A NUMERICAL APPROACH TO SOLVE LANE-EMDEN-TYPE EQUATIONS BY THE FRACTIONAL ORDER OF RATIONAL BERNOULLI FUNCTIONS

K. PARAND1,2, H. YOUSEFI2, M. DELKHOSH2

1Department of Cognitive Modelling, Institute for Cognitive and Brain Sciences, Shahid Beheshti University, G.C., Tehran, Iran
E-mail: k_parand@sbu.ac.ir
2Department of Computer Sciences, Shahid Beheshti University, G.C., Tehran, Iran

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Abstract. In this paper, a numerical method based on the hybrid of the quasi-linearization method (QLM) and the collocation method is suggested for solving well-known nonlinear Lane-Emden-type equations as singular initial value problems, which model many phenomena in mathematical physics and astrophysics. First, by using the QLM method, the nonlinear ordinary differential equation is converted into a sequence of linear differential equations, and then the linear equations by the fractional order of rational Bernoulli collocation (FRBC) method on the semi-infinite interval \([0, \infty)\) are solved. This method reduces the solution of these problems to the solution of a system of algebraic equations. Computational results of several problems are presented to demonstrate the viability and powerfulness of the method. Further, the fractional order of rational Bernoulli functions has also been used for the first time. The first zeros of standard Lane-Emden equation and the approximations of \(y(x)\) for Lane-Emden-type equations are given with unprecedented accuracy.

Key words: Lane-Emden-type equations; Fractional order of rational Bernoulli functions; Quasilinearization method; Collocation method; Isothermal gas spheres; Astrophysics; Nonlinear ODE.

1. INTRODUCTION

Many problems arising in astrophysics, fluid dynamics, quantum mechanics, and other fields are defined on infinite or semi-infinite intervals. In general, most of these problems are not solvable exactly and therefore should be investigated with the help of semi-analytical or numerical approximation methods. These methods include Adomian decomposition method [1, 2], Homotopy perturbation method [3, 4], Variational iteration method [5–7], Exp-function method [8–10], Finite difference approximation method [11, 12], Finite element method (FEM) [12, 13], Meshfree methods [14–18], Spectral methods [19–25], and so on.

Fractional calculus is used to model various different phenomena in nature and is an emerging field drawing much attention in both theoretical and applied disciplines. Fractional differential equations (FDEs) are a branch of differential equations that most of them do not have exact analytic solutions, hence researchers have
tried to find solutions of FDEs using approximate and numerical techniques. Several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives; see for example, Refs. [26]-[42]. In this paper, we have applied a numerical method based on fractional order of rational Bernoulli functions to obtain approximate numerical solutions of Lane-Emden-type differential equations.

In recent years, the studies of singular initial value problems in the ordinary differential equations (ODEs) have attracted the attention of many applied mathematicians and astrophysicists. One class of these equations are the Lane-Emden-type equations that are second order nonlinear ordinary differential equations on semi-infinite interval \([0, \infty)\) and have a singularity at the origin.

MATHMATICAL PRELIMINARIES ON LANE-EMDEN-TYPE EQUATIONS

The Lane-Emden equation describes a variety of phenomena in theoretical physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents. Let \(P(r)\) denote the total pressure at a distance \(r\) from the center of spherical gas cloud. The total pressure is due to the usual gas pressure and a contribution from radiation:

\[
P = \frac{1}{3} \varsigma T^4 + \frac{RT}{v},
\]

where \(\varsigma, T, R\) and \(v\) are, respectively, the radiation constant, the absolute temperature, the gas constant, and the specific volume [43]. Let \(M(r)\) be the mass within a sphere of radius \(r\) and \(G\) the constant of gravitation. The equilibrium equations for the configuration are

\[
\frac{dP}{dr} = -\frac{GM(r)}{r^2},
\]

\[
\frac{dM(r)}{dr} = 4\pi \rho r^2,
\]

where \(\rho\) is the density, at a distance \(r\) from the center of a spherical star. Eliminating \(M\) yields:

\[
\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 dP}{\rho} \right) = -4\pi G \rho.
\]

Pressure and density \(\rho = v^{-1}\) vary with \(r\) and \(P = K \rho^{1+\frac{m}{n}}\), where \(K\) and \(m\) are constants.
We can insert this relation into Eq. (1) for the hydrostatic equilibrium condition and
from this we rewrite the equation to:

\[
\left[ \frac{K(m+1)}{4\pi G} \lambda^{\frac{1}{m}-1} \right] \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dy}{dr} \right) = -y^m,
\]

where \( \lambda \) represents the central density of the star and \( y \) is a dimensionless quantity that are both related to \( \rho \) through the following relation:

\[
\rho = \lambda y^m.
\]

Let:

\[
r = ax,
\]

\[
a = \left[ \frac{K(m+1)}{4\pi G} \lambda^{\frac{1}{m}-1} \right]^{\frac{1}{2}}.
\]

Inserting these relations into our previous relations we obtain the Lane-Emden equation [44]:

\[
\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = -y^m,
\]

and simplifying the previous equation we have:

\[
y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad x > 0,
\]

with the boundary conditions:

\[
y(0) = 1, \quad y'(0) = 0.
\]

It has been claimed in the literature that only for \( m = 0, 1, \) and \( 5 \) the solutions of the Lane-Emden equation could be exact. For the other values of \( m \), the Lane-Emden equation is to be integrated numerically. Thus we decided to present a new and efficient technique to solve it numerically. This problem has been studied by many researchers and has been solved by different techniques; the main numerical methods are listed in Table 1. The Lane-Emden problems have a long history and only a small subset of numerical techniques is given in the bibliography table, see Table 1.

In the present paper, we intend to extend the application of Bernoulli polynomials to solve Lane-Emden-type equations. First, using the quasilinearization method (QLM) these equations convert to a sequence of linear ordinary differential equations to obtain the solution. Then, the equation will be solved on a semi-infinite domain by truncated fractional order of rational Bernoulli series defined as basis functions for the collocation method. The main characteristic of this technique is that it reduces these problems to those of solving a system of linear algebraic equations, thus greatly simplifying the problems. The main idea of this algorithm is to transform
rational basis functions into fractional calculus and then apply the QLM method for
the solution of nonlinear singular initial value problems.

The outline of this paper is as follows: in Sec. 2, the basic definitions and
properties of the Bernoulli functions are expressed. In Sec. 3, the quasilinearization
method and the numerical scheme by using fractional order of rational Bernoulli
collocation method are described briefly. In Sec. 4, we show that the numerical
results demonstrate the high accuracy and efficiency of the method. The conclusions
are given in the last Section.

2. PRELIMINARIES AND NOTATIONS

In this Section, some notations, definitions, and preliminary facts are presented
that will be used further in this work.

2.1. BERNOULLI POLYNOMIALS

Bernoulli polynomials play an important role in various expansions and ap-
proximation formulas, which are useful both in analytic theory of numbers and in
classical and numerical analysis. The Bernoulli polynomials and numbers have been
generalized by Norlund [84] to the Bernoulli polynomials and numbers of higher
order. Also, Vandiver in [85] generalized the Bernoulli numbers. Analogous poly-
nomials and sets of numbers have been defined from time to time, witness the Eu-
ler polynomials and numbers and the so-called Bernoulli polynomials of the second
kind. These polynomials can be defined by various methods depending on the appli-
cations [86]-[93].
The classical Bernoulli polynomials and numbers are defined, respectively, by the
exponential generating functions [94, 95]:

\[
\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x), \quad |t| < 2\pi, \quad x \in \mathbb{R};
\]  

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n, \quad |t| < 2\pi.
\]  

It is known that there is an explicit formula

\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n = 0, 1, \ldots
\]
Table 1

Lane-Emden type equations bibliography

<table>
<thead>
<tr>
<th>Year</th>
<th>Author/Authors</th>
<th>Comment</th>
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<tbody>
<tr>
<td>1977</td>
<td>Pascual [45]</td>
<td>Approximate the equation with Padé series</td>
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<td>1977</td>
<td>Sharma [46]</td>
<td>A non-elliptic solution of the equation for index ( m = 5 )</td>
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<td>Power-series solution of the equation</td>
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<td>( \delta )-perturbation expansion method</td>
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<td>Mandelweig &amp; Tabakin [53]</td>
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<td>Wazwaz [1]</td>
<td>Adomian decomposition method with the modified structure</td>
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<td>He [5]</td>
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<td>2004</td>
<td>Parand &amp; Razzagh [54]</td>
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<td>Classifying the equation and applying the standard Lagrangian</td>
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<td>Rational Legendre pseudospectral approach</td>
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<td>Yuzbasi [69]</td>
<td>A collocation method based on the Bessel polynomials</td>
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<td>Bhrawy &amp; Alofi [71]</td>
<td>Jacobi-Gauss collocation method</td>
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<td>2012</td>
<td>Pandey &amp; Kumar [72]</td>
<td>Using a Bernstein operational matrix</td>
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<td>2012</td>
<td>Mutsa &amp; Shateyi [73]</td>
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<td>2013</td>
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<td>Based on a class of Birkhoff-type interpolation method</td>
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<td>2013</td>
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<td>Mall &amp; Chakaraverty [79]</td>
<td>Chebyshev Neural Network based model</td>
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<td>Azarnavid et al. [82]</td>
<td>Picard-reproducing kernel Hilbert space method</td>
</tr>
<tr>
<td>2016</td>
<td>Parand &amp; Khaledi [83]</td>
<td>Rational Chebyshev of second kind collocation method</td>
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</tbody>
</table>

where \( B_K = B_K(0) \) are the Bernoulli numbers for each \( k = 0, 1, \ldots, n \). Thus, the first four such polynomials are

\[
B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.
\]
They satisfy to the following relations:

the binomial relation

\[ B_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \]

the differential equations

\[ B'_n(x) = n B_{n-1}(x), \]

the reciprocal relations

\[ B_n(1 - x) = (-1)^n B_n(x), \]

as well as the difference equations

\[ B_n(1 + x) - B_n(x) = nx^{n-1}. \]

As a consequence, the relationship between Bernoulli and Euler polynomials is obtained from

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k(0) E_{n-k}(x). \]

2.2. FRACTIONAL ORDER OF RATIONAL BERNOULLI FUNCTIONS

Most important equations in physics, chemistry, mechanics, and mathematical sciences occur in the semi-infinite or infinite intervals. To solve these equations using Spectral methods various strategies have been proposed [96]-[106]. According to [107], Bernoulli polynomials form a complete basis over the interval \([0, 1]\) and Calvert et al. [108] applied the rational Bernoulli functions \((R_n(x; L))\) to solve different problems on semi-infinite intervals. As discussed before, we can apply different Spectral basis that are used to solve problems in the semi-infinite interval. Now by using the algebraic mapping \(x \rightarrow x^\alpha; \alpha > 0\) on the rational Bernoulli functions, we define the fractional order of rational Bernoulli functions in the interval \([0, \infty)\).

The new basis functions that are denoted by \(FRB_n(x)\), are defined as follows:

\[ FRB_n(x; L) = R_n(x^\alpha; L) \]

and an analytical form of \(FRB_n(x; L)\) for \(n = 0, 1, \ldots\) is as follows:

\[ FRB_n(x; L) = \sum_{k=0}^{n} \binom{n}{k} B_k \left( \frac{x^\alpha}{x^\alpha + L} \right)^{n-k}, \]

where the constant parameter \(L\) sets the length scale of the mapping and \(B_k(y)\) is the Bernoulli polynomial of degree \(k\).
2.3. FUNCTION APPROXIMATION

We show how to approximate the unknown function by the rational functions. We determine $w(x) = \frac{\alpha L x^{\alpha-1}}{(x^\alpha + L)^2}$ as a non-negative, integrable, and real-valued weight function for the fractional order of rational Bernoulli functions over the semi-infinite interval. Now, we define $\mathcal{L} = \{x | 0 \leq x \leq \infty\}$ and

$$L^2_w(\Gamma) = \{\mu : \Gamma \rightarrow \mathbb{R} \mid \mu \text{ is measurable and } \|\mu\|_w < \infty\},$$  \hspace{1cm} (7)

where

$$\|\mu\|_w = \left(\int_0^\infty |\mu(x)|^2 w(x) \, dx\right)^{1/2}, \quad w(x) = \frac{\alpha L x^{\alpha-1}}{(x^\alpha + L)^2},$$  \hspace{1cm} (8)

is the norm induced by the inner product of the space $L^2(\Gamma)$,

$$\langle \nu, \mu \rangle_w = \int_0^\infty \nu(x) \mu(x) w(x) \, dx.$$  \hspace{1cm} (9)

Now, we assume $S_n = \{FRB_0(x), FRB_1(x), \cdots, FRB_n(x)\}$, is a finite dimensional subspace, therefore $S_n$ is a complete subspace of $L^2(\Gamma)$. The interpolating function of a smooth function $\nu$ on a semi-infinite interval is denoted by $\epsilon_n \nu$. It is an element of $S_n$ and

$$\epsilon_n \nu = \sum_{k=0}^n a_k FRB_k(x),$$

Here $\epsilon_n \nu$ is the best projection of $\nu$ upon $S_n$ with respect to the inner product Eq. (9) and the norm Eq. (8). Then, we have

$$\langle \xi_n \nu - \nu, FRB_i(x) \rangle = 0 \quad \forall FRB_i(x) \in S_n.$$  

3. DESCRIPTION OF THE QLM-FRBC TECHNIQUE

A numerical method based on hybrid of quasilinearization method (QLM) and fractional order of rational Bernoulli collocation (FRBC) method is employed for solving the Lane-Emden type equations.

3.1. THE QUASILINEARIZATION METHOD

Most algorithms for solving nonlinear equations or systems of equations are variations of Newton’s method. When we can apply the collocation method to solve a nonlinear differential equation, the result is a nonlinear system of algebraic equations. Nonlinearity is not a major complication for Spectral methods but for simplicity, we use the linearization algorithm called the quasilinearization method (QLM).
The method, is constructed to solve a wide variety of nonlinear ordinary differential equations or their systems arising in such different physics, engineering, and biology problems as orbit determination, detection of periodicities, radiative transfer, and cardiology [99, 109]. The aim of the quasilinearization method pioneered by Kalaba and Bellman [110] is to solve a nonlinear $n$th order ordinary differential equation in $N$ dimensions as a limit of a sequence of linear differential equations. For the general properties of the method, particularly its uniform and quadratic convergence, which often is monotonic, see [53]. The QLM is easily understandable since there is no difficult technique for obtaining the general solution of a nonlinear equation in terms of a finite set of particular solutions [53]. Many researchers have used in various papers this method, and it was extended, see Refs. [111]-[114].

For simplicity, we limit our discussion to nonlinear ordinary differential equation in one variable on the interval $[0, b]$, which could be infinite:

$$L^{(n)} y(x) = f(y(x), y^{(1)}(x), \ldots, y^{(n-1)}(x), x),$$

with $n$ boundary conditions

$$g_K = f(y(0), y^{(1)}(0), \ldots, y^{(n-1)}(0)) = 0, \quad k = 1, \ldots, l$$

and

$$g_K = f(y(b), y^{(1)}(b), \ldots, y^{(n-1)}(b)) = 0, \quad k = l + 1, \ldots, n.$$ (12)

Here $L^{(n)}$ is linear $n$th order ordinary differential operator and $n$th $f$ and $g_1, g_2, \ldots, g_n$ are nonlinear functions of $y(x)$ and their $n + 1$ derivatives $y^{(s)}(x), s = 1, \ldots, n$. The QLM prescription [112, 115] determines the $(r + 1)$th iterative approximation $y_{r+1}(x)$ to the solution of Eq. (2.1) as a solution of the linear differential equation

$$L^{(n)} y_{r+1}(x) = f(y_r(x), y_r^{(1)}(x), \ldots, y_r^{(n-1)}(x), x)$$

$$+ \sum_{s=0}^{n-1} (y_{r+1}^{(s)}(x) - y_r^{(s)}(x)) f_{y^{(s)}}(y_r(x), y_r^{(1)}(x), \ldots, y_r^{(n-1)}(x), x),$$

where $y_r^{(0)}(x) = y_r(x)$, with linearized two-point boundary conditions

$$\sum_{s=0}^{n-1} (y_{r+1}^{(s)}(0) - y_r^{(s)}(0)) g_{y^{(s)}}(y_r(0), y_r^{(1)}(0), \ldots, y_r^{(n-1)}(0), 0), \quad k = 1, \ldots, l$$

and

$$\sum_{s=0}^{n-1} (y_{r+1}^{(s)}(b) - y_r^{(s)}(b)) g_{y^{(s)}}(y_r(b), y_r^{(1)}(b), \ldots, y_r^{(n-1)}(b), b), \quad k = l + 1, \ldots, n.$$
A numerical approach to solve Lane-Emden-type equations

3.2. METHOD OF SOLUTION

In this Section, a reliable algorithm that consists of two distinct approaches to handle in a realistic and efficient way the Lane-Emden-type equations is introduced. The proposed approaches depend mainly on the QLM and we apply in transforming an ordinary nonlinear differential equation into a sequence of linear differential equations.

In general, the Lane-Emden-type equations are formulated as

\[ y''(x) + \frac{k}{x}y'(x) + f(x, y) = h(x), \quad k, x \geq 0, \]

with the initial conditions

\[ y(0) = A, \quad y'(0) = B, \]

where \( k, A, \) and \( B \) are real constants and \( f(x, y) \) and \( h(x) \) are some given functions. By using the QLM, the general solution of the Lane-Emden type equations determines the \((r + 1)\)th iterative approximation \( y_{r+1}(x) \) as a solution of the linear differential equation:

\[ y''_{r+1}(x) + \frac{k}{x}y'_{r+1}(x) + f(x, y_{r}(x)) - h(x) - (y_{r+1}(x) - y_{r}(x)) f_y(x, y(x)) + \frac{k}{x} (y'_{r+1}(x) - y'_{r}(x)) = 0, \]

with the boundary conditions:

\[ y_{r+1}(0) = A, \quad y'_{r+1}(0) = B. \]

In the other approaches, the fractional order of rational Bernoulli collocation (FRBC) method is employed for solving the linear differential equations at each iteration in Eq. (15) with the boundary conditions Eq. (16).

We suppose that \( y_0(x) \equiv A \). At the first step, the trial solution for the \((r + 1)\)-th iteration has been constructed as follows:

\[ u_{n,r}(x) = \sum_{i=0}^{n} a_{i,r} FRB_i(x; L), \]

where \( n \) is a positive integer and our goal is to find the coefficients of \( a_{i,r} \).

In the next step, the boundary conditions in Eq. (16) are satisfied.

\[ y_{n,r+1}(x) = A + Bx + x^2 u_{n,r}(x). \]
By substituting Eq. (18) in Eq. (15), we form residual function at each iteration QLM as follows:

\[
\text{Res}_r(x) = y_n^{(n+1)}(x) + \frac{k}{x} y_n^{(n)}(x) + f(x, y_n, r(x)) - h(x) \\
- (y_n^{(n+1)}(x) - y_n^{(n)}(x)) f_g(x, y(x)) + \frac{k}{x} (y_n^{(n+1)}(x) - y_n^{(n+1)}(x)).
\]

(19)

The main goal of the collocation method is to minimize the residual function for calculating unknown coefficients. By putting the arbitrary collocation points \( \{x_j\}_{j=0}^n \) in Eq. (19), a system of \( n + 1 \) linear equations at each iteration will be constructed that can be solved by the Newton method for unknown coefficients of \( a_i, r \). Stability and convergence analysis of Spectral methods have proved in Ref. [21].

The described method is summarized in the following algorithm:

**Algorithm of the method**

**Begin**

1. Input: \( n, L, \text{max\_iteration} \).
2. Selection of an initial guess (According to the initial conditions we suppose that \( y_0(x) \equiv A \)).
3. For \( r = 0, 1, 2, \cdots, \text{max\_iteration} \) do
   3.1 Calculation a linear combination of trial functions
   3.2 Satisfy the boundary conditions by calculating function
   3.3 Calculation of the residual function \( \text{Res}_r(x) \)
   3.4 Creating of the system of equations by putting the collocation points \( \{x_j\}_{j=0}^n \) in \( \text{Res}_r(x) \).
   3.5 Solving the system of linear equations.
4. End For

**End**

**4. PROBLEM REPLACEMENT AND NUMERICAL RESULTS**

In this Section, the approximate solutions of the Lane-Emden-type equations using the QLM-FREC method are presented. In what follows, several numerical problems are given to illustrate the performance and reliability of the present methods of solution. The results are tabulated and compared with accurate results of other methods.

In this study, the roots of the generalized fractional order of the Chebyshev functions of the first kind on the interval \([0, L]\)

\[
x_j = L \left( \frac{1 - \cos \left( \frac{(2j-1)\pi}{(n+1)2} \right)}{2} \right)^\frac{1}{n} \quad j = 1, 2, \cdots, n + 1.
\]
with $\alpha = 1/2$ have been used as collocation points [33] and all of the computations have been done by software Maple 2015.

4.1. PROBLEM 1 (THE STANDARD LANE-EMDEN EQUATION)

For $k = 2$, $f(x,y) = y^m(x)$, $h(x) = 0$, $A = 1$, and $B = 0$, Eq. (14) has been defined the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics.

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0 \quad x \geq 0, \quad (20)$$

subject to the boundary conditions

$$y(0) = 1, \quad y'(0) = 0,$$

where $m$ is a constant. In Eq. (20) for $m = 0$, exact solution is $y(x) = 1 - \frac{1}{3!}x^2$, for $m = 1$ is $y(x) = \frac{\sin(x)}{x}$ and for $m = 5$ is $y(x) = \frac{1}{\sqrt{1+x^2}}$.

In other cases, there is not any exact analytical solution.

By using Eqs. (15) and (16), we have:

$$y_{r+1}''(x) + \frac{2}{x}y_{r+1}'(x) + m y_{r+1}(x)y_r^{(m-1)}(x) - (m-1)y_r^{(m)}(x) = 0, \quad (21)$$

with the boundary conditions:

$$y_{r+1}(0) = 1, \quad y_{r+1}'(0) = 0, \quad r = 0, 1, 2, \ldots, \text{max.iteration}. \quad (22)$$

The QLM iteration requires an initialization or “initial guess” $y_0(x)$. We suppose that $y_0(x) \equiv 1$. According to the algorithm that was presented, the residual function is formed at each iteration QLM is as follows:

$$Res_r(x) = y_{n,r+1}''(x) + \frac{2}{x}y_{n,r+1}'(x)$$

$$+ m y_{n,r+1}(x)y_{n,r}^{(m-1)}(x) - (m-1)y_{n,r}^{(m)}(x). \quad (23)$$

The results show that the method works with proposed basis functions effectively to solve the standard Lane-Emden equation. Accurate numerical results for integer and fractional values of the nonlinear exponent $m$ to 21 and 15 decimal places are reported, respectively. Tables 2 and 3 show comparison of the first zeros of standard Lane-Emden equations, for the QLM-FRBC and numerical results given by Hordet [48], Boyd [70], Calvert et al. [108], Motsa & Shateyi [73], and Seidov [116], for $m = 1.5, 2, 2.5, 3, 3.5, 4$ with $N = 50, 10$th iteration, and $\alpha = 0.5$. Tables 4 and 5 show approximations of $y(x)$ for the standard Lane-Emden equations for $m = 1.5, 2, 2.5, 3, 3.5, 4$ with $N = 50, 10$-th iteration, and $\alpha = 0.5$, respectively.
obtained by Horedt [48] and the methods proposed in this paper. Figure 1 represents the graphs of the standard Lane-Emden type equations for $m = 1.5, 2, 2.5, 3, 3.5, 4$.

### Table 2
Comparison of the first zeros of standard Lane-Emden equations, for $m = 2, 3, 4$, and 10-th iteration.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m = 2$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L = 4.35$</td>
<td>$L = 5.35$</td>
<td>$L = 14.97$</td>
</tr>
<tr>
<td>10</td>
<td>4.3528741319803328205701</td>
<td>6.901388207013917025341</td>
<td>15.627621571181257914388</td>
</tr>
<tr>
<td>20</td>
<td>4.352874595949825746840</td>
<td>6.89684860945193403930</td>
<td>14.971471682958035981497</td>
</tr>
<tr>
<td>30</td>
<td>4.3528745959496124712137</td>
<td>6.896848619377009479001</td>
<td>14.9715463488950645289</td>
</tr>
<tr>
<td>40</td>
<td>4.3528745959467697676</td>
<td>6.8968486193766069021</td>
<td>14.971546348895068327</td>
</tr>
<tr>
<td>50</td>
<td>4.3528745959461267673</td>
<td>6.896848619376606375457</td>
<td>14.971546348895068327</td>
</tr>
</tbody>
</table>

Table 3
Comparison of the first zeros of standard Lane-Emden equations, for $m = 1.5, 2.5, 3.5$, and 10-th iteration.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m = 1.5$</th>
<th>$m = 2.5$</th>
<th>$m = 3.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L = 3.65$</td>
<td>$L = 5.35$</td>
<td>$L = 9.53$</td>
</tr>
<tr>
<td>10</td>
<td>3.6537538092714</td>
<td>5.35487272539126</td>
<td>9.48865890293924</td>
</tr>
<tr>
<td>20</td>
<td>3.653753754218</td>
<td>5.35527545804990</td>
<td>9.53804824098160</td>
</tr>
<tr>
<td>30</td>
<td>3.653753737242</td>
<td>5.35527545899074</td>
<td>9.53805344252336</td>
</tr>
<tr>
<td>40</td>
<td>3.653753736386</td>
<td>5.35527545900911</td>
<td>9.5380534424858</td>
</tr>
<tr>
<td>50</td>
<td>3.653753736116</td>
<td>5.35527545901048</td>
<td>9.5380534424851</td>
</tr>
</tbody>
</table>

Ref. [70] 3.653753736219 | 5.355275459111 | 9.538053442455 |
Ref. [73] 3.653753736219 | 5.355275459111 | 9.538053442455 |
Ref. [108] —— | 5.3552754600 | —— |
Ref. [48] 3.65375374 | 5.35527546 | 9.53805345

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Table 4

The obtained values of \( y(x) \) for standard Lane-Emden equations \( m = 2, 3 \) and \( 4 \) by the present method for the Problem 1 (with \( N = 50, \) 10\(^{th} \) iteration and \( \alpha = 0.5 \)).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( x )</th>
<th>Ref. [83]</th>
<th>Ref. [48]</th>
<th>QLM-FRBC</th>
<th>( \text{Res}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>0.99833499</td>
<td>0.9983350</td>
<td>0.998334998546148173818026</td>
<td>2.872e-17</td>
</tr>
<tr>
<td>0.50</td>
<td>0.95935270</td>
<td>0.9593527</td>
<td>0.9593527158033827008810774</td>
<td>8.404e-19</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.84654110</td>
<td>0.8465411</td>
<td>0.846541111408214967691159877</td>
<td>3.666e-20</td>
<td></td>
</tr>
<tr>
<td>3.00</td>
<td>0.21482408</td>
<td>0.2148241</td>
<td>0.2418340830535409167561426</td>
<td>3.902e-24</td>
<td></td>
</tr>
<tr>
<td>4.00</td>
<td>0.04884013</td>
<td>4.884015e-2</td>
<td>4.88401499759444444956266e-2</td>
<td>1.805e-25</td>
<td></td>
</tr>
<tr>
<td>4.30</td>
<td>0.00681093</td>
<td>6.810943e-3</td>
<td>6.81094327420583009083738e-3</td>
<td>2.083e-24</td>
<td></td>
</tr>
<tr>
<td>4.35</td>
<td>0.00036660</td>
<td>3.660302e-4</td>
<td>3.6603017936128575371457e-4</td>
<td>7.483e-24</td>
<td></td>
</tr>
</tbody>
</table>

| 3 | 0.10 | 0.99833582 | 0.9983358 | 0.998335829569169472981137 | 2.480e-13 |
| 0.50 | 0.95939096 | 0.9593909 | 0.95983969944851709269472 | 1.158e-14 |
| 1.00 | 0.85505756 | 0.8550575 | 0.85505758568826307861273 | 3.379e-15 |
| 5.00 | 0.11081983 | 0.1108198 | 0.1108198351396255985885184 | 2.393e-21 |
| 6.00 | 0.04373798 | 4.373798e-2 | 4.3737983888708140055490e-2 | 2.680e-20 |
| 6.80 | 0.00416779 | 4.1677893654534600127584e-3 | 4.1677893654534600127584e-3 | 8.228e-21 |
| 6.896 | 0.00013365 | 3.601115e-5 | 3.601114543670633177477e-5 | 1.276e-19 |

| 4 | 0.10 | 1.01585088 | 0.9.983367 | 0.9983366593955883440012 | 1.313e-10 |
| 0.20 | 0.99242505 | 0.9933862 | 0.9933862135232638564112 | 8.505e-11 |
| 0.50 | 0.96031089 | 0.9603109 | 0.9603109234225392518492 | 2.111e-12 |
| 1.00 | 0.86081381 | 8.608138 | 0.86081381220840617132238 | 2.003e-11 |
| 5.00 | 0.23592273 | 0.2359227 | 0.23592273104249156716087 | 2.761e-14 |
| 10.0 | 0.05967377 | 5.967274 | 5.967274158948779838014e-2 | 4.519e-14 |
| 14.0 | 0.00833293 | 8.330527e-4 | 8.330526695424819843717e-3 | 6.270e-15 |
| 14.9 | 0.00057667 | 5.764189e-4 | 5.7641886602135426830229e-4 | 6.792e-16 |

Fig. 1 – The obtained graphs of standard Lane-Emden equation solutions for several \( m \) for Problem 1.
Table 5

The obtained values of $y(x)$ for standard Lane-Emden $m = 1.5$, 2.5, and 3.5 by the present method for Problem 1 (with $N = 50$, 10th iteration, and $\alpha = 0.5$).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x$</th>
<th>Ref. [83]</th>
<th>Ref. [48]</th>
<th>QLM-FRBC</th>
<th>Res($x$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.10</td>
<td>0.99833458</td>
<td>0.9983346</td>
<td>0.9983391507397410</td>
<td>5.760e-2</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.95910386</td>
<td>0.9591039</td>
<td>0.9591044963969852</td>
<td>1.097e-3</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.84516976</td>
<td>0.8451698</td>
<td>0.8451697099901000</td>
<td>4.398e-5</td>
</tr>
<tr>
<td></td>
<td>3.00</td>
<td>0.15885061</td>
<td>0.1588576</td>
<td>0.1588576084002291</td>
<td>1.720e-7</td>
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<tr>
<td></td>
<td>3.60</td>
<td>0.01107792</td>
<td>1.1099e-2</td>
<td>1.10909945425596e-2</td>
<td>1.607e-7</td>
</tr>
<tr>
<td></td>
<td>3.65</td>
<td>0.00076275</td>
<td>7.6392e-4</td>
<td>7.63924148072572e-4</td>
<td>1.315e-6</td>
</tr>
<tr>
<td>2.5</td>
<td>0.10</td>
<td>0.99820016</td>
<td>0.9983354</td>
<td>0.9983354163296818</td>
<td>2.157e-5</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.95959775</td>
<td>0.9595978</td>
<td>0.9595977541616153</td>
<td>4.102e-7</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.85194420</td>
<td>0.8519442</td>
<td>0.851941989615764</td>
<td>8.833e-8</td>
</tr>
<tr>
<td></td>
<td>4.00</td>
<td>0.13768075</td>
<td>0.1376807</td>
<td>0.1376807330235731</td>
<td>2.419e-10</td>
</tr>
<tr>
<td></td>
<td>5.00</td>
<td>0.02901986</td>
<td>2.901919E-2</td>
<td>2.90191866495909e-2</td>
<td>1.510e-10</td>
</tr>
<tr>
<td></td>
<td>5.30</td>
<td>0.00425986</td>
<td>4.259544e-3</td>
<td>4.25954353378507e-3</td>
<td>3.688e-10</td>
</tr>
</tbody>
</table>

| 3.5 | 0.10  | ---         | 0.9983362   | 0.9983362416858018 | 2.433e-10 |
|     | 0.50  | ---         | 0.9600768   | 0.9600767558148295 | 9.161e-11 |
|     | 1.00  | ---         | 0.8580096   | 0.858009537895475  | 1.724e-11 |
|     | 5.00  | ---         | 0.1786843   | 0.1786842657489721 | 6.540e-14 |
|     | 9.00  | ---         | 1.180312e-2 | 1.18031215295922e-2 | 2.747e-14 |
|     | 9.50  | ---         | 7.45234e-4  | 7.47234075338948e-4 | 1.430e-13 |
|     | 9.53  | ---         | 1.207723e-4 | 1.2077244475770e-4 | 2.760e-13 |

4.2. PROBLEM 2 (THE ISOTHERMAL GAS SPHERES EQUATION)

According to Eq. (14), if $k = 2$, $f(x,y) = e^{y(x)}$, $h(x) = 0$, $A = 0$, and $B = 0$, the isothermal gas spheres equation has been defined as follows:

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0 \quad x \geq 0,$$  \hspace{1cm} (24)

subject to the boundary conditions

$$y(0) = 0, \quad y'(0) = 0,$$  \hspace{1cm} (25)

Davis [117] and Van Gorder [118] have discussed about Eq. (24) that can be used to view the isothermal gas spheres, where the temperature remains constant.

This equation has been solved by some researchers, for example Wazwaz [1] and Chowdhury and Hashim [19] by using ADM and HPM, respectively, Parand et al. [76, 103] by using Hermite collocation method, and Bessel orthogonal functions collocation method, respectively. A series solution has been investigated by Wazwaz [1], Liao [120] and Singh et al. [64] by using ADM, ADM, and MHAM, respectively:

$$y(x) \simeq -\frac{1}{6}x^2 + \frac{1}{5!}x^4 - \frac{8}{21.6!}x^6 + \frac{122}{81.8!}x^8 - \frac{61.67}{459.10!}x^{10} + \ldots.$$ \hspace{1cm} (26)
By using Eqs. (15) and (16), we have:

\[ y_{r+1}''(x) + \frac{2}{x} y_{r+1}'(x) + e^{y_r(x)}(y_{r+1}(x) - y_r(x) + 1) = 0 \]  \hspace{1cm} (27)

with the boundary conditions:

\[ y_{r+1}(0) = 0, \quad y_{r+1}'(0) = 0, \quad r = 0, 1, 2, \ldots \]  \hspace{1cm} (28)

We suppose that \( y_0(x) \equiv 0 \), i.e. the initial guess is satisfying the boundary conditions. According to the algorithm that was presented, the residual function is formed at each iteration QLM as follows:

\[ \text{Res}_r(x) = y_{n,r+1}''(x) + \frac{2}{x} y_{n,r+1}'(x) + e^{y_n(x)}(y_{n,r+1}(x) - y_n(x) + 1) \]  \hspace{1cm} (29)

In this equation, the QLM-FRBC method with \( n = 40 \), 10th iteration, and \( L = 2.5 \) are considered.

Table 6 shows the comparison of \( y(x) \) obtained by the proposed method in this paper and those obtained by Wazwaz [1]. The resulting graph of the isothermal gas spheres equation in comparison to the presented method and the Log graph of the residual error of approximate solution of the isothermal gas spheres equation are shown in Figs. 2 and 3.

### Table 6

<table>
<thead>
<tr>
<th>( x )</th>
<th>ADM</th>
<th>Ref. [103]</th>
<th>QLM-FRBC</th>
<th>( \text{Res}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.0016658339</td>
<td>-0.0016664188</td>
<td>-0.00166583386320605875</td>
<td>7.705e-21</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.006533671</td>
<td>-0.006539713</td>
<td>-0.006533671004231439</td>
<td>5.738e-22</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.0411539568</td>
<td>-0.0411545150</td>
<td>-0.0411539572927138819</td>
<td>6.117e-24</td>
</tr>
<tr>
<td>1.0</td>
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<td>-0.158821737</td>
<td>-0.1588276775243942141</td>
<td>5.155e-26</td>
</tr>
<tr>
<td>1.5</td>
<td>-0.3380131103</td>
<td>-0.3380198308</td>
<td>-0.338019424760796012</td>
<td>1.092e-27</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.5599626601</td>
<td>-0.5598233120</td>
<td>-0.559823004335377849</td>
<td>1.401e-30</td>
</tr>
<tr>
<td>2.5</td>
<td>-0.8100196713</td>
<td>-0.8063410846</td>
<td>-0.8063408705983917220</td>
<td>7.758e-30</td>
</tr>
</tbody>
</table>

### 4.3. PROBLEM 3

According to Eq. (14), if \( f(x, y) = \sinh(y(x)) \), \( A = 1 \) and \( B = 0 \) the equation has been defined as follows:

\[ y''(x) + \frac{2}{x} y'(x) + \sinh(y(x)) = 0 \quad x \geq 0, \]  \hspace{1cm} (30)

subject to the boundary conditions

\[ y(0) = 1, \quad y'(0) = 0, \]
A series solution that has been investigated by Wazwaz [1] by using Adomian Decomposition Method (ADM) is

\[
y(x) \simeq 1 - \frac{(e^2 + 1)x^2}{12e} + \frac{1}{480} \frac{(e^4 - 1)x^4}{e^2} - \frac{1}{30240} \frac{(2e^6 + 3e^2 - 3e^4 - 2)x^6}{e^3} \\
+ \frac{1}{26127360} \frac{(61e^8 + 104e^6 - 104e^2 - 61)x^8}{e^4} + \cdots.
\]
By using the QLM, the solution of the equation determines the \((r + 1)\)th iterative approximation \(y_{r+1}(x)\) as a solution of the linear differential equation:

\[
y''_{r+1}(x) + \frac{2}{x} y'_{r+1}(x) + \sinh(y_r(x)) + (y_{r+1}(x) - y_r(x)) \sinh(y_r(x)) = 0
\]

with the boundary conditions:

\[
y_{r+1}(0) = 1, \quad y'_{r+1}(0) = 0, \quad r = 0, 1, 2, \ldots
\]

We suppose that \(y_0(x) \equiv 1\), i.e. the initial guess is satisfying the boundary conditions.

According to the algorithm that was presented, the residual function is formed at each iteration QLM as follows:

\[
\text{Res}_r(x) = y''_{n,r+1}(x) + \frac{2}{x} y'_{n,r+1}(x) + \sinh(y_{n,r}(x)) + (y_{n,r+1}(x) - y_{n,r}(x)) \sinh(y_{n,r}(x))
\]

In this equation, the QLM-FRBC method with \(n = 40\), 10th iteration, and \(dL = 2\) is used.

Table 7 shows the comparison of \(y(x)\) obtained by the proposed method in this paper and the results obtained by Wazwaz [1]. The resulting graph of the isothermal gas spheres equation in comparison to the presented method and the Log graph of the residual error of approximate solution of the equation are shown in Figs. 4 and 5.

**Table 7**

The obtained values of \(y(x)\) for the Lane-Emden equation by the present method for Problem 3.

<table>
<thead>
<tr>
<th>(x)</th>
<th>ADM</th>
<th>Ref. [103]</th>
<th>QLM-FRBC</th>
<th>(\text{Res}(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9980428414</td>
<td>0.9980428414</td>
<td>0.9980428414444807299</td>
<td>3.095e-23</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9921894348</td>
<td>0.9921894348</td>
<td>0.992189434812197457</td>
<td>8.626e-25</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9519610925</td>
<td>0.9519610925</td>
<td>0.951961092744912401</td>
<td>6.488e-27</td>
</tr>
<tr>
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<td>0.8182516669</td>
<td>0.8182429282</td>
<td>0.818242928490522158</td>
<td>1.052e-32</td>
</tr>
<tr>
<td>1.5</td>
<td>0.6258916077</td>
<td>0.6254387632</td>
<td>0.625438763484943800</td>
<td>1.252e-34</td>
</tr>
<tr>
<td>2.0</td>
<td>0.4136691039</td>
<td>0.4066228877</td>
<td>0.406622887545649827</td>
<td>8.598e-38</td>
</tr>
</tbody>
</table>

4.4. PROBLEM 4

According to Eq. (14), if \(f(x, y) = \sin(y(x))\), \(A = 1\) and \(B = 0\) the equation has been defined as follows:

\[
y''(x) + \frac{2}{x} y'(x) + \sin(y(x)) = 0 \quad x \geq 0,
\]

subject to the boundary conditions

\[
y(0) = 1, \quad y'(0) = 0,
\]
The series solution investigated by Wazwaz [1] by using the Adomian Decomposition Method (ADM) is

\[ y(x) \simeq 1 - \frac{1}{6}k_1 x^2 + \frac{1}{120} k_1 k_2 x^4 + k_1 \left( \frac{1}{3024} k_1^2 - \frac{1}{5040} k_2^2 \right) x^6 + k_1 k_2 \left( \frac{113}{3265920} k_1^2 - \frac{1}{3628800} k_2^2 \right) x^8 + \frac{1781}{82892800} k_1^2 k_2^2 - \frac{1}{399168000} k_4 - \frac{19}{23950080} k_1^4 x^{10} + \cdots. \]
By using the QLM, the solution of the equation determines the \((r + 1)\)th iterative approximation \(y_{r+1}(x)\) as a solution of the linear differential equation:

\[
y''_{r+1}(x) + \frac{2}{x} y'_{r+1}(x) + \sin(y_r(x)) + \cos(y_r(x)) (y_{r+1}(x) - y_r(x)) = 0
\]  
(35)

with the boundary conditions:

\[
y_{r+1}(0) = 1, \quad y'_{r+1}(0) = 0, \quad r = 0, 1, 2, \ldots, \text{max}_\text{Iteration}.
\]  
(36)

We suppose that \(y_0(x) \equiv 1\), i.e. the initial guess is satisfying the boundary conditions. According to the algorithm that was presented, the residual function is formed at each iteration QLM as follows:

\[
\text{Res}_r(x) = y''_{n,r+1}(x) + \frac{2}{x} y'_{n,r+1}(x) + \sin(y_n,r(x)) + \cos(y_n,r(x)) (y_{n,r+1}(x) - y_n,r(x)) 
\]  
(37)

In this equation, the QLM-FRBC method with \(n = 40\), 10th iteration, and \(L = 2\) is considered.

![Graph of residual error for Problem 4.](image)

Table 8 shows the comparison of \(y(x)\) obtained by the proposed method in this paper and the results obtained by Wazwaz [1]. The resulting graph of the equation in comparison to the presented method and the Log graph of the residual error of approximate solution of the isothermal gas spheres equation are shown in Figs. 6 and 7.
5. CONCLUSIONS

The main purpose of this paper was to propose a powerful numerical method for solving Lane-Emden-type equations arising in astrophysics. The proposed method is based on converting an ordinary nonlinear differential equation into a sequence of linear differential equations through the quasilinearization method (QLM), which then are solved using the collocation method. The useful properties of QLM and fractional order of rational Bernoulli collocation (FRBC) method make it a computationally efficient method to approximate the solution on the semi-infinite interval \([0, \infty)\). The present method has several advantages, such as:

1. For the first time, to the best of our knowledge, the fractional order of rational Bernoulli functions (FRB) is introduced as a new basis for Spectral methods and this basis can be used to develop a framework or theory in Spectral methods. One
can see that some equations are solved more accurately using the fractional basis.

2. For the first time, to the best of our knowledge, the fractional basis was used for solving an ordinary differential equation (nonlinear singular Lane-Emden-type differential equation) and it provided insight into an important issue.

3. The convergence of the obtained results is illustrated.

4. Accurate numerical results for the first root are achieved for the integer and fractional values of the nonlinear exponent $m$ of the standard Lane-Emden equations to 21 and 15 decimal places respectively, after only ten iterations. Moreover a very good approximation solution of $y(x)$ for Lane-Emden type equations is given.

5. This paper tells a quite complete history of different methods used in the literature for solving Lane-Emden-type equations.

6. The method is valid for large interval calculations and it was also shown that the accuracy can be improved by increasing the number of collocation points.

7. The approximate solution obtained by the proposed method shows its superiority on the other method.

8. The presented results show that this approach can be used for effectively solving other initial and boundary value problems.

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REFERENCES