SOLUTION OF MAXWELL'S WAVE EQUATIONS IN BICOMPLEX SPACE

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Abstract. The concept of bicomplex numbers is introduced in Electromagnetics, with direct applications to the solution of Maxwell's equations. In this paper, we discuss the technique to find the analytic solution of the electromagnetic wave equation in vacuum with the help of bicomplex analysis as tool. Also, we find the solution of Gaussian pulse wave using bicomplex vector field.

Key words: Bicomplex functions, Maxwell's equations, Bicomplex Fourier transform.

1. INTRODUCTION

Solving linear and nonlinear partial differential equations describing real world phenomena is still an interesting problem, see, for example, Refs. [1–7]. In this paper, we discuss the solution of electromagnetic wave equation in vacuum by using the concept of bicomplex numbers. In 1892, Segre [8] defined bicomplex numbers as

$$C_2 = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + jx_3 |\ x_0, x_1, x_2, x_3 \in C_0\},$$

or

$$C_2 = \{\xi : \xi = z_1 + i_2z_2 |\ z_1, z_2 \in C_1\},$$

where i_1 and i_2 are imaginary units such that $i_1^2 = i_2^2 = -1,\ i_1i_2 = i_2i_1 = j$, $j^2 = 1$. The sets of real numbers, complex numbers, and bicomplex numbers are represented by C_0, C_1, and C_2, respectively. Non-invertible elements $\{(i_1 \pm i_2)z : z \in C_1\}$ of bicomplex numbers are situated on the null cone. Bicomplex numbers are commutative while quaternions are not.

In 1928 and 1932, Futagawa developed the concept of holomorphic functions of a bicomplex variable in a series of papers [9], [10]. In 1934, Dragoni [11] discussed some basic results of bicomplex holomorphic functions while Price [12] and Rönn [13] have developed the bicomplex algebra and function theory.

In recent developments, a series of authors have done efforts to extend Polygamma function [14], inverse Laplace transform, its convolution theorem [15], Stieltjes transform [16], Tauberian theorem of Laplace-Stieltjes transform [17], and Bochner theorem of Fourier-Stieltjes transform in the bicomplex variable from their complex counterparts. In their procedure, the idempotent representation of bicomplex functions plays a vital role.

Idempotent Representation: Every bicomplex number can be uniquely expressed as a complex combination of e_1 and e_2, viz.

$$\xi = (z_1 + i_2 z_2) = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2,$$

where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}; e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$, which is the idempotent representation of a bicomplex number ξ. The coefficients $(z_1 + i_1 z_2)$ and $(z_1 - i_1 z_2)$ are called the idempotent components of the bicomplex number $\xi = z_1 + i_2 z_2$ and $\{e_1, e_2\}$ is called the idempotent basis.

Cartesian Set: The auxiliary complex spaces A_1 and A_2 are defined as follows:

$$A_1 = \{z_1 - i_1 z_2, \forall\, z_1, z_2 \in C_1\},\ A_2 = \{z_1 + i_1 z_2, \forall\, z_1, z_2 \in C_1\}.$$

X_1 and X_2 determines a Cartesian set in A_1 and A_2, respectively, which is denoted as $X_1 \times_e X_2$ and is defined as:

$$X_1 \times_e X_2 = \{\xi \in C_2 : \xi = \xi_1 e_1 + \xi_2 e_2,\ \xi_1 \in X_1, \xi_2 \in X_2\}.$$

We define functions $P_1 : C_2 \to A_1 \subseteq C_1$, $P_2 : C_2 \to A_2 \subseteq C_1$ with the help of idempotent representation as follows:

$$P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1,\ \forall\, z_1 + i_2 z_2 \in C_2,$$

$$P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2,\ \forall\, z_1 + i_2 z_2 \in C_2.$$

In the following theorem, Price [12] discussed the convergence of bicomplex function with respect to its idempotent complex component functions. This theorem is useful in proving our results.

Theorem 1.1 (Price [12]) *$F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $D \subseteq C_2$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$, the projections under the functions $P_1 : D \to D_1 \subseteq C_1$ and $P_2 : D \to D_2 \subseteq C_1$, are convergent in domains D_1 and D_2, respectively.*

The contour integral in bicomplex space is defined in the following way:

$$\int_\Gamma F(\xi)d\xi = \int_{\Gamma_1} F_{e_1}(\xi_1)d\xi_1 e_1 + \int_{\Gamma_2} F_{e_2}(\xi_2)d\xi_2 e_2$$

where $\Gamma = (\Gamma_1, \Gamma_2)$ and Γ_1 and Γ_2 are closed contours in complex space along a line in which functions are meromorphic.

James Maxwell published his first paper in 1853 after obtaining his graduate degree. In that paper, he published Faraday's concept *lines of force*. He gave the mathematical explanation of Faraday's work (see, *e.g.* Guilmette [18]). Maxwell's equations describe how electric and magnetic fields are generated and influenced by each other and by charges and currents. These equations are named after the mathematician and physicist Maxwell, who published these equations between 1861 and 1862. In 1864, Maxwell [19] discussed that an electromagnetic disturbance travels in free space with the velocity of light. In 1873, Maxwell [20] records the transformation of Maxwell's complete theory of electromagnetism.

Hertz discussed electromagnetic waves in the year 1888 [21, Chapter 7, p. 107-123], in which Hertz confirmed Maxwell's prediction and helped in the acceptance of Maxwell's electromagnetic theory. By the efforts of Hertz, George Francis Fitzgerald (1851-1901), Oliver Lodge (1851-1940), and Oliver Heaviside (1850-1925) (see, *e.g.* Sengupta and Sarkar [22]) Maxwell's ideas and equations were made understandable. These developments are well documented in Refs. [23] and [24].

Motivated by the work of Anastassiu *et al.* [25] for finding solution of Maxwell's equations in source free domain for electric and magnetic fields using quaternions, we have made efforts to solve the Maxwell's equations in vacuum using bicomplex analysis. The method discussed here has the advantage of dealing both the vector fields (electric and magnetic) together as a single vector field in bicomplex space.

The organization of this paper is as follows. In Sec. 2, we establish bicomplex vector field, defined by combining electric and magnetic fields and then find the solution of electromagnetic wave equation by using concept of bicomplex numbers. In Sec. 3, we find complete solution of bicomplex Gaussian travelling electromagnetic wave equation and the last section contains the conclusions.

2. BICOMPLEX SOLUTION FOR ELECTROMAGNETIC WAVE EQUATION IN VACUUM

The Maxwell's equations in vacuum for electromagnetic field are (see, *e.g.* Lonngren and Savov [26, Chapter 7])

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{4}$$

where electric field $\mathbf{E}$ and magnetic field intensity $\mathbf{H}$ are complex-valued vectors, μ_0 is permeability and ϵ_0 is the permittivity of free space. Let us define the bicomplex vector field $\mathbf{F}$ as

$$\mathbf{F} \equiv \sqrt{\epsilon_0}\mathbf{E} + i_2\sqrt{\mu_0}\mathbf{H} \tag{5}$$

with the intimation that each directional component of $\mathbf{F}$ is a scalar bicomplex function, obtaining by combining the corresponding field directional components. Now, taking curl of (5) on both sides,

$$\begin{aligned}\nabla\times\mathbf{F} &= \sqrt{\epsilon_0}\nabla\times\mathbf{E} + i_2\sqrt{\mu_0}\nabla\times\mathbf{H}\\ &= -\mu_0\sqrt{\epsilon_0}\frac{\partial\mathbf{H}}{\partial t} + i_2\sqrt{\mu_0}\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\\ &= i_2\sqrt{\mu_0\epsilon_0}\frac{\partial}{\partial t}\left(\sqrt{\epsilon_0}\mathbf{E} + i_2\sqrt{\mu_0}\mathbf{H}\right)\\ &= i_2\sqrt{\mu_0\epsilon_0}\frac{\partial\mathbf{F}}{\partial t}\end{aligned}$$

Therefore, we obtain the bicomplex Maxwell's vector equations as

$$\nabla\times\mathbf{F} = i_2\frac{1}{c}\frac{\partial\mathbf{F}}{\partial t}, \quad \left[\text{where } c = \frac{1}{\sqrt{\mu_0\epsilon_0}}\right] \tag{6}$$

$$\nabla\cdot\mathbf{F} = 0 \tag{7}$$

Assuming that the wave is travelling in x-direction, *i.e.*, a vanishing x-component, then (6) and (7) are reduced to the following system of bicomplex differential equations,

$$-\frac{\partial F_z}{\partial x} = i_2\frac{1}{c}\frac{\partial F_y}{\partial t} \tag{8}$$

$$\frac{\partial F_y}{\partial x} = i_2\frac{1}{c}\frac{\partial F_z}{\partial t} \tag{9}$$

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0 \tag{10}$$

$$\frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0 \tag{11}$$

Put $Q_z = i_2F_z$. The equations (8) and (9) become

$$\frac{\partial Q_z}{\partial x} = \frac{1}{c}\frac{\partial F_y}{\partial t}, \tag{12}$$

$$\text{and} \quad \frac{\partial F_y}{\partial x} = \frac{1}{c}\frac{\partial Q_z}{\partial t}, \text{ respectively.} \tag{13}$$

Differentiating (12) and (13) and using respectively (13) and (12) therein, we get

$$\frac{\partial^2}{\partial x^2}F_y(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}F_y(x,t) \tag{14}$$

$$\frac{\partial^2}{\partial x^2}Q_z(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}Q_z(x,t) \tag{15}$$

Due to (10) and (11) the initial conditions of F_y and Q_z are functions of the variable x only. Let the initial conditions $F_y(x,0) = Af_1(x)$, $\frac{\partial}{\partial t}F_y(x,0) = Bg_1(x)$, $Q_z(x,0) = Df_1(x)$ and $\frac{\partial}{\partial t}Q_z(x,0) = Gg_1(x)$, where $f_1(x)$, $g_1(x)$ are bicomplex-valued functions and A, B, D, G are bicomplex constants.

Taking the bicomplex Fourier transform (for details, refer [27] and [28]) of (14) with respect to x, we get

$$\frac{d^2}{dt^2}\bar{F}_y(\xi,t) + c^2\xi^2\bar{F}_y(\xi,t) = 0 \tag{16}$$

Solving (16) and applying initial conditions therein, we get

$$\bar{F}_y(\xi,t) = \frac{A}{2}\bar{f}_1(\xi)\left(e^{i_1c\xi t} + e^{-i_1c\xi t}\right) - i_1B\frac{\bar{g}_1(\xi)}{2c\xi}\left(e^{i_1c\xi t} - e^{-i_1c\xi t}\right). \tag{17}$$

[where $\bar{f}_1(\xi) = \mathcal{F}[f_1(x)](\xi)$ and $\bar{g}_1(\xi) = \mathcal{F}[g_1(x)](\xi)$]

Remark 2.1 *There is no major reason to prefer i_1 instead of i_2, however i_1 is more appropriate than i_2 for the decomposition of bicomplex form in idempotent components.*

Taking the inverse bicomplex Fourier transform [29, Eq. 11] of (17) with respect to ξ, we have

$$F_y(x,t) = \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi x}\bar{F}_y(\xi,t)d\xi$$

where $\Gamma = (\Gamma_1,\Gamma_2)$ is a closed contour in the bicomplex space, where Γ_1 and Γ_2 are closed contours in the complex space along the horizontal lines $\{-\alpha < \mathrm{Im}(P_1 : \xi) <$

$\beta\}$ and $\{-\alpha < \mathrm{Im}(P_2 : \xi) < \beta\}$, respectively.

$$\begin{aligned}F_y(x,t) = & \frac{A}{2}\left[\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x-ct)}\bar{f}_1(\xi)d\xi + \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x+ct)}\bar{f}_1(\xi)d\xi\right] \\ & + \frac{B}{2c}\left\{\frac{1}{2\pi}\int_\Gamma \frac{\bar{g}_1(\xi)}{i_1\xi}\left(e^{-i_1\xi(x-ct)} - e^{-i_1\xi(x+ct)}\right)d\xi\right\} \\ = & \frac{A}{2}\left[\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x-ct)}\bar{f}_1(\xi)d\xi + \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x+ct)}\bar{f}_1(\xi)d\xi\right] \\ & + \frac{B}{2c}\left\{\frac{1}{2\pi}\int_\Gamma \bar{g}_1(\xi)\int_{x-ct}^{x+ct} e^{-i_1\xi p}dpd\xi\right\} \\ = & \frac{A}{2}\left[\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x-ct)}\bar{f}_1(\xi)d\xi + \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x+ct)}\bar{f}_1(\xi)d\xi\right] \\ & + \frac{B}{2c}\left\{\int_{x-ct}^{x+ct} dp\left(\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi p}\bar{g}_1(\xi)d\xi\right)\right\}. \end{aligned} \tag{18}$$

By simplifying (18), we get

$$F_y(x,t) = \frac{A}{2}\left[f_1(x-ct) + f_1(x+ct)\right] + \frac{B}{2c}\int_{x-ct}^{x+ct} g_1(p)dp. \tag{19}$$

Similarly,

$$F_z(x,t) = -i_2Q_z(x,t) = -i_2\frac{D}{2}\left[f_1(x-ct) + f_1(x+ct)\right] - i_2\frac{G}{2c}\int_{x-ct}^{x+ct} g_1(p)dp. \tag{20}$$

Therefore, wave traveling in x-direction with vector field $\mathbf{F_x}$ is

$$\mathbf{F_x} = F_y(x,t)\hat{y} + F_z(x,t)\hat{z} \tag{21}$$

Since (21) is the solution of bicomplex Maxwell's equations, it satisfies the equations (6-7). So we obtain the values of D and G in terms of A and B.

Similarly, the wave traveling in y-direction with vector field $\mathbf{F_y}$ with initial conditions $F_x(y,0) = Rf_2(y)$, $Q_z(y,0) = Mf_2(y)$, $\frac{\partial}{\partial t}F_x(y,0) = Sg_2(y)$ and $\frac{\partial}{\partial t}Q_z(y,0) = Ng_2(y)$ are

$$\mathbf{F_y} = F_x(y,t)\hat{x} + F_z(y,t)\hat{z} \tag{22}$$

where

$$F_x(y,t) = \frac{R}{2}\left[f_2(y-ct) + f_2(y+ct)\right] + \frac{S}{2c}\int_{y-ct}^{y+ct} g_2(p)dp,$$

and

$$F_z(y,t) = -i_2\frac{M}{2}\left[f_2(y-ct) + f_2(y+ct)\right] - i_2\frac{N}{2c}\int_{y-ct}^{y+ct} g_2(p)dp.$$

Again, since (22) is the solution of bicomplex Maxwell's equations, it satisfies the equations (6-7). So we obtain the values of M and N in terms of R and S.

Also, wave traveling in z-direction with vector field $\mathbf{F_z}$ and initial conditions $F_x(z,0) = Lf_3(z)$, $Q_y(z,0) = If_3(z)$, $\frac{\partial}{\partial t}Q_y(z,0) = Jg_3(z)$ and $\frac{\partial}{\partial t}F_x(z,0) = Gg_3(z)$ are

$$\mathbf{F_z} = F_x(z,t)\hat{x} + F_y(z,t)\hat{y} \tag{23}$$

where

$$F_x(z,t) = \frac{L}{2}\left[f_3(z-ct) + f_3(z+ct)\right] + \frac{G}{2c}\int_{z-ct}^{z+ct} g_3(p)dp,$$

and

$$F_y(z,t) = -i_2\frac{I}{2}\left[f_3(z-ct) + f_3(z+ct)\right] - i_2\frac{J}{2c}\int_{z-ct}^{z+ct} g_3(p)dp.$$

Again, since (23) is the solution of bicomplex Maxwell's equations, it satisfies the equations (6-7). So we obtain the values of I and J in terms of L and G.

Now, by applying the superposition principle on equations (21), (22), and (23), we obtain the solution of equations (6) and (7)as

$$\mathbf{F} = \left[F_x(y,t) + F_x(z,t)\right]\hat{x} + \left[F_y(x,t) + F_y(z,t)\right]\hat{y} + \left[F_z(x,t) + F_z(y,t)\right]\hat{z}. \tag{24}$$

By separating bi-real and bi-imaginary part we obtain the electric and magnetic fields in all three dimensions, which satisfy the Maxwell's equations.

3. BICOMPLEX GAUSSIAN PULSE WAVE

In this Section, we find the complete solution of the bicomplex Gaussian pulse travelling electromagnetic wave equation. For Gaussian pulse wave function is a solution of Gaussian pulse travelling wave (see, *e.g.* Lonngren and Savov [26, p. 345-346]). A Two-Pulse Synthesis model presented by Goswami *et al.* [30] successfully reconstructed digital volume pulse waveforms using Rayleigh functions with small Mean Square Error. In [31], Wang Lu *et al.* presented a multi-Gaussian model to fit real pulse waveforms using an adaptive number of Gaussian waves.

Consider the bicomplex Gaussian pulse travelling electromagnetic wave equations are

$$\nabla \times \mathbf{F} = i_2\frac{1}{c}\frac{\partial \mathbf{F}}{\partial t}$$
$$\nabla \cdot \mathbf{F} = 0$$

For the bicomplex Gaussian pulse wave travelling in x- direction, the initial conditions (see, *e.g.* Lonngren and Savov [26, p. 345-346]) are of the form

$$F_y(x,0) = Ae^{-x^2}, \quad \frac{\partial}{\partial t}F_y(x,0) = Bxe^{-x^2}, \quad A, B \in C_2 \tag{25}$$

$$Q_z(x,0) = De^{-x^2}, \quad \frac{\partial}{\partial t}Q_z(x,0) = Gxe^{-x^2}, \quad D, G \in C_2 \tag{26}$$

Since (21) satisfies bicomplex Maxwell's equations and using (25) and (26) in (19) and (20), respectively. We get

$$F_y(x,t) = \left(\frac{A}{2} + \frac{B}{4c}\right)e^{-(x-ct)^2} + \left(\frac{A}{2} - \frac{B}{4c}\right)e^{-(x+ct)^2}$$

and

$$F_z(x,t) = i_2\left(\frac{A}{2} + \frac{B}{4c}\right)e^{-(x-ct)^2} - i_2\left(\frac{A}{2} - \frac{B}{4c}\right)e^{-(x+ct)^2}$$

Let $\frac{A}{2} + \frac{B}{4c} = \alpha \in C_2$ and $\frac{A}{2} - \frac{B}{4c} = \beta \in C_2$, then (21) becomes

$$\mathbf{F_x} = \left[\alpha e^{-(x-ct)^2} + \beta e^{-(x+ct)^2}\right]\hat{y} + i_2\left[\alpha e^{-(x-ct)^2} - \beta e^{-(x+ct)^2}\right]\hat{z}. \tag{27}$$

Similarly, for the bicomplex Gaussian pulse wave travelling in y- direction with initial conditions

$$F_x(y,0) = Re^{-y^2}, \quad \frac{\partial}{\partial t}F_x(y,0) = Sye^{-y^2}, \quad R, S \in C_2 \tag{28}$$

$$Q_z(y,0) = Me^{-y^2}, \quad \frac{\partial}{\partial t}Q_z(y,0) = Nye^{-y^2}, \quad M, N \in C_2 \tag{29}$$

is

$$\mathbf{F_y} = \left[\delta e^{-(y-ct)^2} + \gamma e^{-(y+ct)^2}\right]\hat{x} - i_2\left[\delta e^{-(y-ct)^2} - \gamma e^{-(y+ct)^2}\right]\hat{z}, \tag{30}$$

where $\delta = \frac{R}{2} + \frac{S}{4c}$ and $\gamma = \frac{R}{2} - \frac{S}{4c}$. And the bicomplex Gaussian pulse wave travelling in z- direction with initial conditions

$$F_x(z,0) = Le^{-z^2}, \quad \frac{\partial}{\partial t}F_x(z,0) = Gze^{-y^2}, \quad L, G \in C_2, \tag{31}$$

$$Q_y(z,0) = Ie^{-z^2}, \quad \frac{\partial}{\partial t}Q_y(z,0) = Jze^{-z^2}, \quad I, J \in C_2 \tag{32}$$

is

$$\mathbf{F_z} = \left[\phi e^{-(z-ct)^2} + \psi e^{-(z+ct)^2}\right]\hat{x} + i_2\left[\phi e^{-(z-ct)^2} - \psi e^{-(z+ct)^2}\right]\hat{y}, \tag{33}$$

where $\phi = \frac{L}{2} + \frac{G}{4c}$, $\psi = \frac{L}{2} - \frac{G}{4c}$ and $\alpha, \beta, \phi, \psi, \delta$ and γ are bicomplex constants. Now, by applying the superposition principle on equations (27), (30), and (33), we get vector field as

$$\begin{aligned}\mathbf{F} = &\left[\delta e^{-(y-ct)^2} + \gamma e^{-(y+ct)^2} + \phi e^{-(z-ct)^2} + \psi e^{-(z+ct)^2}\right]\hat{x} \\ &+ \left[\alpha e^{-(x-ct)^2} + \beta e^{-(x+ct)^2} + i_2\phi e^{-(z-ct)^2} - i_2\psi e^{-(z+ct)^2}\right]\hat{y} \\ &+ i_2\left[\alpha e^{-(x-ct)^2} - \beta e^{-(x+ct)^2} - \delta e^{-(y-ct)^2} + \gamma e^{-(y+ct)^2}\right]\hat{z}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{F} \equiv \sqrt{\epsilon_0}\mathbf{E} + i_2\sqrt{\mu_0}\mathbf{H} = &\left[\delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2} + \phi_1 e^{-(z-ct)^2} + \psi_1 e^{-(z+ct)^2}\right]\hat{x} \\ &+ \left[\alpha_1 e^{-(x-ct)^2} + \beta_1 e^{-(x+ct)^2} - \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{y} \\ &+ \left[-\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \delta_2 e^{-(y-ct)^2} - \gamma_2 e^{-(y+ct)^2}\right]\hat{z} \\ &+ i_2\left\{\left[\delta_2 e^{-(y-ct)^2} + \gamma_2 e^{-(y+ct)^2} + \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{x}\right. \\ &+ \left[\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \phi_1 e^{-(z-ct)^2} - \psi_1 e^{-(z+ct)^2}\right]\hat{y} \\ &+ \left.\left[\alpha_1 e^{-(x-ct)^2} - \beta_1 e^{-(x+ct)^2} - \delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2}\right]\hat{z}\right\} \qquad (34)\end{aligned}$$

where $\delta_1, \delta_2, \gamma_1, \gamma_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \phi_1, \phi_2, \psi_1, \psi_2 \in C_1$.

By separating bi-real and bi-imaginary part of (34), we get

$$\begin{aligned}\mathbf{E} = \frac{1}{\sqrt{\epsilon_0}}&\left\{\left[\delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2} + \phi_1 e^{-(z-ct)^2} + \psi_1 e^{-(z+ct)^2}\right]\hat{x}\right. \\ &+ \left[\alpha_1 e^{-(x-ct)^2} + \beta_1 e^{-(x+ct)^2} - \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{y} \\ &+ \left.\left[-\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \delta_2 e^{-(y-ct)^2} - \gamma_2 e^{-(y+ct)^2}\right]\hat{z}\right\} \qquad (35)\end{aligned}$$

$$\begin{aligned}\mathbf{H} = \frac{1}{\sqrt{\mu_0}}&\left\{\left[\delta_2 e^{-(y-ct)^2} + \gamma_2 e^{-(y+ct)^2} + \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{x}\right. \\ &+ \left[\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \phi_1 e^{-(z-ct)^2} - \psi_1 e^{-(z+ct)^2}\right]\hat{y} \\ &+ \left.\left[\alpha_1 e^{-(x-ct)^2} - \beta_1 e^{-(x+ct)^2} - \delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2}\right]\hat{z}\right\}. \qquad (36)\end{aligned}$$

Therefore, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ satisfy the Maxwell's equations (1-4). Therefore, the electric and magnetic fields of Gaussian pulse wave propagating in positive direction are

$$\mathbf{E} = \frac{1}{\sqrt{\epsilon_0}} \left\{ \left[\delta_1 e^{-(y-ct)^2} + \phi_1 e^{-(z-ct)^2} \right] \hat{x} + \left[\alpha_1 e^{-(x-ct)^2} - \phi_2 e^{-(z-ct)^2} \right] \hat{y} + \left[-\alpha_2 e^{-(x-ct)^2} + \delta_2 e^{-(y-ct)^2} \right] \hat{z} \right\} \quad (37)$$

$$\mathbf{H} = \frac{1}{\sqrt{\mu_0}} \left\{ \left[\delta_2 e^{-(y-ct)^2} + \phi_2 e^{-(z-ct)^2} \right] \hat{x} + \left[\alpha_2 e^{-(x-ct)^2} + \phi_1 e^{-(z-ct)^2} \right] \hat{y} + \left[\alpha_1 e^{-(x-ct)^2} - \delta_1 e^{-(y-ct)^2} \right] \hat{z} \right\}. \quad (38)$$

Similarly, the electric and magnetic fields of Gaussian pulse wave propagating in negative direction are

$$\mathbf{E} = \frac{1}{\sqrt{\epsilon_0}} \left\{ \left[\gamma_1 e^{-(y+ct)^2} + \psi_1 e^{-(z+ct)^2} \right] \hat{x} + \left[\beta_1 e^{-(x+ct)^2} + \psi_2 e^{-(z+ct)^2} \right] \hat{y} + \left[\beta_2 e^{-(x+ct)^2} - \gamma_2 e^{-(y+ct)^2} \right] \hat{z} \right\} \quad (39)$$

$$\mathbf{H} = \frac{1}{\sqrt{\mu_0}} \left\{ \left[\gamma_2 e^{-(y+ct)^2} + \psi_2 e^{-(z+ct)^2} \right] \hat{x} + \left[\beta_2 e^{-(x+ct)^2} - \psi_1 e^{-(z+ct)^2} \right] \hat{y} + \left[-\beta_1 e^{-(x+ct)^2} + \gamma_1 e^{-(y+ct)^2} \right] \hat{z} \right\}. \quad (40)$$

Also, the electric and magnetic fields of Gaussian pulse wave in equations (37-40) satisfy the Maxwell's equations (1-4). The bicomplex approach is advantageous than the quaternion approach due to the commutativity property of bicomplex numbers. The authors have also discussed the application of bicomplex Mellin transform [32] in RLC circuit.

4. CONCLUSION

The concept of bicomplex numbers has been applied for finding the solution of Maxwell's equations. In this paper, we find the solution of bicomplex electromagnetic Maxwell's equations in vacuum by defining bicomplex vector field. We conclude that the bicomplex analysis has great advantage of dealing both the vector fields (electric and magnetic) together as a single vector field in the bicomplex space. Instead of finding solutions of four equations (1-4), now we are required to solve only two equations (6-7). This approach is also advantageous than the quaternion approach due to the commutativity property of bicomplex numbers (see, Anastassiu *et al.* [25]).

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