \lambda_2 f_2 g_2}} \begin{pmatrix} T_2_{11} & T_2_{12} \\ T_2_{21} & T_2_{22} \end{pmatrix}. \tag{21}
\]

3.3. THE \( n \)-FOLD DT

Let us consider the \( n \)-fold DT for the coupled CLL-NLS equation with the similar method as above. Since
\[
T_n(\lambda) = T_n(\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{k=1}^{n} T_2^{[k-1]}(\lambda, \lambda_k),
\]
let
\[
T_n(\lambda) = T_n(\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{l=1}^{n} P_l \sigma_1^{n-l} \lambda^l + P_0, \tag{22}
\]
where \( P_l \) and \( \sigma_1 \) are defined by
\[
P_l = \begin{pmatrix} P_{l1} & 0 \\ 0 & P_{l2} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Furthermore, \( P_0 \) is determined by
\[
P_0 = \prod_{k=1}^{n} \begin{pmatrix} 0 & -\lambda_k \sqrt{H_{k-1}} \frac{f_i^{[k-1]}}{g_i^{[k-1]}} \\ -\lambda_k \sqrt{H_{k-1}} \frac{g_i^{[k-1]}}{f_i^{[k-1]}} & 0 \end{pmatrix}. \tag{23}
\]
Here, \( H_0 = H, f_1^{[0]} = f_1, \) and \( g_1^{[0]} = g_1 \). According to Lemma 3.2, then
\begin{itemize}
  \item if \( n \) is odd,
\end{itemize}
\[
P_0 = \begin{pmatrix} -\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n \sqrt{H} \frac{f_i^{[n-1]}}{g_i^{[n-1]}} \\ -\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n \sqrt{H} \frac{g_i^{[n-1]}}{f_i^{[n-1]}} \end{pmatrix}, \tag{24}
\]
if $n$ is even,

$$P_0 = \left( \begin{array}{c} \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n \frac{1}{\sqrt{H}} \sqrt{\frac{n-1}{\mu_{n-1}}} \\ \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n \frac{1}{\sqrt{H}} \sqrt{\frac{n-1}{\mu_{n-1}}} \end{array} \right).$$  \tag{25}

**Lemma 3.3** After the action of $k$-fold DT, the eigenfunction $\Phi_j$ ($j > k$) related to $\lambda_j$ becomes

- if $k$ is odd

$$\Phi_j^{[k]} = \frac{1}{\sqrt{|\Delta_k^1||\Delta_k^2|}} \left( \sqrt{H} \frac{1}{H} \left( \det(A_{k+1}^1, A_{k+1}^2, A_{k+1}^3, \ldots, A_{k+1}^j)^T \right) \det(B_{k+1}^1, B_{k+1}^2, B_{k+1}^3, \ldots, B_{k+1}^j)^T \right),$$

- if $k$ is even

$$\Phi_j^{[k]} = \frac{1}{\sqrt{|\Delta_k^1||\Delta_k^2|}} \left( \sqrt{H} \frac{1}{H} \left( \det(A_{k+1}^1, A_{k+1}^2, A_{k+1}^3, \ldots, A_{k+1}^j)^T \right) \det(B_{k+1}^1, B_{k+1}^2, B_{k+1}^3, \ldots, B_{k+1}^j)^T \right).$$

**Remark:** this Lemma is obtained with the inductive method, and the detailed proof is omitted.

Therefore, the explicit expression of $P_0$ is obtained as follows based on Lemma 3.3.

$$P_0 = \begin{cases} 
\left( \begin{array}{c} \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_k \sqrt{\frac{\Delta_k^1}{\Delta_k^1}} \\ \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_k \sqrt{\frac{\Delta_k^2}{\Delta_k^1}} \end{array} \right) & \text{if } k \text{ even,} \\
\left( \begin{array}{c} \sqrt{H} \\ \frac{1}{\sqrt{H}} \end{array} \right) \left( \begin{array}{c} -\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_k \sqrt{\frac{\Delta_k^1}{\Delta_k^1}} \\ -\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_k \sqrt{\frac{\Delta_k^2}{\Delta_k^1}} \end{array} \right) & \text{if } k \text{ odd.} 
\end{cases}$$  \tag{26}

**Proof of Theorems 2.1 and 2.2:**

Note that the kernel of $T_n$ consists of $\Phi_k (k = 1, 2, \ldots, n)$, i.e.,

$$T_n(\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n) \Phi_k |_{\lambda = \lambda_k} = 0.$$  

Substituting (26) into these algebraic equations, the elements of $P_k$ ($k = 1, 2, \ldots, n$) in $n$-fold DT are obtained by the Cramer’s rule. This proves Theorem 2.1. Then, Theorem 2.2 is derived by comparing the coefficient of $\lambda^{n-1}$ in $T_{nk} + T_n U = U^{[n]} T_n$.  


4. EXACT SOLUTIONS OF THE CLL-NLS EQUATION

In this Section, we consider the DT and solution of the coupled CLL-NLS equation (3) under the reduction condition \( r = -\overline{q} \), which leads to the DT and solutions of the CLL-NLS equation.

Theorem 4.1 Let
\[
\begin{align*}
\lambda_k &= -\overline{\lambda}_k & \text{if } n \text{ odd}, \\
\lambda_{2k} &= -\overline{\lambda}_{2k-1} & \text{if } n \text{ even},
\end{align*}
\] (27)
then solution \((q_n, r_n)\) in Theorem 2.2 preserves the reduction condition \( r_n = -\overline{q}_n \). This means that \( T_n \) in Theorem 2.1 is a \( n \)-fold DT of the CLL-NLS equation (1), and correspondingly, \( r_n \) is an \( n \)-th order solution of the CLL-NLS equation (1).

Let \( q = -\overline{r} \). From the \( x \)-part of the Lax pair, we have
\[
\begin{align*}
\overline{f}_x &= -(i\lambda^2 - \frac{1}{2}i - \frac{1}{4}i|\overline{r}|^2)\overline{f} + \overline{\lambda}\overline{r}g, & \overline{g}_x &= -\overline{\lambda}\overline{r}\overline{f} + (i\lambda^2 - \frac{1}{2}i - \frac{1}{4}i|\overline{r}|^2)\overline{g}.
\end{align*}
\] (28)
That is
\[
\begin{pmatrix}
\overline{g}_x \\
\overline{f}_x
\end{pmatrix} = \begin{pmatrix}
(i\lambda^2 - \frac{1}{2}i - \frac{1}{4}i|\overline{r}|^2) & -\overline{\lambda}\overline{r} \\
\overline{\lambda}\overline{r} & (i\lambda^2 - \frac{1}{2}i + \frac{1}{4}i|\overline{r}|^2)
\end{pmatrix}
\begin{pmatrix}
\overline{g} \\
\overline{f}
\end{pmatrix}.
\] (29)

The same property can be obtained from the \( t \)-part of the Lax pair. Thus, it is obvious that \( \begin{pmatrix}
\overline{g} \\
-\overline{f}
\end{pmatrix} \) is a new eigenfunction for \( \lambda = \overline{\lambda} \) or \( \begin{pmatrix}
\overline{f} \\
\overline{g}
\end{pmatrix} \) for \( \lambda = -\overline{\lambda} \). For example, \( \begin{pmatrix}
\overline{g}_j \\
-\overline{f}_j
\end{pmatrix} \) is a new eigenfunction related to \( \lambda_k \) when \( \lambda_k = \overline{\lambda}_j \), and \( \begin{pmatrix}
\overline{f}_j \\
\overline{g}_j
\end{pmatrix} \) is another one when \( \lambda_k = -\overline{\lambda}_j \).

Based on the above property of the eigenfunctions, we prove that the potentials \( q_n \) and \( r_n \) will satisfy the reduction condition, if the choices in (27) are adopted in the \( n \)-fold DT.

Note that \( \frac{H}{H} = \frac{1}{H} \). For \( n = 1 \), let \( \lambda_1 = -\overline{\lambda}_1 \), then
\[
\begin{align*}
r_1 &= H \left( \frac{q_1}{f_1} r + 2i\lambda_1 \right) = H \left( \frac{f_1}{f_1} r + 2i\lambda_1 \right), \\
\tau_1 &= \frac{1}{H} \left( \frac{f_1}{f_1} r - 2i\lambda_1 \right) = -\frac{1}{H} \left( \frac{f_1}{f_1} q - 2i\lambda_1 \right) = -q_1.
\end{align*}
\]
For \( n = 2 \), let \( \lambda_2 = -\overline{\lambda}_1 \), then \( f_2 = \overline{g}_1 \) and \( g_2 = \overline{f}_1 \). Therefore,
\[
\begin{align*}
r_2 &= \frac{(\lambda_2 g_2 f_1 - \lambda_1 g_1 f_2) r + 2i(\lambda_2^2 - \lambda_1^2) f_1 f_2}{\lambda_2 g_1 f_2 - \lambda_1 f_1 g_2} = \frac{(-\overline{\lambda}_1|f_1|^2 - \lambda_1|g_1|^2) r + 2i(\overline{\lambda}_1^2 - \lambda_1^2) f_1 g_1}{-\overline{\lambda}_1|f_1|^2 - \lambda_1|g_1|^2}, \\
q_2 &= \frac{(\lambda_2 g_2 g_1 - \lambda_1 f_1 g_2) q - 2i(\lambda_2^2 - \lambda_1^2) g_1 g_2}{\lambda_2 f_1 g_2 - \lambda_1 f_1 g_2} = \frac{(-\overline{\lambda}_1|g_1|^2 - \lambda_1|f_1|^2) q - 2i(\overline{\lambda}_1^2 - \lambda_1^2) g_1 \overline{f}_1}{-\overline{\lambda}_1|f_1|^2 - \lambda_1|g_1|^2} = -\tau_2,
\end{align*}
\]
When \( n > 2 \), the reduction condition \( q_n = -r_n \) can also be obtained by iteration.

Next, we provide the solutions of the CLL-NLS, and then discuss their localization properties. In order to achieve this purpose, the eigenfunctions associated with the “seed” solution depend on the determinant representation of DT.

4.1. EIGENFUNCTIONS FOR THE LAX PAIR

In this Subsection, we consider the solution for the Lax pair. Let the seed solution be

\[
q = -r = c \exp(i\rho), \quad \rho = ax + bt, \quad b = a^2 + (a - 1)c^2, \quad a, c \in \mathbb{R}
\]  

We substitute (30) into the Lax pair equations (4) and obtain the eigenfunction as follows:

\[
\psi = \begin{pmatrix} \psi_1(x, t, \lambda) \\ \psi_2(x, t, \lambda) \end{pmatrix} = \begin{pmatrix} D \exp\left(i\left(\frac{\sqrt{S}}{4}x + \frac{\sqrt{S}}{8}(-4\lambda^2 + 2 + c^2 + 2a)t - \frac{\rho}{2}\right)\right) \\ D(-i\sqrt{S} + 4i\lambda^2 - 21 - i\lambda^2 + 2ia)_\exp\left(i\left(\frac{\sqrt{S}}{4}x + \frac{\sqrt{S}}{8}(-4\lambda^2 + 2 + c^2 + 2a)t + \frac{\rho}{2}\right)\right) \end{pmatrix},
\]

where \( S \) is defined by

\[
S = 16\lambda^4 + (16a - 16 + 8c^2)\lambda^2 - 4c^2a + 4 + 4c^2 - 8a + c^4 + 4a^2,
\]

and \( D \) is a constant. Note that \( \begin{pmatrix} \overline{\psi}_2(x, t, -\lambda) \\ \overline{\psi}_1(x, t, -\lambda) \end{pmatrix} \) is also an eigenfunction under the reduction condition \( r = -\overline{q} \). Thus, we can induce a new eigenfunction by use of the superposition principle:

\[
\Phi = \begin{pmatrix} f(x, t, \lambda) \\ g(x, t, \lambda) \end{pmatrix} = \begin{pmatrix} \psi_1(x, t, \lambda) + \overline{\psi}_2(x, t, -\lambda) \\ \psi_2(x, t, \lambda) + \overline{\psi}_1(x, t, -\lambda) \end{pmatrix}.
\]  

Let \( \lambda = \lambda_j \), then \( \Phi \) (see (31)) leads to the eigenfunction \( \Phi_j \) related to \( \lambda_j \). Furthermore, when \( q = -r = c \exp(i\rho) \), the explicit expression of \( H \) is given in (8). Now, with the help of theorems 2.2 and 4.1, we can present the solutions of the CLL-NLS equation.

4.2. SOLITON, BREATHER, AND FIRST-ORDER ROGUE WAVE SOLUTIONS

For \( n = 1 \), let \( \lambda_1 = i\beta \) and \( D = 1 \) in Theorem 2.2, then the first-order solution of the CLL-NLS equation is

\[
r_1 = \left(\frac{L_1 \cos \theta + iL_2 \sin \theta}{L_1 \cos \theta + iL_2 \sin \theta} e^{-2\beta}\right) H,
\]  

(32)
with
\[
\theta = \left( \frac{1}{2} \beta^2 + \frac{1}{4} + \frac{1}{8} c^2 + \frac{1}{4} a \right) \sqrt{S_1 t} + \frac{1}{4} \sqrt{S_1 x},
\]
\[
L_1 = -4 \beta c + \sqrt{S_1 + 4 \beta^2 + 2 + c^2 - 2a},
\]
\[
L_2 = 4 \beta c + \sqrt{S_1 + 4 \beta^2 + 2 + c^2 - 2a},
\]
\[
S_1 = 16 \beta^4 + (-8 c^2 - 16 a + 16) \beta^2 - 4 c^2 a + 4 + 4 c^2 - 8 a + c^4 + 4 a^2.
\]

It is obvious that \( r_1 \) leads to a periodic solution, if \( S_1 > 0 \) and gives a solitonic solution if \( S_1 < 0 \). When \( x \) and \( t \) tend both to infinity, \( |r_1|^2 \) tends to \( 2a - 2 \) (here \( a > 1 \)). When \( S_1 < 0 \), \( |r_1| \) reaches to its amplitude of \( |2 \beta + c| \) at \( x = -\frac{1}{2} (4 \beta^2 + 2 + c^2 + 2a) t \). Thus, if \( 2a - 2 > |2 \beta + c| \), it gives a dark soliton. Otherwise, it leads to a bright solitonic localization with a non-vanishing value at the boundary. That is, the CLL-NLS equation can give both bright and dark solitons. This is different from the case of NLS equation; depending on the signs of dispersion and nonlinear parameters the NLS equation admit either dark or bright soliton solutions. The typical bright and dark soliton solutions of the CLL-NLS equation are shown in Fig. 1.

For \( n = 2 \), let \( D = 1 \), \( \lambda_1 = \alpha_1 + i \beta_1 \) and \( \lambda_2 = -\alpha_1 + i \beta_1 \) in Theorem 2.2, then
\[
| r_2 | = \frac{(-\bar{\lambda}_1 |g_1|^2 - \lambda_1 |f_1|^2) r + 2i(\bar{\lambda}_1^2 - \lambda_1^2) f_1 \bar{g}_1}{-\bar{\lambda}_1 |g_1|^2 - \lambda_1 |f_1|^2}
\]  
(33)
gives the second-order solution of the CLL-NLS equation.
Fig. 2 – (Color online). The breather solutions of the CLL-NLS equation with parameters: (a) $c = 1,$ $\alpha = 0.75,$ and $\beta = 0.4;$ (b) $c = 1,$ $\alpha = 0.8,$ and $\beta = 0.55;$ (c) $c = 1,$ $\alpha_1^2 = \beta_1^2 + \frac{1}{2},$ and $\beta_1 = 0.52.$

Fig. 3 – (Color online). The first-order RW solution of the CLL-NLS equation with parameters: $\alpha_1 = -1,$ $\beta_1 = 0.5.$ The right panel is the density plot of the left one.
For convenience, let \( a = 2\beta_1^2 - \frac{1}{2}c^2 - 2\alpha_1^2 + 1 \), then

\[
r_2 = \frac{-K_1 c \cos \theta_1 + iK_2 c \sin \theta_1 + (K_3 c + K_5) \cosh \theta_2 + i(K_6 - K_4 c) \sinh \theta_2}{-K_1 \cos \theta_1 + iK_2 \sin \theta_1 + K_3 \cosh \theta_2 + iK_4 \sinh \theta_2} \exp(-i\rho) \tag{34}
\]

with

\[
\theta_1 = ((4\alpha_1^2 - 4\beta_1^2 - 2)t - x)K_0, \quad \theta_2 = 4\alpha_1 \beta_1 tK_0, \quad K_0 = \sqrt{(c^2 + 4\alpha_1^2)(c^2 - 4\beta_1^2)},
\]

\[
K_1 = 8\alpha_1^2 \beta_1^2 + 2c^2 \alpha_1 \beta_1^2 + 2\alpha_1 \beta_1 K_0, \quad K_2 = 2c^2 \alpha_1 \beta_1 - 8\alpha_1 \beta_1^2 + 2\alpha_1 \beta_1 K_0,
\]

\[
K_3 = c\beta_1 + 4c\alpha_1^3 + c\alpha_1 K_0, \quad K_4 = 4c\beta_1^3 - c^3 \beta_1 - c\beta_1 K_0,
\]

\[
K_5 = -8c^2 \alpha_1 \beta_1^2 - 32\alpha_1^3 \beta_1^2 - 8\alpha_1 \beta_1^2 K_0, \quad K_6 = 8c^2 \alpha_1^2 \beta_1^3 - 32\alpha_1^3 \beta_1^2 + 8\alpha_1^2 \beta_1 K_0.
\]

Note that the trajectory of \( r_2 \) is defined by \((4\alpha_1^2 - 4\beta_1^2 - 2)t - x = 0\), if \( K_0^2 < 0 \), and by \( 4\alpha_1 \beta_1 t = 0 \), if \( K_0^2 > 0 \). Thus, we can get both the spatial periodic breather solution (similar to the NLS Akhmediev breather [85]) and the temporal periodic breather solution (similar to the NLS Kuznetsov-Ma breather [86, 87]). In fact, this solution \( r_2 \) can travel periodically with an additional velocity in the \((x, t)\)-plane. Three kinds of breather solutions, propagating along the \((x, t)\)-plane with different angles, are shown in Fig. 2.

After a simple analysis, we emphasize that the periodicity of the breather solution is proportional to \( \frac{1}{K_0} \), i.e. when \( K_0 \) tends to zero, the distance of two peaks goes to infinity, which leaves only one peak locating on the \((x, t)\)-plane. Thus, let \( c \to 2\beta_1 \), then \( r_2 \) in (34) leads to a new solution, having the property to possess only one local peak and surrounding two holes, which is very similar to the Peregrine solution and therefore, being appropriate to model RWs. This kind of doubly-localized rational solution is described by

\[
r_{2r} = \frac{2\beta_1 (L_{11} + iL_{12})}{L_{11} + iL_{13} + 4} \exp(-i\rho), \tag{35}
\]

with

\[
L_{11} = (16\alpha_1^2 \beta_1^2 + 16\beta_1^4)x^2 - (128\alpha_1^4 \beta_1^2 - 64\beta_1^4 - 64\alpha_1^2 \beta_1^2 - 128\beta_1^6)xt
\]

\[
- (256\alpha_1^4 \beta_1^2 - 64\alpha_1^2 \beta_1^2 - 256\beta_1^8 - 256\alpha_1^6 \beta_1^2 - 64\beta_1^6) t^2 - 3,
\]

\[
L_{12} = 8\beta_1^2 x + 16\beta_1^2 t - 96\alpha_1^2 \beta_1^2 t, \quad L_{13} = (32\alpha_1^2 \beta_1^2 - 64\beta_1^4 - 16\beta_1^2) t - 8\beta_1^2 x.
\]

When \( x \) and \( t \) tend to infinity, \( |r_{2r}| \) tends to \( 2\beta_1 \). Moreover, the maximum peak amplitude is equal to \( 6\beta_1 \), which is three times the background amplitude. The profiles are shown in Fig. 3, and this solution is the same as presented in [73]. The latter has been derived using the Hirota bilinear method, while difficulties using the DT have been also discussed in [73].
4.3. HIGHER-ORDER ROGUE WAVE SOLUTIONS

Inspired by the above method, we consider the higher-order RWs of the CLL-NLS equation in this Subsection. Generally, it is difficult to derive higher-order RWs from multi-breather solutions, since the explicit expression of \( n \)-th order breather is very challenging to calculate when \( n > 4 \). Similarly for the NLS equation, for which the formulae are given by Theorem 2.2, an indeterminate form \( 0 \quad 0 \) is a consequence, when eigenvalues \( \lambda_k \) tend to a limit point (from a breather to a doubly-localized RW solution).

Thus, we derive the higher-order RWs directly from the determinant expressions of solutions in Theorem 2.2 by adopting a Taylor expansion [16, 49–51].

**Theorem 4.2** Let \( n = 2N \), \( \lambda_{2k-1} = \sqrt{1 - \frac{a^2}{2}} + i c + \epsilon^2 \) (\( a < 1 \)) and \( \lambda_{2k} = -\lambda_{2k-1} \), by applying the Taylor expansion, then a determinant expression of the \( N \)-th order RW is given as

\[
\hat{r}_{nr} = \frac{|\hat{\Delta}_1^n|}{|\hat{\Delta}_2^n|} r - 2i \frac{|\hat{\Delta}_4^n|}{|\hat{\Delta}_2^n|},
\]

where \( \hat{\Delta}_n^k \) (\( k = 1, 2, 3, 4 \)) are defined by

\[
\hat{\Delta}_n^k = \left( \frac{\partial n_i}{\partial c} \right)_{\epsilon=0} (\Delta_n^k)_{ij} \text{ for } n \times n.
\]

Here, \( n_i = i \) if \( i \) is odd and \( n_i = i - 1 \) if \( i \) is even.

Note that \( D \) is a constant in (31); it is reasonable to choose \( D = \exp \left( i \sqrt{\mathcal{B}} \sum_{l=1}^{N-1} s_l \epsilon^{2l} \right) \).

Next, we derive RWs with these parameters, and consider their dynamical evolution. For convenience, let \( a = -1 \) and \( c = 1 \). For \( N = 2 \), the second-order RW of the CLL-NLS equation is

\[
r_{4r} = \frac{L_{21}}{L_{22}} \exp(-i\rho),
\]

where

\[
L_{21} = 125 x^6 + 150 ix^5 - 750 tx^5 - 285 x^4 + 3375 t^2 x^4 - 2100 ix^3 - 156 ix^3 - 8500 t^3 x^3
\]

\[
+ 2220 tx^3 + 8100 it^2 x^3 - 24000 it^3 x^2 - 14850 t^2 x^2 + 333 x^2 + 16875 t^4 x^2
\]

\[
+ 2160 it x^2 - 270 tx - 18750 t^5 x + 33750 it^4 x + 28500 t^3 x - 8100 it x^2 - 90 ix + 45 + 15625 t^6 + 1044 it + 600 it^3 - 2205 t^2 - 375000 t^5 + 26625 t^4 + (-300 x^3 -
\]

\[
1800 ix^2 + 4500 tx^2 + 1800 it x + 180 x - 4500 t^2 x - 4500 t^3 - 432 i + 540 t + 7200 it^2 \right) s_1
\]

\[
\frac{L_{21}}{L_{22}} \exp(-i\rho).
\]
where $s_1$ is a free complex parameter. The maximum amplitude of $|r_{4r}|$ is equal to 5 when $s_1 = 0$, which is reached at $(x = 0, t = 0)$.

This solution is shown in Fig. 4(a). Allocating different values to $s_1$, we can obtain RWs that are distinct from the above ones. For example, RWs with $s_1 = 100 - 100i$ and $s_1 = 100 + 100i$ are shown in Fig. 4(b) and Fig. 4(c), respectively. Both of them possess three intensity peaks, located at different time and space values. Each peak is similar to a first-order RW, shown in Fig. 3(a). Moreover, solution $r_{4r}$ in Fig. 4(b) is different from the one in Fig. 4(c), since three peaks in each solution are arrayed in different directions.

Fig. 4 – (Color online). The second-order RW of the CLL-NLS equation with parameters: (a) $s_1 = 0$; (b) $s_1 = 100 - 100i$; (c) $s_1 = 100 + 100i$. 
Fig. 5 – (Color online). The third-order RW of the CLL-NLS equation with parameters \((s_1, s_2)\) as: (a) \((0, 0)\); (b) \((100, 0)\); (c) \((0, 5000)\); (d) \((100, 13000)\).

Fig. 6 – (Color online). The fourth-order RW of the CLL-NLS equation with parameters \((s_1, s_2, s_3)\) as: (a) \((0, 0, 0)\); (b) \((500, 0, 0)\); (c) \((0, 50000, 0)\).
Fig. 7 – (Color online). The fourth-order RW in a circular pattern of the CLL-NLS equation with parameters \(s_1, s_2, s_3\) as: (a) \((0, 0, 500000)\); (b) \((500, 0, 5000000)\).

For \(N = 3\), according to Theorem 4.2, we can obtain the third-order RW solution of the CLL-NLS equation. However, its expression, with two non-zero parameters \(s_1\) and \(s_2\), is very cumbersome. We just provide the exact expression in the case \(s_1 = s_2 = 0, \ i.e.

\[
 r_{0r} = \frac{L_{31}}{L_{32}} \exp(-i\rho), \tag{39}
\]

with

\[
 L_{31} = -3125.3^{12} + 375000.3^{11} - 75000.3^{11} - 268250.3^{10} + 1500000.3^{10} + 187500.3^{10} + 1437500.3^{9} - 1237500.3^{9} + 2500300.3^{8} - 3225000.3^{8} + 3386250.3^{7} + 6975000.3^{7} - 5671875.3^{6} - 49000.3^{6} + 317250.3^{5} + 530000.3^{5} + 63720.3^{4} + 1747500.3^{4} + 846000.3^{4} - 1930000.3^{3} + 27975000.3^{3} + 88724000.3^{3} + 82887500.3^{3} + 116676.3^{2} - 8925600.3^{2} + 3000000.3^{2} - 33810000.3^{2} - 436375000.3^{2} + 22302000.3^{2} + 87375000.3^{2} + 26429000.3^{2} + 313200.3^{2} + 133155000.3^{2} + 2099775000.3^{2} + 2458350000.3^{2} + 1239845.3^{2} + 40125000.3^{2} + 550312500.3^{2} + 146475.3^{2} + 17172000.3^{2} + 366750000.3^{2} + 9381250.3^{2} + 607500000.3^{2} + 141796875.3^{2} + 1330000000.3^{2} + 1315980.3^{2} + 5280000.3^{2} - 5800375000.3^{2} + 6064500000.3^{2} + 42228000.3^{2} + 97650000.3^{2} + 1796875000.3^{2} + 9022500000.3^{2} + 2099250000.3^{2} + 9762000000.3^{2} - 8460000000.3^{2} - 1242945000.3^{2} + 17519125000.3^{2} + 6562500000.3^{2} + 1078537500.3^{2} - 198720000.3^{2} + 12969000000.3^{2} + 32322150000.3^{2} + 17347500000.3^{2} - 6325000000.3^{2} + 887750000.3^{2} + 1171875000.3^{2} - 113400.3^{2} + 258250000.3^{2} - 2994750000.3^{2} - 286740.3^{2} + 5224500000.3^{2} + 6215625000.3^{2} - 50460000000.3^{2} - 4921875000.3^{2} + 31387500000.3^{2} - 12899250000.3^{2} - 83905250000.3^{2} - 190350000000.3^{2} - 468750000000.3^{2} - 715230000.3^{2} - 4882812500.3^{2} + 2835 + 576180000.3^{2} + 2223281250.3^{2} - 478338751.3^{2} + 5355000000.3^{2} + 23437500000.3^{2} + 3632812500.3^{2} + 29193750000.3^{2} + 158760000.3^{2}.
The maximum amplitude is equal to 7 that occurs at (0, 0), see Fig. 5(a). Let \( s_1 \neq 0 \) and \( s_2 \neq 0 \), we obtain other solutions which are different from the one given in Fig. 5(a). In each of these solutions, the third-order RW is split into six intensity peaks, which are similar to a first-order RW. These six peaks, located at different points of time and space, make up different profiles. As examples, three such solutions are displayed in Fig. 5(b-d) with \((s_1, s_2) = (100, 0), (s_1, s_2) = (0, 5000), \) and \((s_1, s_2) = (100, 13000), \) respectively. In Fig. 5(b), the six intensity peaks form a triangle. In Fig. 5(c), they compose a pentagon with five peaks locating on the shell and the other one locating on the center. In Fig. 5(d), three peaks compose a triangle and the other three peaks compose a part of a circular arc.

Let \( N = 4 \) in Theorem 4.2, then \( r_{8r} \) gives a fourth-order RW of the CLL-NLS equation with three parameters \( s_1, s_2, \) and \( s_3 \). Let \((s_1, s_2, s_3) = (0, 0, 0)\), \( r_{8r} \) leads to a solution with a highest peak surrounded by several gradually decreasing peaks in two sides along \( t \)-direction, which is the fundamental pattern and is shown in Fig 6(a). The amplitude of this solution is 9, located at the origin of coordinate. Furthermore, allocating different values to \((s_1, s_2, s_3)\), we obtain a hierarchy of solutions, which have a triangle pattern, a pentagon pattern, a circular pattern with a inner second-order fundamental pattern or a triangle pattern. These solutions are shown in Fig. 6(b-c) and Fig. 7.

All the results are derived as a consequence of Theorem 4.2 and can be trivially extended to the higher-order RWs. That is, the explicit expressions of other higher-order solutions can be obtained in a straightforward manner. However, we will omit this, since expressions are too cumbersome to be explicitly written here. All solutions, presented above, have been verified analytically by symbolic computation through a Maple computer software.
5. LOCALIZATION CHARACTERISTICS OF CLL-NLS ROGUE WAVES

In this Section, we consider the localization features of the RW of the CLL-NLS equation as well as the influence of SSE on this localization. First, we need to define the length and width of the RW solution as described in [58]. In order to compare the latter properties with localization of NLS RWs [16, 58], we replace the parameters $\alpha_1$ and $\beta_1$ with $a$ and $c$ in (35). That is, we substitute $\alpha_1 = \frac{1-a}{2} (a < 1)$ and $\beta_1 = \frac{c}{2}$ into (35). In this case, the first-order RW of the CLL-NLS equation is expressed as the following

$$r_{2r} = \frac{L_n}{L_d} c \exp(-i\rho),$$

with

$$L_n = 3 - c^4 t^2 - 4 t^2 c^4 - 4 c^8 t^2 - c^8 t^2 - 2 c^6 x t + 8 i c^2 - 12 i c^2 t a - 4 c^4 t x + 8 c^2 t^2 a^2 - 2 c^2 x^2 + 2 a c^2 x^2 - 8 c^2 x t a - 2 i c^2 x - 8 c^2 t^2 a^2 + 8 c^2 x t a^2,$$

$$L_d = -1 - c^4 t^2 - 4 t^2 c^4 - 4 c^8 t^2 - c^8 t^2 - 2 c^6 x t + 4 i c^2 t a + 4 i c^4 t - 4 c^4 t x + 8 c^2 t^2 a^2 - 2 c^2 x^2 + 2 a c^2 x^2 - 8 c^2 x t a + 2 i c^2 x - 8 c^2 t^2 a^2 + 8 c^2 x t a^2.$$

As it is known, there exist two holes near the peak in the first-order RW. These two holes are located at $P_1 = (\sqrt{-24 a + 24 + 3 c^2 (2 a - 2 - c^2)} c, \sqrt{-24 a + 24 + 3 c^2 (2 a - 2 - c^2)} c)$ and $P_2 = (\frac{18 a - 12}{\sqrt{-24 a + 24 + 3 c^2 (2 a - 2 - c^2)} c}, \frac{-3}{\sqrt{-24 a + 24 + 3 c^2 (2 a - 2 - c^2)} c})$. It is obvious that $P_1$ and $P_2$ are on the line $l_1 : x = -2(3a - 2)t$. On the background plane with height $|r_{2r}| = c$, the contour line is a hyperbola

$$(4 c^6 - 4 c^4 - 8 c^2 a^3 + 3 c^8 + 24 a c^4 - 16 a^2 c^4 + 8 c^2 a^2 + 4 a c^6) t^2 + (4 a c^4 + 4 c^6 + 8 c^4 + 8 c^2 a - 8 c^2 a^2) x t - 1 + (c^4 + 2 c^2 - 2 c^2 a) x^2 = 0,$$ (41)

which intersects with the line $l_1$ at two points $P_3 = (\frac{6 a - 4}{\sqrt{-8 a + 3 c^2 + 8 (2 a - 2 - c^2)} c}, 1)$ and $P_4 = (\frac{-6 a - 4}{\sqrt{-8 a + 3 c^2 + 8 (2 a - 2 - c^2)} c}, -1)$. We define the tangential direction of hyperbola to be at two points $P_3$ and $P_4$, which is the length-direction, as described by a line $l_2 : x = -(2a + 3/2c^3)t$. The density plot for $|r_{2r}|^2$ combined with the hyperbola and the length-direction is displayed in Fig. 8.

Since the contour line is not closed on the background in the length direction, we have to select a contour $|r_{2r}|^2 - 2c^2 = 0$ with height twice the background such that it is closed. The closed contour is useful to discuss the localization features of the first-order RW. It intersects with the length-direction at two points. We define the distance $d^L$ of these two points as the length of the first-order RW, and we determine the projection $d^W$ of $|P_1P_2|$ on the width direction, which is perpendicular to the
Fig. 8 – (Color online). The density plots of the first-order RW $|r_{2r}|^2$ with hyperbola ($|r_{2r}|^2 = c^2$) and length direction. (a) $a = -1, c = 1$, (b) $a = -2, c = 1$. The black solid line is the hyperbola, the yellow dashed line is the length direction.

length direction, to be the width of the first-order RW. Through a simple calculation, we obtain

$$d_L = \frac{dL_n}{dL_d},$$  \hspace{1cm} (42)

with

$$dL_n = 2\sqrt{(4 + 16a^2 + 24c^2a + 9c^4)(48 + 9c^4 + 48a^2 + 46c^2a - 96a + 46c^2a + 2\sqrt{M_1})},$$

$$dL_d = (8a - 8 - c^2)(2a - 2 - c^2)c^2,$$

$$M_1 = 1024 - 4096a + 22c^8 - 4992c^2a + 1024a^4 + 242c^6 - 1024a^4 - 952a^2c^4 - 242ac^6 + 976a^2c^4 - 1952ac^4 + 1024a^2c^2 + 1664c^2 + 6144a^2 + 976c^4,$$

and

$$dW = \frac{dW_n}{dW_d},$$ \hspace{1cm} (43)

while

$$dW_n = 6(8a - 3c^2 - 8),$$

$$dW_d = \sqrt{(-24a + 24 + 3c^2)(4 + 16a^2 + 24c^2 + 9c^4)}(2a - 2 - c^2)c.$$

The length $d_L$ and width $d_W$ are related to $a$ and $c$, and their profiles are plotted in Fig. 9.

If one fixes the parameter $c$, the length decreases with the increase of $a$ at first and then increases until $a = 1$. On the other hand, the width increases first, decreases, and then increases again until $a = 1$. For example, when $c = 1$, the length decreases with $a$ if $a \in (-\infty, -0.88)$ and increases with $a$ if $a \in (-0.88, 1)$. At the same time, the width increases with $a$ if $a \in (-\infty, -0.69)$ or $a \in (0.52, 1)$ and
decreases with \( a \) if \( a \in (-0.69, 0.52) \). Furthermore, when \( a \) tends to \(-\infty\), \( d_L \) tends to \( 2\sqrt{7} \) and \( d_W \) tends to \( 0 \), \( d_L \) reaches to the minimum \( 1.32 \) when \( a = -0.88 \) and gets to the maximum \( 62.42 \) when \( a \to 1 \), and \( d_W \) reaches to the maximum \( 1.70 \) when \( a = -0.69 \). In order to provide a visual support of above analysis on the trend with respect to \( a \) of two localization characteristics of the RW for the CLL-NLS equation, the two curves for \( d_L \) and \( d_W \) with fixed \( c = 1 \) are given in Fig. 10(a).

In order to consider the contribution of the SSE on the localization properties of the RW, we define the length and width of the RW as mentioned above for the NLS equation \( i{r_t} + r_{xx} + |r|^2 r = 0 \), which is trivially given by ignoring the SSE term in the CLL-NLS equation. After applying a scaling transformation, due to the different coefficient of the nonlinear term, the first-order RW of the NLS equation can be obtained from the results, reported in [16]. Then, the length and width of the first-order RW of the NLS equation are expressed by

\[
d_L^{\text{NLS}} = \sqrt{\frac{7(1+4a^2)}{2c^2}}, \quad d_W^{\text{NLS}} = \sqrt{\frac{3}{(1+4a^2)c}},
\]

and the length direction is described by a line \( l_{2\text{NLS}}: x = 2at \).

We set \( c = 1 \), then \( d_L^{\text{NLS}} \) and \( d_W^{\text{NLS}} \) reach the minimum \( \frac{\sqrt{7}}{2} \) and the maximum \( \sqrt{3} \) at \( a = 0 \), respectively. It implies that the maximum of width and the minimum of length of the RW for the NLS equation are roughly equal to the corresponding values of the RW for the CLL-NLS equation. The width of the RW for the NLS equation also tends to \( 0 \) when \( a \to \pm\infty \). However, the length tends to \( +\infty \), when \( a \to \pm\infty \). There is no oscillation interval for the width of the RW solution of the NLS equation. This is different from the analogous CLL-NLS equation. The profiles of \( d_L^{\text{NLS}} \) and \( d_W^{\text{NLS}} \) with \( c = 1 \) are given in Fig. 10(b).
Furthermore, we notice that $d_L < d_{L_{NLS}}$ if $a \in (-\infty, -0.47)$ and $d_L > d_{L_{NLS}}$ if $a \in (-0.47, 1)$, $d_W < d_{W_{NLS}}$, if $a \in (-\infty, -2.53)$ or $a \in (-0.33, 0.67)$, and $d_W > d_{W_{NLS}}$ if $a \in (-2.53, -0.33)$ or $a \in (0.67, 1)$ in the case of $c = 1$. These detailed comparisons on localization features of the first-order RWs are given in Table 1 and Fig. 11.

This analysis is visually verified by contours of $|r|^2$ for heights, being twice higher than the background in Fig. 12. Furthermore, since the length and the width of the first-order RW of CLL-NLS equation are smaller than the corresponding NLS
Fig. 12 – (Color online). The contours of first-order RWs at height \((2c^2)\) twice background with \(c = 1\) and different values of \(\alpha\). The red solid line is for the NLS equation and the blue dashed line is for the CLL-NLS equation. (a) \(\alpha = -3\), (b) \(\alpha = -1\), (c) \(\alpha = -0.4\), (d) \(\alpha = 0\), (e) \(\alpha = 0.7\).
Table 1

The localization features for the RW in the NLS and CLL-NLS equation. NLS<CLL-NLS means the localization of RW in the CLL-NLS is better than in the NLS, since the width and length of the CLL-NLS RW are smaller compared to those of the NLS. NLS>CLL-NLS is the opposite case.

<table>
<thead>
<tr>
<th>Values of ( a )</th>
<th>Length</th>
<th>Width</th>
<th>Localization</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a &lt; -2.53 )</td>
<td>( d^L_{NLS} &gt; d^L )</td>
<td>( d^W_{NLS} &gt; d^W )</td>
<td>NLS&lt;CLL-NLS</td>
</tr>
<tr>
<td>( -2.53 &lt; a &lt; -0.47 )</td>
<td>( d^L_{NLS} &gt; d^L )</td>
<td>( d^W_{NLS} &lt; d^W )</td>
<td>Indeterminate</td>
</tr>
<tr>
<td>( -0.47 &lt; a &lt; -0.33 )</td>
<td>( d^L_{NLS} &lt; d^L )</td>
<td>( d^W_{NLS} &lt; d^W )</td>
<td>NLS&gt;CLL-NLS</td>
</tr>
<tr>
<td>( 0.33 &lt; a &lt; 0.67 )</td>
<td>( d^L_{NLS} &lt; d^L )</td>
<td>( d^W_{NLS} &gt; d^W )</td>
<td>Indeterminate</td>
</tr>
<tr>
<td>( 0.67 &lt; a &lt; 1 )</td>
<td>( d^L_{NLS} &lt; d^L )</td>
<td>( d^W_{NLS} &lt; d^W )</td>
<td>NLS&gt;CLL-NLS</td>
</tr>
</tbody>
</table>

equation case when \( a < -2.53 \), CLL-NLS is therefore better than the corresponding NLS one. From an experimental point of view, having a smaller localization, we emphasize therefore a simpler set-up, since the propagating distance of first-order CLL-NLS RW is considerably smaller compared to the NLS case. The opposite case is valid for \( -0.47 < a < -0.33 \) and \( 0.67 < a < 1 \), where the CLL-NLS RW is worse. Unfortunately, we have not been able to compare the localization properties, when \( a \) belongs to one of the other two intervals, shown in the third and fifth columns of Table 1. This is due to the fact that the width and length of the corresponding localization is alternatively smaller or bigger for the CLL-NLS equation compared to the NLS equation. In other words, the SSE in the CLL-NLS equation gives a remarkable change of the localization properties of the first-order RW, although we are not able to claim, if the RW localization for this equation is rather improved or destroyed by this term at different points \((a,c)\), in the parameter space. This is the first impact of the SSE on RW solutions of the CLL-NLS equation. As a second impact we emphasize that the SSE induces a strong rotation of the direction length on the RW of the CLL-NLS equation by comparing the two lines \( l_2 \) and \( l_{2NLS} \). These two impacts are demonstrated intuitively by contours at a height \( 2c^2 \) of the modulus square for the first-order RWs of the CLL-NLS equation and of the NLS equation in Fig. 12.

6. CONCLUSIONS

We have shown explicitly the determinant representation \( T_n \) of the \( n \)-fold DT for the CLL-NLS equation. To our best knowledge, this kind of DT is unusual by comparing with known DT for soliton equations, because the \( n \)-fold DT depends on overall factors \( H_k \), which are involved with integrals of previous potentials \((q,r)\), \((q_1,r_1)\), \((q_2,r_2)\), \ldots, \((q_{n-1},r_{n-1})\). This induces difficulties to calculate the analytic expressions with increase of iterative terms. Especially, for the case of multi-Peregrine solutions. After a rigorous analysis, we managed to find in Lemma 3.2 that
$\frac{d^{[k]}H_{k+1}}{dt^{[k]}}$ is a constant. By using Lemma 3.2, there exists only one integral in $T_n$ which depends on the “seed” solution in the odd order DT, which can be easily calculated. Furthermore, we have provided the fundamental and higher-order solution of the CLL-NLS equation by $T_n$. These solutions may describe the accurate propagation of localized structures in nonlinear dispersive media, since dispersion, nonlinearity, and SSE have been taken into account. In particular, we provided exact analytical expressions for doubly-localized RW solutions. Furthermore, we discussed the influence of the SSE on the localization characteristics of CLL-NLS RWs using visualization contour method. This work may motivate similar studies for higher-order evolution equations of this kind, e.g. for higher-order generalized nonlinear Schrödinger-type equations. In particular, new experiments in several nonlinear dispersive media, for example in nonlinear optical fibers or in water wave flumes might be a consequence of these studies.

Acknowledgements. This work is supported by the NSF of China under Grant No. 11271210 and the K. C. Wong Magna Fund in Ningbo University. J.S. He acknowledges sincerely Prof. A. S. Fokas for arranging the visit to Cambridge University in 2012-2014 and for many useful discussions. A. Chabchoub acknowledges support from the Isaac Newton Institute for Mathematical Sciences.

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