GAUSSIAN QUANTUM STEERING OF TWO BOSONIC MODES IN A THERMAL ENVIRONMENT

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\textit{Received November 27, 2016}

Abstract. Einstein-Podolsky-Rosen steerability of quantum states is a property that is different from entanglement and Bell nonlocality. We describe the time evolution of a measure that quantifies steerability for arbitrary bipartite Gaussian states in a system consisting of two bosonic modes embedded in a common thermal environment. We work in the framework of the theory of open quantum systems. If the initial state of the subsystem is taken of Gaussian form, then the evolution under completely positive quantum dynamical semigroups assures the preservation in time of the Gaussian form of the state. We study the Gaussian quantum steering in terms of the covariance matrix under the influence of the thermal environment and find out that the thermal noise and dissipation destroy the steerability between the two parts. We make a comparison with other quantum correlations for the same system, and show that, unlike Gaussian quantum discord, which is decreasing asymptotically in time, the Gaussian quantum steerability suffers a "sudden death" behaviour - it vanishes in a finite time, like quantum entanglement.

\textit{Key words:} Quantum steering, squeezed thermal states, open systems.

\textit{PACS:} 03.65.Yz, 03.67.Bg, 03.67.Mn.

1. INTRODUCTION

In the past decades the field of quantum information theory has developed mainly due to the great advances in implementation of information processing tasks using states of quantum nature allowing for impressing groundbreaking applications in quantum technologies, like sensing, imaging and metrology. The genuine properties of quantum states in comparison with classical ones consist in quantum correlations, such as entanglement, Bell nonlocality, quantum discord and quantum steering [1–3].

Quantum entanglement has attracted a major interest since the discovery of quantum teleportation, based on the property of nonseparability of quantum systems, in the sense that the subsystems cannot be completely described by only local states
Another much studied manifestation of quantum correlations is the Bell nonlocality, which cannot be explained by the local hidden variable model, and this lead to disproving the local causality as a true feature of the world. While these two types of quantum correlations coincide for pure states, the rigorous study in the case of mixed states shows that the violation of Bell inequalities takes place in a restricted set of states, which form the subset of the entangled states.

Steering is also a type of quantum nonlocality first identified in the Einstein - Podolsky - Rosen paper [5], which is distinct from both nonseparability and Bell nonlocality, allowing for new practical applications such as one-sided device-independent quantum key distribution [6]. To infer the steerability between two parties is equivalent with verifying the shared entanglement distribution by an untrusted party. That is to say, Alice has to convince Bob (who does not trust Alice) that the state they share is entangled, by performing local measurements and classical communications. Thus, the operational definition of steering is obtained if we weaken the condition in the definition of Bell nonlocality, namely, in bipartite setting the protocol of attesting entanglement allows just one of the parties to be untrusted [7].

In the present paper we work with continuous variable states [2, 8], in particular with Gaussian states, which are easily produced and controlled in laboratories. During their evolution, the systems interact with the environment, and the phenomena of decoherence and dissipation take place. In some previous papers [9–24], it was studied decoherence and dynamics of quantum entanglement and quantum discord in continuous variable open systems. In order to study the evolution of quantum systems in weak interaction with the environment we use the Kossakowski - Lindblad master equation, which is valid in the Markovian approximation of the theory of open systems based on completely positive dynamical semigroups. Within this model we consider the system of two uncoupled bosonic modes interacting with a common thermal reservoir and describe the behaviour of the Gaussian quantum steering, when the initial state of the system is a squeezed thermal state. We show that the suppression of the Gaussian steering takes place in a finite time, for all temperatures of the thermal reservoir and all values of the squeezing parameter, this behaviour being similar to the well-known phenomenon of entanglement sudden death. This kind of evolution of Gaussian steering and entanglement dynamics is in contrast with the dynamics of quantum discord, which decreases to zero asymptotically in time.

2. EVOLUTION OF A TWO-MODE BOSONIC SYSTEM INTERACTING WITH A THERMAL RESERVOIR

We consider a system of two bosonic modes with the vector \( \mathbf{R} = \{x, p_x, y, p_y\} \) of canonical variables of coordinates and momenta, in a Gaussian state defined by Gaussian characteristic function and quasi-probability distributions. The canonical
commutation relations can be expressed in the form $[R_i, R_j] = i\hbar \Omega_{ij}$, $i, j = 1, ..., 4$, with the symplectic matrix

$$\Omega = \bigoplus_{k=1}^{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1)$$

A Gaussian state is completely characterized by the first and second moments of canonical variables, however for the study of quantum entanglement and steering only the $4 \times 4$ covariance matrix $\sigma$ of second moments $\sigma_{ij} = \text{Tr}[(R_i R_j + R_j R_i) \rho] / 2$ is sufficient to know, where $\rho$ is the density operator. The bipartite covariance matrix can be written in the following block form:

$$\sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad (2)$$

where $A$ and $B$ are the $2 \times 2$ uni-modal covariance matrices and $C$ is the correlation matrix. The Robertson-Schrödinger uncertainty principle imposes the following restriction on the covariance matrix:

$$\sigma + \frac{i}{2} \Omega \geq 0. \quad (3)$$

In the Markovian approximation and for a weak interaction with the environment, the most general equation which describes an irreversible evolution of an open quantum system is the following Kossakowski-Lindblad master equation for the density operator $\rho$ [25–27]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_k (2B_k \rho(t) B_k^\dagger - \{\rho(t), B_k^\dagger B_k\} + ) , \quad (4)$$

where $H$ is the Hamiltonian of the system and $B_k$ are operators which describe the interaction with the environment. The Hamiltonian of the two uncoupled bosonic modes is

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{m}{2} (\omega_1^2 x^2 + \omega_2^2 y^2), \quad (5)$$

where $m$ is the mass, $\omega_1$, $\omega_2$ are the frequencies of the two modes, and the operators $B_k$ are taken polynomials of the first degree in coordinates and momenta, in order to assure the preservation in time of the Gaussianity of the considered states [28].

By using the master equation (4), we obtain the following time evolution of the covariance matrix [28]:

$$\sigma(t) = M(t)(\sigma(0) - \sigma(\infty))M^T(t) + \sigma(\infty), \quad (6)$$
with \( M(t) = \exp(Yt) \), where \([28]\)

\[
Y = \begin{pmatrix}
-\lambda & \frac{1}{m} & 0 & 0 \\
-\lambda m & 0 & 0 & 0 \\
0 & 0 & -\lambda & \frac{1}{m} \\
0 & 0 & -m \lambda & -\lambda \\
\end{pmatrix}
\]

(7)

and \( \lambda \) is the dissipation coefficient. \( \sigma(\infty) \) is the covariance matrix corresponding to the asymptotic Gibbs state of the two bosonic modes in thermal equilibrium at temperature \( T \). Its diagonal elements are given by (we take \( \hbar = 1 \) \([27]\)):

\[
m\omega_1 \sigma_{xx}(\infty) = \frac{\sigma_{pp_x}(\infty)}{m\omega_1} = \frac{1}{2} \coth \frac{\omega_1}{2kT},
\]

\[
m\omega_2 \sigma_{yy}(\infty) = \frac{\sigma_{pp_y}(\infty)}{m\omega_2} = \frac{1}{2} \coth \frac{\omega_2}{2kT},
\]

(8)

where \( k \) is the Boltzmann constant, while all the non-diagonal elements are 0.

3. Dynamics of Gaussian Quantum Steering

In the case of bipartite Gaussian states one restricts the local measurements performed on Alice’s side to the set of Gaussian measurements. Such measurements can be described by a positive operator with covariance matrix \( \Gamma^{RA} \), satisfying

\[
\Gamma^{RA} + \frac{i}{2} \Omega_A \geq 0, \quad \Omega_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

(9)

and are generally implemented via symplectic transformations and homodyne detection \([29]\). When measuring the part of Alice’s state we obtain the outcome \( a \) and the conditional state of Bob \( \rho_B^a \) is described by the covariance matrix \( B - C(A + \Gamma^{RA})^{-1}C^T \).

It has been shown in Ref. \([7, 30]\) that the steerability \( A \rightarrow B \) is present if and only if the following relation does not hold:

\[
\sigma + \frac{i}{2} 0_A \oplus \Omega_B \geq 0,
\]

(10)

or equivalently

\[
A > 0, \quad \text{and} \quad \Delta^B_{\sigma} + \frac{i}{2} \Omega_B \geq 0,
\]

(11)

where \( \Delta^B_{\sigma} = B - CA^{-1}C^T \) is the Schur complement of \( A \) in covariance matrix \( \sigma \). Since the covariance matrix of a reduced state is always positive, the only condition for certifying the absence of steering is to check whether \( \Delta^B_{\sigma} \) is a genuine covariance matrix. For this, one has to compute the Williamson form of \( \Delta^B_{\sigma} \) using appropriate symplectic transformation such that \( S^T \Delta^B_{\sigma} S = \text{Diag}(\nu^B, \nu^B) \), and verify the uncertainty relation for its symplectic eigenvalue \( \nu^B \geq 1/2 \). Accordingly, a measure can
be proposed of how much a state with covariance matrix $\sigma$ is $A \rightarrow B$ steerable with Gaussian measurements, by quantifying the amount by which the condition (10) is violated, as follows [31]:

$$G^{A\rightarrow B}(\sigma) := \max\{0, -\ln 2\nu^B\}. \quad (12)$$

Clearly, also the $B \rightarrow A$ steerability can be quantified in an equable manner by computing the symplectic eigenvalues of the Schur complement of $B$ in the covariance matrix $\sigma$. The quantity in Eq. (12) vanishes if and only if the state $\sigma$ is not steerable by Gaussian measurements and is invariant under local symplectic transformations.

Note the similarity of this quantity with the negativity which measures the degree of entanglement of a state. In that situation the symplectic eigenvalues of the partially transposed covariance matrix $\sigma$ are relevant, which quantify the violation of the positive partial transpose (PPT) criterion [32, 33]. However, steering is fundamentally different from entanglement, being in general asymmetric with respect to the interchange between steerable parties.

The general quantity proposed in Ref. [31, 34], while easily computable for an arbitrary number of modes, has a particularly simple form when the steered party has one mode:

$$G^{A\rightarrow B}(\sigma) = \max\{0, \frac{1}{2} \ln \frac{\det A}{4 \det \sigma}\} = \max\{0, S(A) - S(\sigma)\}, \quad (13)$$

where $S$ is the Renyi-2 entropy, which for Gaussian states reads $S = \frac{1}{2} \ln(16 \det \sigma)$.

Now we study the dynamics of the Gaussian quantum steering for the considered system, by taking an initial squeezed thermal state, with the covariance matrix of the form:

$$\sigma(0) = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{pmatrix}, \quad (14)$$

with

$$a = n_1 \cosh^2 r + n_2 \sinh^2 r + \frac{1}{2} \cosh 2r,$$

$$b = n_1 \sinh^2 r + n_2 \cosh^2 r + \frac{1}{2} \cosh 2r,$$

$$c = \frac{1}{2}(n_1 + n_2 + 1) \sinh 2r,$$

where $n_1, n_2$ are the mean photon numbers of the two modes and $r$ is the squeezing parameter. The state is entangled if $r > r_s$, where $\cosh^2 r_s = (n_1 + 1)(n_2 + 1)/(n_1 + n_2 + 1)$ [35]. If $\omega_1 = \omega_2$ and $n_1 = n_2$, then we have a symmetry between the two modes and therefore $A \rightarrow B$ and $B \rightarrow A$ steerability are equivalent in the sense of above mentioned measure.

In Figs.1 it is illustrated the evolution of the Gaussian $A \rightarrow B$ quantum steering $G$ for an initial squeezed vacuum state ($n_1 = n_2 = 0$) and a squeezed thermal state, respectively, for a non-zero equilibrium temperature of the thermal bath, as a function
of time $t$ and squeezing parameter $r$. While the initial squeezed vacuum state is always steerable, the initial squeezed thermal state is not steerable for all $r > 0$, like in the case of the entanglement. We notice that the Gaussian steering is increasing with the squeezing parameter $r$ and it is suppressed in a finite time.

In Fig. 2 it is shown the dynamics of the Gaussian $A \rightarrow B$ steering for an initial squeezed vacuum state as a function of time $t$ and temperature $C$ of the thermal reservoir. We observe that the Gaussian steering decreases with increasing the temperature. Thermal noise and dissipation destroy in a finite time the steerability between the two parts.
Gaussian quantum steering of two bosonic modes in a thermal environment

The asymmetry in Gaussian steering is captured in Figs. 3 and this feature is the result of the asymmetry between the modes in the initial squeezed thermal state.

Compared to the Gaussian quantum discord, which is decreasing asymptotically in time, the Gaussian quantum steering suffers a sudden death behaviour like quantum entanglement. This can be seen in Fig. 4 where it is represented the dependence on time and squeezing parameter of the logarithmic negativity (red plot) and Gaussian $A \rightarrow B$ quantum steering (green plot) for an initial squeezed vacuum state.
4. SUMMARY AND CONCLUSIONS

We investigated the dynamics of the Gaussian quantum steering of two uncoupled bosonic modes in interaction with a common thermal bath in the Markovian approximation. For initial squeezed vacuum states and squeezed thermal states we have shown that a sudden suppression of quantum steering takes place, for all temperatures of the thermal reservoir and all values of the squeezing parameter, a phenomenon similar to the entanglement sudden death. This behaviour is different from the time evolution of the Gaussian quantum discord, which tends to zero asymptotically in time [36]. An initial squeezed thermal state is not steerable for all values of the squeezing parameter \( r \), while a squeezed vacuum state is steerable for all values of \( r \). The Gaussian steering is increasing with the squeezing parameter and is decreasing with increasing the temperature of the thermal bath. We mention that the suppression of steering in a finite time takes place for all temperatures, including zero temperature. The asymmetric Gaussian steering between the two modes has been illustrated in the case of a squeezed thermal state with different photon numbers of the modes, in particular it was shown that for definite values of the squeezing parameter, the system exhibits only an one-way steering.

REFERENCES