LIE SYMMETRY, FULL SYMMETRY GROUP, AND EXACT SOLUTIONS TO THE (2+1)-DIMENSIONAL DISSIPATIVE AKNS EQUATION

ZHENG-YI MA[1,2,*], HUI-LIN WU[1], QUAN-YONG ZHU[1]

[1]Department of Mathematics,
Lishui University, Lishui 323000, P.R. China
[2]Department of Mathematics,
Zhejiang Sci-Tech University, Hangzhou 310018, P.R. China
E-mail[*]: ma-zhengyi@163.com (corresponding author)

Received January 13, 2017

Abstract. In this paper, a detailed Lie symmetry analysis of the (2+1)-dimensional dissipative AKNS equation is performed. The general finite transformation group is given *via* a simple direct method, which is in fact equivalent to Lie point symmetry group. The similarity reductions are considered from the general Lie symmetry and some exact solutions of the (2+1)-dimensional dissipative AKNS equation are obtained.

Key words: (2+1)-dimensional dissipative AKNS equation, Lie symmetry, full symmetry group, similarity reduction, invariant solutions.

PACS: 02.30Jr, 02.20.Sv, 05.45.Pq, 02.20.Tw.

1. INTRODUCTION

The study of the symmetries of partial differential equations (PDEs) is a central issue in nonlinear mathematical physics. During the nineteenth century, Sophus Lie [1, 2] proposed a standard technique, the Lie symmetry method, to find the classical Lie point symmetry groups and algebras for differential systems. A lot of universal applications of Lie symmetry groups in differential equation were discussed in the literature [1–19], such as the reduction of order of ordinary differential equations and dimension of PDEs, the construction of invariant solutions, the mapping solutions to other solutions and the detection of linearizing transformations, the derivation of conservation laws and so on. Several generalizations of the classical Lie group method for symmetry reduction were developed. Since all solutions of the classical determining equations need to satisfy the nonclassical determining equations and the solution set may be larger in the nonclassical case, Bluman and Cole [3] presented the method of conditional symmetries. These methods were further extended by Olver and Rosenau [4] to include weak symmetries and, even more generally, side conditions or differential constraints. Motivated by the fact that the known symmetry reductions of the Boussinesq equation are not obtained using the classical Lie group method, Clarkson and Kruskal [5] (CK) introduced a simple direct method to find all the possible similarity reductions of a nonlinear system without using any group theory. Most recently, inspired by the CK direct method, Lou and Ma [6] modified the CK direct method to find the generalized Lie and non-Lie symmetry groups for some well-known nonlinear evolution equations. Such kind of full symmetry group can obtain not only the Lie point symmetry group but also the non-Lie symmetry group for the given nonlinear system.

In this paper, we consider the following (2+1)-dimensional dissipative AKNS equation [20–23]

$$4u_{xt} + u_{xxxy} + 8u_{xy}u_x + 4u_yu_{xx} + ru_{xx} = 0, \tag{1}$$

where r is a constant. Especially, the coefficient $r \neq 0$ shows that the system has dissipative effects. When $r = 0$, Eq. (1) reduces to the (2+1)-dimensional AKNS equation [20]. Based on the Bell polynomial theory, Liu *et al.* studied the complete integrability of Eq. (1) such as the Bäcklund transformation, the Lax pair, and the infinite conservation laws [21]. Some exact traveling wave solutions were derived for the (2+1)-dimensional AKNS equation by using the extended homoclinic test approach [24] and the improved tanh method [25]. Wazwaz [22] investigated the soliton solutions by using the simplified form of the bilinear method. By means of the multidimensional Riemann theta function, the periodic wave solutions were explicitly constructed [23].

The outline of this paper is as follows. In Sec. 2, the Lie point symmetry of the (2+1)-dimensional dissipative AKNS equation (1) and the commutator relations of the generators associated with the symmetry are provided by means of the classical method. In Sec. 3, with the help of a simple direct method, the finite transformation groups of Eq. (1) is given, which is equivalent to Lie point symmetry groups generated through the standard approach. Section 4 is devoted to performing the (1+1)-dimensional similar reductions from the general Lie point symmetry. In Sec. 5, exact solutions of the dissipative AKNS equation (1) are constructed from the reduced equations. Some conclusions and a brief discussion of the results are given in the last section.

2. LIE POINT SYMMETRY VIA THE CLASSICAL SYMMETRY METHOD

In this Section we present the Lie symmetry analysis for the (2+1)-dimensional dissipative AKNS equation (1). In order to apply the classical method to Eq. (1) (i.e., Δ=r.h.s. of (1) $= 0$), we consider a one-parameter Lie group of infinitesimal transformation in (x,y,t,u) given by

$$\begin{aligned}
&x \to x+\epsilon X(x,y,t,u),\\
&y \to y+\epsilon Y(x,y,t,u),\\
&t \to t+\epsilon T(x,y,t,u),\\
&u \to u+\epsilon U(x,y,t,u),
\end{aligned} \tag{2}$$

where ϵ is the group parameter. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\underline{\mathbf{V}} = X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+T\frac{\partial}{\partial t}+U\frac{\partial}{\partial u}. \tag{3}$$

Then the invariance of Eq. (1) under the transformation (2) provides that

$$\mathrm{pr}^{(4)}\mathcal{V}(\Delta)\mid_{\Delta=0}=0, \tag{4}$$

where $\mathrm{pr}^{(4)}\mathcal{V}$ is the fourth prolongation of the vector field (3). This yields an overdetermined, linear system of differential equations for the infinitesimals X,Y,T, and U. With the help of the symbolic computation system MAPLE, by solving these determining equations, we find that the infinitesimals can be written as follows:

$$\begin{aligned}
&X=\frac{1}{4}(c_6t+2c_4-2c_5)x+g,\\
&Y=\frac{1}{2}(c_6y+2c_3)t+c_5y+c_2,\\
&T=\frac{1}{2}c_6^2t^2+c_4t+c_1,\\
&U=\frac{1}{4}(c_6y+2c_3)x+[\dot{g}-\frac{r}{16}(3c_6t+2c_4+2c_5)]y+f-\frac{1}{4}(c_6t+2c_4+2c_5)u,
\end{aligned} \tag{5}$$

where f and g are arbitrary functions of t, c_i $(i=1,2,\ldots,6)$ are arbitrary constants, and the dot over the function means its derivative with respect to time t. The presence of the arbitrary function and constants leads to an infinite-dimensional Lie algebra of symmetries. A general element of this algebra is written as

$$\underline{\mathbf{V}} = c_1\underline{\mathbf{V}}_1+c_2\underline{\mathbf{V}}_2+c_3\underline{\mathbf{V}}_3+c_4\underline{\mathbf{V}}_4+c_5\underline{\mathbf{V}}_5+c_6\underline{\mathbf{V}}_6+\underline{\mathbf{V}}_7(f)+\underline{\mathbf{V}}_8(g), \tag{6}$$

where

$$
\begin{aligned}
&\underline{\mathbf{V}}_1=\frac{\partial}{\partial t},\ \ \underline{\mathbf{V}}_2=\frac{\partial}{\partial y},\\
&\underline{\mathbf{V}}_3=t\frac{\partial}{\partial y}+\frac{x}{2}\frac{\partial}{\partial u},\\
&\underline{\mathbf{V}}_4=\frac{x}{2}\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{1}{8}(ry+4u)\frac{\partial}{\partial u},\\
&\underline{\mathbf{V}}_5=-\frac{x}{2}\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-\frac{1}{8}(ry-4u)\frac{\partial}{\partial u},\\
&\underline{\mathbf{V}}_6=\frac{xt}{4}\frac{\partial}{\partial x}+\frac{yt}{2}\frac{\partial}{\partial y}+\frac{t^2}{2}\frac{\partial}{\partial t}-\frac{1}{16}(3ryt-4xy+4tu)\frac{\partial}{\partial u},\\
&\underline{\mathbf{V}}_7(f)=f\frac{\partial}{\partial u},\\
&\underline{\mathbf{V}}_8(g)=g\frac{\partial}{\partial x}+\dot{g}y\frac{\partial}{\partial u},
\end{aligned}
\tag{7}
$$

construct a basis for the vector space. The associated Lie algebra among these vector fields are given by Table 1, where the entry in the j-th row and the k-th column represents the commutator $[\underline{\mathbf{V}}_j,\underline{\mathbf{V}}_k]$.

Table 1

The commutation relation between infinitesimal generators of point symmetries.

$[\underline{\mathbf{V}}_j,\underline{\mathbf{V}}_k]$	$\underline{\mathbf{V}}_1$	$\underline{\mathbf{V}}_2$	$\underline{\mathbf{V}}_3$	$\underline{\mathbf{V}}_4$	$\underline{\mathbf{V}}_5$	$\underline{\mathbf{V}}_6$
$\underline{\mathbf{V}}_1$	0	0	$\underline{\mathbf{V}}_2$	$\underline{\mathbf{V}}_1$	0	$\underline{\mathbf{V}}_4+\frac{1}{2}\underline{\mathbf{V}}_5$
$\underline{\mathbf{V}}_2$		0	0	$-\frac{1}{8}\underline{\mathbf{V}}_7(r)$	$\underline{\mathbf{V}}_2-\frac{1}{8}\underline{\mathbf{V}}_7(r)$	$\underline{\mathbf{V}}_3-\frac{3}{16}\underline{\mathbf{V}}_7(rt)$
$\underline{\mathbf{V}}_3$			0	$-\underline{\mathbf{V}}_3-\frac{1}{8}\underline{\mathbf{V}}_7(rt)$	$\underline{\mathbf{V}}_3-\frac{1}{8}\underline{\mathbf{V}}_7(rt)$	$-\frac{3}{16}\underline{\mathbf{V}}_7(rt^2)$
$\underline{\mathbf{V}}_4$				0	0	$\underline{\mathbf{V}}_6$
$\underline{\mathbf{V}}_5$					0	0
$\underline{\mathbf{V}}_6$						0

$[\underline{\mathbf{V}}_j,\underline{\mathbf{V}}_k]$	$\underline{\mathbf{V}}_7(f)$	$\underline{\mathbf{V}}_8(g)$
$\underline{\mathbf{V}}_1$	$\underline{\mathbf{V}}_7(\dot{f})$	$\underline{\mathbf{V}}_8(\dot{g})$
$\underline{\mathbf{V}}_2$	0	$\underline{\mathbf{V}}_7(\dot{g})$
$\underline{\mathbf{V}}_3$	0	$\underline{\mathbf{V}}_7(t\dot{g}-\frac{1}{2}g)$
$\underline{\mathbf{V}}_4$	$\underline{\mathbf{V}}_8(t\dot{g}-\frac{1}{2}g)$	
$\underline{\mathbf{V}}_5$	$-\frac{1}{2}\underline{\mathbf{V}}_7$	$\frac{1}{2}\underline{\mathbf{V}}_8$
$\underline{\mathbf{V}}_6$	$\underline{\mathbf{V}}_7(\frac{1}{2}t^2\dot{f}+\frac{1}{4}tf)$	$\underline{\mathbf{V}}_8(\frac{1}{2}t^2\dot{g}-\frac{1}{4}tg)$
$\underline{\mathbf{V}}_7(f)$	0	0
$\underline{\mathbf{V}}_8(g)$		0

We now consider a point transformation

$$G : (x,y,t,u) \mapsto (\chi,\zeta,\gamma,P). \tag{8}$$

From the transformation (2), we have the corresponding one-parameter group of symmetries of the (2+1)-dimensional dissipative AKNS equation

$$\begin{aligned}
&G_1 : (x,y,t,u) \mapsto (x,y,t+\epsilon,u),\\
&G_2 : (x,y,t,u) \mapsto (x,y+\epsilon,t,u),\\
&G_3 : (x,y,t,u) \mapsto (x,y+\epsilon t,t,\frac{1}{2}\epsilon x+u),\\
&G_4 : (x,y,t,u) \mapsto (xe^{\frac{\epsilon}{2}},y,te^{\epsilon},ue^{-\frac{\epsilon}{2}}+\frac{r}{4}(e^{-\frac{\epsilon}{2}}-1)y),\\
&G_5 : (x,y,t,u) \mapsto (xe^{-\frac{\epsilon}{2}},ye^{\epsilon},t,ue^{\frac{\epsilon}{2}}-\frac{r}{4}(e^{\epsilon}-e^{\frac{\epsilon}{2}})y),\\
&G_6 : (x,y,t,u) \mapsto \Big(\frac{\mathrm{i}}{2}\sqrt{2\epsilon t-4}x,-\frac{2y}{\epsilon t-2},-\frac{2t}{\epsilon t-2},\frac{\mathrm{i}}{2}\sqrt{2\epsilon t-4}u+\frac{\mathrm{i}}{4t}\sqrt{2\epsilon t-4}\\
&\qquad\qquad \ln(\frac{2}{2-\epsilon t})xy-\frac{r\mathrm{i}}{8\epsilon t-16}[\sqrt{2\epsilon t-4}(2-\epsilon t)+4\mathrm{i}]y\Big),\\
&G_7 : (x,y,t,u) \mapsto (x,y,t,u+\epsilon f),\\
&G_8 : (x,y,t,u) \mapsto (x+\epsilon g,y,t,u+\epsilon \dot{g}y).
\end{aligned} \tag{9}$$

So the entire symmetry group is obtained by composing one-dimensional subgroups G_i $(i=1,2,\ldots,8)$. When G is an element of this group, if $u(x,y,t)$ is a solution of the dissipative AKNS equation, then $P(\chi,\zeta,\gamma)$ is also a solution of the dissipative AKNS equation.

3. SYMMETRY GROUP *VIA* A SIMPLE DIRECT METHOD

According to the symmetry group direct method [6], we can take the simplified symmetry transformation ansatz as

$$u = \alpha + \beta U(\xi,\eta,\tau), \tag{10}$$

where $\alpha,\beta(i=1,2,3),\xi,\eta$ and τ are functions of $\{x,y,t\}$ to be determined. We require that $U(\xi,\eta,\tau)$ also satisfies the (2+1)-dimensional dissipative AKNS equation but with different independent variables $\{\xi,\eta,\tau\}$

$$4U_{\xi\tau}+U_{\xi\xi\xi\eta}+8U_{\xi\eta}U_{\xi}+4U_{\eta}U_{\xi\xi}+rU_{\xi\xi}=0. \tag{11}$$

Substituting (10) into Eqs. (1), eliminating $U_{\xi\tau}$ and their higher-order derivatives by Eqs. (11), then setting the coefficients of the polynomials of U and its derivatives to be zero, we obtain a huge numbers of nonlinear PDEs with respect to differentiable functions: $\{\alpha,\beta,\xi,\eta,\tau\}$. After thorough analysis and quite tedious calculations, the general results read:

$$\begin{aligned}
\alpha &= \sigma\left[-\frac{2s_5(s_1t+s_2)}{s_1}\right]^{\frac{1}{2}}\left[\frac{r(s_2s_3-s_1s_4)}{4(s_1t+s_2)^2}-\frac{d}{dt}\xi_0(t)-\frac{1}{4}\sigma r\right]y\\
&+\left[\frac{s_1(y+2s_7)}{2(s_1t+s_2)}\right]x+\alpha_0(t),\\
\beta &= \sigma\left[-\frac{s_1}{4s_5(s_1t+s_2)}\right]^{\frac{1}{2}},\\
\xi &= \sigma\left[-\frac{s_1}{4s_5(s_1t+s_2)}\right]^{\frac{1}{2}}x+\xi_0(t),\\
\eta &= \frac{2s_5(s_1s_4-s_2s_3)}{s_1(s_1t+s_2)}(y+2s_7)+s_6,\\
\tau &= \frac{s_3t+s_4}{s_1t+s_2},\ \ \sigma^2=1,
\end{aligned} \tag{12}$$

where $\xi_0 \equiv \xi_0(t)$ and $\alpha_0 \equiv \alpha_0(t)$ are arbitrary functions of t and $s_i(i=1,2,3,4,5,6,7)$ are arbitrary constants. It is necessary to point out that the independent variable t possess invariant property under the Möbious (conformal) transformation.

In summary, we can arrive at the following final transformation group theorem of Eq. (1).

Theorem 3.1 If $U=U(x,y,t)$ is a solution of the (2+1)-dimensional dissipative AKNS equation then so is u

$$\begin{aligned}
u =& \sigma\left[-\frac{2s_5(s_1t+s_2)}{s_1}\right]^{\frac{1}{2}}\left[\frac{r(s_2s_3-s_1s_4)}{4(s_1t+s_2)^2}-\frac{d}{dt}\xi_0(t)-\frac{1}{4}\sigma r\right]y\\
&+\left[\frac{s_1(y+2s_7)}{2(s_1t+s_2)}\right]x+\alpha_0(t)+\sigma\left[-\frac{s_1}{4s_5(s_1t+s_2)}\right]^{\frac{1}{2}}U(\xi,\eta,\tau),
\end{aligned} \tag{13}$$

where ξ,η, and τ are determined by (12).

In order to see the equivalence between the Lie point symmetry group obtained in **Theorem 3.1** and the known one from classical Lie group method, we need to take the arbitrary functions $\xi_0(t),\alpha_0(t)$ and arbitrary constants $s_i(i=1,2,3,4,5,6,7)$ to be different forms with respect to an infinitesimal parameter ϵ

$$\begin{aligned}
&\sigma=1,\ \ s_1=-\frac{\epsilon}{2}c_6,\ \ s_2=1+\epsilon(c_5-c_4),\ \ s_4=\epsilon c_1,\ \ s_5=\frac{\epsilon}{4}c_6,\ \ s_6=-\frac{2c_3}{c_6},\\
&s_7=\frac{c_3}{c_6}+\epsilon\left(\frac{c_2}{2}-\frac{c_3c_5}{c_6}\right),\ \ \xi_0(t)=\epsilon g(t),\ \ \alpha_0(t)=-\epsilon f(t),
\end{aligned}$$

then (13) can be written as

$$
\begin{aligned}
&u = u + \epsilon\sigma = u + \epsilon\sigma(u),\\
&\sigma(u) = \left[\frac{1}{4}(c_6 t + 2c_4 - 2c_5)x + g\right]u_x + \left[\frac{1}{2}(c_6 y + 2c_3)t + c_5 y + c_2\right]u_y\\
&\qquad + (\frac{1}{2}c_6^2 t^2 + c_4 t + c_1)u_t - \frac{1}{4}(c_6 y + 2c_3)x - [\dot{g} - \frac{r}{16}(3c_6 t + 2c_4 + 2c_5)]y\\
&\qquad - f + \frac{1}{4}(c_6 t + 2c_4 + 2c_5)u,
\end{aligned}
$$

which is exactly the same as the one obtained by the standard Lie approach.

4. (1+1)-DIMENSIONAL SIMILAR REDUCTION OF THE DISSIPATIVE AKNS EQUATION

After determining the infinitesimal generators in Sec. 2, the similarity variables can be obtained by solving the characteristic equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dt}{T} = \frac{du}{U}.$$

Here we only list the following cases.

Case 1. For the generator $\underline{\mathbf{V}}_1 + a_1\underline{\mathbf{V}}_2$, one has the following invariants

$$\chi = x,\ \ \gamma = t - \frac{y}{a_1},\ \ u = P(\chi,\gamma). \tag{14}$$

Substituting Eq. (14) into Eq.(1) yields the corresponding reduced equation

$$-P_{\chi\chi\chi\gamma} + a_1 P_{\chi\gamma} - 8P_{\chi\gamma}p_\chi - 4P_{\chi\chi}P_\gamma + a_1 r P_{\chi\chi} = 0. \tag{15}$$

Case 2. For the generator $\underline{\mathbf{V}}_1 + a_2\underline{\mathbf{V}}_2 + a_3\underline{\mathbf{V}}_3$, we have the following invariants

$$\chi = x,\ \ \gamma = \frac{a_3}{2}t^2 + a_2 t - y,\ \ u = \frac{1}{2}(a_3 t + a_2)x + P(\chi,\gamma). \tag{16}$$

Substituting Eq. (16) into Eq. (1) leads to the corresponding reduced equation

$$-P_{\chi\chi\chi\eta} - 8P_{\chi\gamma}p_\chi - 4P_{\chi\chi}P_\gamma + rP_{\chi\chi} + 2a_3 = 0. \tag{17}$$

Case 3. For the generator $\underline{\mathbf{V}}_4$, the following invariants can be derived

$$\chi = y,\ \ \gamma = \frac{t}{x^2},\ \ u = -\frac{1}{4}ry + P(\chi,\gamma). \tag{18}$$

Substituting Eq. (18) into Eq. (1), then the reduced equation reads

$$
\begin{aligned}
&-8\gamma^3 P_{\chi\gamma\gamma\gamma} - 48\gamma^2 P_{\chi\gamma\gamma} + 2(8P - 27)\gamma P_{\chi\gamma} - 8\gamma P_{\gamma\gamma}\\
&+ (32\gamma^2 P_{\chi\gamma} + 56\gamma P_\chi - 12)P_\eta + (16\gamma^2 P_{\gamma\gamma} + 16P - 6)P_\chi = 0.
\end{aligned} \tag{19}
$$

Case 4. For the generator $\underline{\mathbf{V}}_5$, one can get the following invariants

$$\chi = yx^2,\ \ \gamma = t,\ \ u = -\frac{1}{4}ry + P(\chi,\gamma). \tag{20}$$

Substituting Eq. (20) into Eq. (1), it follows the reduced equation

$$8\chi^3 P_{\chi\chi\chi\chi} + 24\chi^2 P_{\chi\chi\chi} - 2\chi(8P-3)P_{\chi\chi}8\chi P_{\chi\gamma} + 48\chi^2 P_{\chi\chi}P_\chi - 4P_\gamma = 0. \quad (21)$$

Case 5. For the generator $\underline{\mathbf{V}}_6$, the corresponding invariants are

$$\chi = \frac{y}{x^2}, \ \gamma = \frac{t}{x^2}, \ u = -\frac{1}{4}ry + \frac{xy}{2t} + P(\chi,\gamma). \quad (22)$$

Substituting Eq. (22) into Eq. (1) yields the reduced equation

$$\begin{aligned} &-8\chi^3 P_{\chi\chi\chi\chi} - 8\gamma^3 P_{\chi\gamma\gamma\gamma} - 24\chi^2\gamma P_{\chi\chi\chi\gamma} - 24\chi\gamma^2 P_{\chi\chi\gamma\gamma} - 72\chi^2 P_{\chi\chi\chi} \\ &+ 2\gamma(32\chi P_\chi + 16\gamma P_\gamma + 8P - 75)P_{\chi\gamma} + 2\chi(24\chi P_\chi + 16\gamma P_\gamma + 8P - 75)P_{\chi\chi} \\ &+ 88\chi P_\chi^2 + (88\gamma P_\gamma + 32P - 60)P_\chi + 16\gamma^2 P_\chi P_{\gamma\gamma} - 144\chi\gamma P_{\chi\chi\gamma} - 72\gamma^2 P_{\chi\gamma\gamma} = 0. \end{aligned}$$

Case 6. For the generator $\underline{\mathbf{V}}_1 + \underline{\mathbf{V}}_7(f) + \underline{\mathbf{V}}_8(g)$, the corresponding invariants read

$$\chi = x - \tilde{g}, \ \gamma = y, \ u = \frac{d}{dt}\tilde{g}y + \tilde{f} + P(\chi,\gamma), \ g = \frac{d}{dt}\tilde{g}, \ f = \frac{d}{dt}\tilde{f}. \quad (23)$$

Substituting Eq. (23) into Eq. (1), the initial equation becomes the reduced equation

$$P_{\chi\chi\chi\gamma} + 8P_{\chi\gamma}P_\chi + 4P_{\chi\chi}P_\gamma + rP_{\chi\chi} = 0. \quad (24)$$

5. EXACT SOLUTIONS OF THE DISSIPATIVE AKNS EQUATION

In this Section, we provide some exact solutions of the dissipative AKNS equation (1) from three kinds of reduction transformation given in the previous Section.

A. Solutions from $\underline{\mathbf{V}}_1 + a_1\underline{\mathbf{V}}_2$ reduction

Here we are looking for the traveling wave solutions for the dissipative AKNS equation (1), therefore we consider a similarity variable $\theta = \chi + k_1\gamma$, where k_1 is a constant. The reduced equation becomes an ordinary differential equation

$$-k_1\phi_{\theta\theta\theta} - 6k_1\phi_\theta^2 + a_1(r + 4k_1)\phi_\theta + c_0 = 0, \quad (25)$$

where c_0 is an integral constant.

In fact, the ordinary differential equation that the function ϕ_θ in Eq.(25) need to satisfy is an elliptic equation. With different choices of the constants, we have exact solutions listed below. Hereafter, sn, cn, and dn represent the Jacobi elliptic functions and *Ei* is the second kind incomplete elliptic integral.

Case 1

$$u_1 = \frac{A}{h}\omega + \frac{h\,\mathrm{cn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{sn}(\omega,m)} + hEi(\mathrm{sn}(\omega,m),m), \quad (26)$$

with

$$\begin{aligned}&\omega=h\left[x-\frac{k_1(y-a_1t)}{a_1}\right],\ \ A=\frac{1}{3}(m^2-2)h^2+\frac{a_1(r+4k_1)}{12k_1},\\&c_0=\frac{2}{3}k_1(m^4-m^2+1)h^4-\frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned}\tag{27}$$

$$u_2=\frac{A}{h}\omega+\frac{h\mathrm{cn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{sn}(\omega,m)}+2hEi(\mathrm{sn}(\omega,m),m),\tag{28}$$

with

$$\begin{aligned}&\omega=h\left[x-\frac{k_1(y-a_1t)}{a_1}\right],\ \ A=\frac{1}{3}(m^2-5)h^2+\frac{a_1(r+4k_1)}{12k_1},\\&c_0=\frac{2}{3}k_1(m^4+14m^2+1)h^4-\frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned}\tag{29}$$

Especially, when $m=1$, the elliptic functions degenerate to the hyperbolic functions, then the above solutions become

$$\begin{aligned}&\tilde{u}_1=A\left[x-\frac{k_1(y-a_1t)}{a_1}\right]+h\tanh^{-1}\left\{h\left[x-\frac{k_1(y-a_1t)}{a_1}\right]\right\},\\&A=-\frac{h^2}{3}+\frac{a_1(r+4k_1)}{12k_1},\ \ c_0=\frac{2}{3}k_1h^4-\frac{a_1^2(r+4k_1)^2}{24k_1},\end{aligned}\tag{30}$$

and

$$\begin{aligned}&\tilde{u}_2=A\left[x-\frac{k_1(y-a_1t)}{a_1}\right]+2h\tanh^{-1}\left\{2h\left[x-\frac{k_1(y-a_1t)}{a_1}\right]\right\},\\&A=-\frac{4h^2}{3}+\frac{a_1(r+4k_1)}{12k_1},\ \ c_0=\frac{32}{3}k_1h^4-\frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned}\tag{31}$$

Case 2

$$u_3=\frac{A}{h}\omega+\frac{h\mathrm{sn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{cn}(\omega,m)}+hEi(\mathrm{sn}(\omega,m),m),\tag{32}$$

with

$$\begin{aligned}&\omega=h\left[x-\frac{k_1(y-a_1t)}{a_1}\right],\ \ A=\frac{1}{3}(m^2-2)h^2+\frac{a_1(r+4k_1)}{12k_1},\\&c_0=\frac{2}{3}k_1(m^4-m^2+1)h^4-\frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned}\tag{33}$$

$$u_4=\frac{A}{h}\omega+\frac{h\mathrm{sn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{cn}(\omega,m)}+2hEi(\mathrm{sn}(\omega,m),m),\tag{34}$$

with

$$\begin{aligned}&\omega = h\left[x - \frac{k_1(y-a_1t)}{a_1}\right], \ \ A = \frac{1}{3}(4m^2-5)h^2 + \frac{a_1(r+4k_1)}{12k_1},\\ &c_0 = \frac{2}{3}k_1(m^4+14m^2+1)h^4 - \frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned} \tag{35}$$

Especially, when $m=1$, the elliptic functions degenerate to the hyperbolic functions, then the above solutions become

$$\begin{aligned}&\tilde{u}_3 = A\left[x - \frac{k_1(y-a_1t)}{a_1}\right],\\ &A = -\frac{h^2}{3} + \frac{a_1(r+4k_1)}{12k_1}, \ \ c_0 = \frac{2}{3}k_1h^4 - \frac{a_1^2(r+4k_1)^2}{24k_1},\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\tilde{u}_4 = A\left[x - \frac{k_1(y-a_1t)}{a_1}\right] + h\tanh\left\{h\left[x - \frac{k_1(y-a_1t)}{a_1}\right]\right\},\\ &A = -\frac{h^2}{3} + \frac{a_1(r+4k_1)}{12k_1}, \ \ c_0 = \frac{2}{3}k_1h^4 - \frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned} \tag{37}$$

Case 3

$$u_5 = \frac{A}{h}\omega + \frac{hm^2\mathrm{sn}(\omega,m)\mathrm{cn}(\omega,m)}{\mathrm{dn}(\omega,m)} + hEi(\mathrm{sn}(\omega,m),m), \tag{38}$$

with

$$\begin{aligned}&\omega = h\left[x - \frac{k_1(y-a_1t)}{a_1}\right], \ \ A = \frac{1}{3}(m^2-2)h^2 + \frac{a_1(r+4k_1)}{12k_1},\\ &c_0 = \frac{2}{3}k_1(m^4-m^2+1)h^4 - \frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned} \tag{39}$$

$$u_6 = \frac{A}{h}\omega + \frac{hm^2\mathrm{sn}(\omega,m)\mathrm{cn}(\omega,m)}{\mathrm{dn}(\omega,m)} + 2hEi(\mathrm{sn}(\omega,m),m), \tag{40}$$

with

$$\begin{aligned}&\omega = h\left[x - \frac{k_1(y-a_1t)}{a_1}\right], \ \ A = \frac{1}{3}(m^2-2)h^2 + \frac{a_1(r+4k_1)}{12k_1},\\ &c_0 = \frac{2}{3}k_1(m^4-16m^2+16)h^4 - \frac{a_1^2(r+4k_1)^2}{24k_1}.\end{aligned} \tag{41}$$

Especially, when $m=1$, the elliptic functions degenerate to the hyperbolic functions, then the above solutions become

$$\begin{aligned} &\tilde{u}_5 = A\left[x - \frac{k_1(y-a_1t)}{a_1}\right], \\ &A = -\frac{h^2}{3} + \frac{a_1(r+4k_1)}{12k_1}, \ \ c_0 = \frac{2}{3}k_1h^4 - \frac{a_1^2(r+4k_1)^2}{24k_1}, \end{aligned} \tag{42}$$

and

$$\begin{aligned} &\tilde{u}_6 = A\left[x - \frac{k_1(y-a_1t)}{a_1}\right] + h\tanh\left\{h\left[x - \frac{k_1(y-a_1t)}{a_1}\right]\right\}, \\ &A = -\frac{h^2}{3} + \frac{a_1(r+4k_1)}{12k_1}, \ \ c_0 = \frac{2}{3}k_1h^4 - \frac{a_1^2(r+4k_1)^2}{24k_1}. \end{aligned} \tag{43}$$

B. Solutions from $\underline{\mathbf{V}}_1 + a_2\underline{\mathbf{V}}_2 + a_3\underline{\mathbf{V}}_3$ reduction

For the reduced equation (17), it is straightforward to verify that it has one special solution,

$$P(\chi,\gamma) = \frac{a_3\chi^2}{4\lambda_1 - r} + \lambda_2\chi + \lambda_1\gamma + \lambda_3, \tag{44}$$

where $\lambda_i (i=1,2,3)$ are arbitrary constants. Therefore, we can get the following exact solution of the dissipative AKNS equation,

$$u_7 = \frac{1}{2}(a_3t+a_2)x + \frac{a_3x^2}{4\lambda_1 - r} + \lambda_2x + \lambda_1\left(\frac{a_3}{2}t^2 + a_2t - y\right). \tag{45}$$

C. Solutions from $\underline{\mathbf{V}}_1 + \underline{\mathbf{V}}_7(f) + \underline{\mathbf{V}}_8(g)$ reduction

For the reduced equation (24), we still seek for traveling wave reduction with a similarity variable $\theta = \chi + k_3\gamma$, where k_3 is a constant. Then the reduced equation (24) becomes an ordinary differential equation

$$k_3\phi_{\theta\theta\theta} + 6k_3\phi_\theta^2 + r\phi_\theta + c_0 = 0, \tag{46}$$

where c_0 is an integral constant. This equation is same as Eq. (25), thus we only list the final results as follows:

Case 1

$$\begin{aligned} u_8 =& \frac{d}{dt}\tilde{g}y + \tilde{f} + [\frac{1}{3}(m^2-2)h - \frac{r}{12k_3h}]\omega + \frac{h\mathrm{cn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{sn}(\omega,m)} \\ &+ h\boldsymbol{Ei}(\mathrm{sn}(\omega,m),m), \\ u_9 =& \frac{d}{dt}\tilde{g}y + \tilde{f} + [\frac{1}{3}(m^2-5)h - \frac{r}{12k_3h}]\omega \\ &+ \frac{h\mathrm{cn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{sn}(\omega,m)} + 2h\boldsymbol{Ei}(\mathrm{sn}(\omega,m),m), \end{aligned} \tag{47}$$

with $\omega = h(x+k_3y-\tilde{g})$, $c_0 = -\frac{2}{3}k_3(m^4-m^2+1)h^4+\frac{r^2}{24k_1}$, and $c_0 = -\frac{2}{3}k_3(m^4+14m^2+1)h^4+\frac{r^2}{24k_1}$, respectively.

Case 2

$$
\begin{aligned}
u_{10} =& \frac{d}{dt}\tilde{g}y+\tilde{f}+[\frac{1}{3}(m^2-2)h-\frac{r}{12k_3h}]\omega-\frac{h\mathrm{sn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{cn}(\omega,m)} \\
&+hEi(\mathrm{sn}(\omega,m),m), \\
u_{11} =& \frac{d}{dt}\tilde{g}y+\tilde{f}+[\frac{1}{3}(4m^2-5)h-\frac{r}{12k_3h}]\omega-\frac{h\mathrm{sn}(\omega,m)\mathrm{dn}(\omega,m)}{\mathrm{cn}(\omega,m)} \\
&+2hEi(\mathrm{sn}(\omega,m),m),
\end{aligned} \tag{48}
$$

with $\omega = h(x+k_3y-\tilde{g})$, $c_0 = -\frac{2}{3}k_3(m^4-m^2+1)h^4+\frac{r^2}{24k_1}$, and $c_0 = -\frac{2}{3}k_3(16m^4-16m^2+1)h^4+\frac{r^2}{24k_1}$, respectively.

Case 3

$$
\begin{aligned}
u_{12} =& \frac{d}{dt}\tilde{g}y+\tilde{f}+[\frac{1}{3}(m^2-2)h-\frac{r}{12k_3h}]\omega-\frac{hm^2\mathrm{sn}(\omega,m)\mathrm{cn}(\omega,m)}{\mathrm{dn}(\omega,m)} \\
&+hEi(\mathrm{sn}(\omega,m),m), \\
u_{13} =& \frac{d}{dt}\tilde{g}y+\tilde{f}+[\frac{1}{3}(m^2-2)h-\frac{r}{12k_3h}]\omega-\frac{hm^2\mathrm{sn}(\omega,m)\mathrm{cn}(\omega,m)}{\mathrm{dn}(\omega,m)} \\
&+2hEi(\mathrm{sn}(\omega,m),m),
\end{aligned} \tag{49}
$$

with $\omega = h(x+k_3y-\tilde{g})$, $c_0 = -\frac{2}{3}k_3(m^4-m^2+1)h^4+\frac{r^2}{24k_1}$, and $c_0 = -\frac{2}{3}k_3(m^4-16m^2+16)h^4+\frac{r^2}{24k_1}$, respectively.

In addition, with the help of **Theorem 3.1**, one can obtain more new types of exact solutions for the (2+1)-dimensional dissipative AKNS equation from the solutions (26)-(43), (45), and (47)-(49). However, we do not list them here.

6. SUMMARY AND DISCUSSIONS

In summary, we have carried out a detailed Lie symmetry analysis of the (2+1)-dimensional dissipative AKNS equation and we have investigated the algebraic structure of the symmetry groups. Meanwhile, we have applied a simple direct method to derive the general finite transformation groups of the dissipative AKNS equation, which is equivalent to Lie point symmetry groups generated *via* the standard approaches. Based on general Lie point symmetry, we have derived the corresponding similarity reduction by solving the characteristic equations and have obtained some exact solutions of the (2+1)-dimensional dissipative AKNS equation. The other important properties such as the Hamiltonian structure, the conservation laws and the generalized (nonlocal) symmetry of the the (2+1)-dimensional dissipative AKNS equation deserves a further study.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Grant No. 11447017), the Natural Science Foundation of Zhejiang Province (Grant No. LY14A010005) and the Applied Nonlinear Science and Technology from the Most Important Among all the Top Priority Disciplines of Zhejiang Province.

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