PARALLEL LINE ROGUE WAVES OF A \((2+1)\)-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION DESCRIBING THE HEISENBERG FERROMAGNETIC SPIN CHAIN

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Abstract. Under investigation in this paper is a \((2+1)\)-dimensional nonlinear Schrödinger equation describing the Heisenberg ferromagnetic spin chain. A general explicit form of rogue wave solutions for the \((2+1)\)-dimensional nonlinear Schrödinger equation is given in terms of the Gram determinants by employing the bilinear method. The relevant formulas involve determinants whose matrix elements are simple polynomials. The fundamental rogue wave is called line rogue wave in the \((x, y)\)-plane, which arises from a constant background with a line profile and then disappears into the same background. High-order rogue waves consist of several parallel line rogue waves, and describe the interaction of several fundamental rogue waves. Besides, their dynamical behaviors in the \((x, t)\)-plane are also investigated by three-dimensional plots.

Key words: Heisenberg ferromagnetic spin chain, \((2+1)\)-dimensional nonlinear Schrödinger equation, Parallel line rogue waves, Bilinear transformation method.

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1. INTRODUCTION

Nonlinear evolution equations (NLEEs) are well used to describe various kinds of nonlinear phenomena in fields such as fluids, plasmas, optics, particle physics, biophysics, and condensed matter physics [1–11]. Exact solutions of NLEEs have been explored to investigate these nonlinear phenomena, because they can give more insight into the physical aspects and then lead to further applications. Indeed, various effective methods have been developed to obtain exact solutions of NLEEs, such as the Darboux transformation method [12, 13], the inverse scattering method [14], the Hirota bilinear method [15], the homogeneous balance method [16, 17], the Lie group method [18, 19], and so on [20–22].

Originally, the rogue waves were observed in a relatively calm sea, where they suddenly appear from nowhere as localized and isolated surface waves, and then make a sudden hole in the sea just before attaining surprisingly high amplitudes, and
finally disappear without a trace. Recently, efforts devoted to study such rare extreme events have ranged from oceanography [23] to various areas, such as nonlinear optics [24–26], Bose-Einstein condensates [27], plasma physics [28], and so on [29–32]. Mathematically, rogue waves are expressed in rational form, and are localized in both space and time [33, 34]. Recently, a variety of nonlinear soliton equations including nonlocal systems satisfying parity-time (PT) symmetry have been verified possessing rogue wave solutions [35–54]. Two recent articles [55, 56] have provided comprehensive overviews on various rogue-wave phenomena from the physical point of view.

Compared with one-dimensional nonlinear systems, the higher dimensional nonlinear systems are less studied mainly because of the non-availability of analytical methods like in one dimension. However, from the realistic and physical point of views, the extension of rogue waves studies to higher dimensions is essential because the physical fields are modeled by multi-dimensional systems, such as in the studies of ocean waves and ultrafast nonlinear optics. In particular, ultrafast optical rogue waves are also higher dimensional, because the spatial and temporal degrees of freedom cannot be treated separately in the theoretical and experimental studies of self-focusing of intense ultrashort pulses [57–64]. Up to now, different types of rogue waves occurring in a series of multi-dimensional systems have been investigated [65–72].

Nonlinear magnetization dynamics of the Heisenberg ferromagnetic spin chain with different magnetic interactions in classical and semiclassical limit has been associated with soliton theory and condensed matter physics [73–75]. Nonlinear spin excitations in the magnetic materials have their applications in the microwave communication systems and nonlinear signal processing devices [76, 77]. The dynamics of the nonlinear spin excitations in the Heisenberg ferromagnetism can be described by the nonlinear Schrödinger-type equations [78]. Theoretically, the nonlinear dynamics of the (2 + 1)-dimensional ferromagnetic spin systems with bilinear and bi-quadratic interactions in the semiclassical limit has been discussed, which has been described by integrable (2 + 1)-dimensional nonlinear Schrödinger (NLS) equations [79–84]. In this paper, we will consider a (2 + 1)-dimensional NLS equation

\[ iu_t - iu_x + u_{xx} + u_{yy} - 2u_{xy} + 2\vert u \vert^2u = 0, \]  

(1)

for the (2 + 1)-dimensional Heisenberg ferromagnetic spin chain with bilinear and anisotropic interactions in the semiclassical limit. Here \( u(x, y, t) \) is a complex-valued function, \( x, y, \) and \( t \), respectively, denote the scaled spatial and time coordinates.

For the (2 + 1)-dimensional NLS equation (1), the Lax pair has been constructed, and the bright soliton solutions have been obtained by virtue of the Hirota method [78] and Darboux transformation [82], and dark soliton interactions have also been discussed [84]. To the best of our knowledge, general high-order rogue waves
have not been investigated for the $(2+1)$-dimensional NLS equation (1).

In this work, we obtain parallel line rogue waves for the $(2+1)$-dimensional NLS equation (1), which are expressed in term of determinants by employing the Hirota’s bilinear method [15]. The basic idea is to treat the $(2+1)$-dimensional NLS equation (1) as a constrained KP hierarchy, then rational solutions of the equation (1) are reduced from tau functions of the KP hierarchy [18, 85].

The outline of the paper is organized as follows. In Sec. 2, the exact and explicit rational solutions of the $(2 + 1)$-dimensional NLS equation (1) are presented in the determinant form by using the Hirota bilinear method. In Sec. 3, the dynamics of rogue wave solutions is analyzed and illustrated graphically. Section 4 contains a summary and discussion.

2. ROGUE WAVE SOLUTIONS VIA DETERMINANTS OF $N \times N$ MATRICES

In this Section, we derive an explicit form of the rogue wave solutions for the $(2 + 1)$-dimensional NLS equation (1). Using the dependent variable transformation

$$u = e^{2it} \frac{g}{f},$$

the $(2 + 1)$-dimensional NLS equation can be transformed into the bilinear forms

$$(iD_t + D_x^2 + D_y^2 - 2D_x D_y)g \cdot f = 0,$$

$$(D_x^2 + D_y^2 - 2D_x D_y)f \cdot f = 2(2g^* - f^2).$$

Here, $f$ is a real function, $g$ is a complex function, the asterisk denotes complex conjugation, and the operator $D$ is the Hirota’s bilinear differential operator [15] defined by

$$P(D_x, D_y, D_t) F(x, y, t, \cdots) \cdot G(x, y, t, \cdots) = P(\partial_x - \partial_{x'}, \partial_y - \partial_{y'}, \partial_t - \partial_{t'}, \cdots) F(x, y, t, \cdots) G(x', y', t', \cdots)|_{x=x', y'=y', t'=t'},$$

where $P$ is a polynomial of $D_x, D_y, D_t, \cdots$.

**Theorem 1.** The $(2 + 1)$-dimensional NLS equation (1) has rogue wave solutions (2) with $f$ and $g$ given by $N \times N$ determinants

$$f = \tau_0, \quad g = \tau_1,$$

where $\tau_n = \det_{1 \leq i,j \leq N}(m_{2i-1,2j-1}^{(n)})$, and the matrix elements are given by

$$m_{i,j}^{(n)} = \sum_{k=0}^{i} \frac{a_k}{(i-k)!} (p \partial_p + \xi + n)^{i-k} \sum_{l=0}^{j} \frac{a_l^*}{(j-l)!} (p^* \partial_{p^*} + \xi^* - n)^{j-l} \frac{1}{p + p^*}|_{p=1},$$

(5)
and $\xi' = \frac{2}{p}y + px + 2(ip^2 + p)t$, $i, j$ are arbitrary positive integers, $a_k$ and $a_l$ are arbitrary complex constants.

By a scaling of $m_{ij}$, we can normalize $a_0 = 1$ without loss of generality, thus hereafter we set $a_0 = 1$ in this paper. Note that these rational solutions can also be expressed in terms of Schur polynomials as discussed in [37, 65]. Next we give a short proof of Theorem 1.

**Lemma 1.** The bilinear equation in the KP hierarchy

\[
\begin{align*}
(D_{x_1}^2 - D_{x_2})\tau_{n+1} \cdot \tau_n &= 0, \\
(D_{x_{n+1}}D_{x_1} - 2)\tau_n \cdot \tau_n &= -2\tau_{n+1} \cdot \tau_n, \\
\end{align*}
\]

has the Gram determinant solutions

\[
\tau_n = \det_{1 \leq i, j \leq N} (m_{ij}^{(n)}),
\]

with the matrix element $m_{ij}^{(n)}$ satisfying the following differential and difference relations,

\[
\begin{align*}
\partial_{x_1} m_{ij}^{(n)} &= \psi_i^{(n)} \phi_j^{(n)}, \\
\partial_{x_2} m_{ij}^{(n)} &= \psi_i^{(n+1)} \phi_j^{(n)} + \psi_i^{(n)} \phi_j^{(n-1)}, \\
\partial_{x_{n+1}} m_{ij}^{(n)} &= -\psi_i^{(n-1)} \phi_j^{(n+1)}, \\
m_{ij}^{(n+1)} &= m_{ij}^{(n+1)} + \psi_i^{(n)} \phi_j^{(n+1)}, \\
\partial_{x_v} \psi_i &= \psi_i^{(n+v)}, \\
\partial_{x_v} \phi_j &= \phi_j^{(n-v)} \quad (v = -1, 1, 2).
\end{align*}
\]

This Lemma can be proved in a similar way as the proof of the Lemma 1 of Refs. [36, 85], thus we omit the proof of this Lemma in this paper. Next we use this Lemma to prove Theorem 1.

**Proof of Theorem 1.** To get rational solutions for the $(2 + 1)$-dimensional NLS equation, we choose functions $m_{ij}^{(n)}$, $\psi_i^{(n)}$ and $\phi_j^{(n)}$ as the following formulas

\[
\begin{align*}
\psi_i^{(n)} &= A_i p^n e^\xi, \\
\phi_j^{(n)} &= B_j (-q)^{-n} e^n, \\
m_{ij}^{(n)} &= A_i B_j \frac{1}{p + q} (-p/q)^n e^{\xi + \eta},
\end{align*}
\]

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where
\[ A_i = \sum_{k=0}^{i} \frac{a_k}{(i-k)!} (p\partial_p)^{i-k}, \quad B_j = \sum_{l=0}^{j} \frac{b_l}{(j-l)!} (q\partial_q)^{j-l}, \]
\[ \xi = \frac{1}{p} x_{-1} + px_1 + p^2 x_2, \quad \eta = \frac{1}{q} x_{-1} + qx_1 - q^2 x_2, \]

and for simplicity, the functions \( m_{ij}^{(n)} \) can be rewritten as
\[ m_{ij}^{(n)} = e^{\xi + \eta} \frac{1}{p^2} \sum_{k=0}^{n_i} \frac{a_k}{(i-k)!} (p\partial_p + \xi + n)^{n_i-k} \sum_{l=0}^{n_j} \frac{b_l}{(j-l)!} (q\partial_q + \eta + n)^{n_j-l} \frac{1}{p+q}. \] (10)

where
\[ \xi' = -\frac{1}{p} x_{-1} + px_1 + 2p^2 x_2, \quad \eta' = -\frac{1}{q} x_{-1} + qx_1 - 2q^2 x_2. \]

Here \( p, q, a_k, b_l \) are arbitrary complex constants, and \( i, j, n_i, N \) are arbitrary positive integers.

Further, taking the parameter constraints
\[ q = p^* = 1, \quad b_k = a_k^*, \] (11)
and assuming \( x_1, x_3 \) are real, \( x_2 \) is pure imaginary, we have
\[ \eta' = \xi'^*, \quad m_{ij}^*(n) = m_{ji}(-n), \quad \tau^*_n = \tau_{-n}. \] (12)

What is more, under parameter constraints (11), the \( \tau_n \) satisfies the reduction condition
\[ (\partial_{x_1} + \partial_{x_{-1}})\tau_n = 4N\tau_n \] (13)
Applying the change of independent variables
\[ x_{-1} = -y, x_1 = 2x + 2t, x_2 = it, \]
and taking
\[ \tau(0) = f, \tau(1) = g, \tau(-1) = g^*, \]
the bilinear equation (6) can be transformed into the bilinear equation (3). Finally, using the gauge freedom of \( \tau_n \), it is easy to get the rational solutions of the \((2+1)\)-dimensional NLS equation (1) from the solutions of equation (6), which are given in Theorem 1. Thus the Theorem 1 has been proved.

Below we concentrate on commenting that the obtained solutions are nonsingular by using equation (8), (9), and (11). We note that \( f = \tau_0 \) is given by the determinant
\[ f = \det_{1 \leq i, j \leq N} (m_{2i-1,2j-1}(0)). \] (14)
Indeed, for any non-zero column vector \( \mu = (\mu_1, \mu_2, \ldots, \mu_N)^T \) and \( \overline{\mu} \) being its complex transpose, we have

\[
p f \mu = \sum_{i,j=1}^{N} \overline{\mu_i} \mu_j A_{2i-1} B_{2j-1} - \frac{1}{p + q} e^{\xi + \eta} |_{\eta = p^*}
\]

\[
= \sum_{i,j=1}^{N} \overline{\mu_i} \mu_j A_{2i-1} B_{2j-1} \int_{-\infty}^{\infty} e^{\xi + \eta} dx |_{\eta = p^*}
\]

\[
= \int_{-\infty}^{\infty} (\sum_{i,j=1}^{N} \overline{\mu_i} \mu_j A_{2i-1} B_{2j-1} e^{\xi + \eta} |_{\eta = p^*}) dx
\]

\[
= \int_{-\infty}^{\infty} (\sum_{i=1}^{N} \overline{\mu_i} e^{\xi} |^2 dx > 0,
\]

thus we have proved that \( f \) is positive definite. Therefore, the rational solutions \( u \) given in Theorem 1 are non-singular.

3. DYNAMICS OF ROGUE WAVES IN THE \((2 + 1)\)-DIMENSIONAL NLS EQUATION

In this Section, we present the analysis of the dynamics of the rogue wave solutions of the \((2 + 1)\)-dimensional NLS equation (1) given in Theorem 1.

3.1. FUNDAMENTAL ROGUE WAVE SOLUTIONS

As the simplest rational solution, the one-rational solution of first order is given by taking \( N = 1 \) in Theorem 1,

\[
f = \left( \sum_{k=0}^{1} \frac{a_k}{(1-k)!} (p \partial p + \xi')^{1-k} \sum_{l=0}^{1} \frac{a_k^*}{(1-k)!} (p^* \partial p^* + \xi'^*)^{1-l} \right) \frac{1}{p + p^*}
\]

\[
= (p \partial p + \xi + a_1)(p^* \partial p^* + \xi'^* + a_1^*) \frac{1}{p + p^*},
\]

\[
g = \left( \sum_{k=0}^{1} \frac{a_k}{(1-k)!} (p \partial p + \xi' + 1)^{1-k} \sum_{l=0}^{1} \frac{a_k^*}{(1-k)!} (p^* \partial p^* + \xi'^* - 1)^{1-l} \right) \frac{1}{p + p^*},
\]

\[
= (p \partial p + \xi + a_1 + 1)(p^* \partial p^* + \xi'^* + a_1^* - 1) \frac{1}{p + p^*},
\]

\[
= \frac{1}{p + p^*} [(\xi' - \frac{p}{p + p^*} + a_1 + 1)(\xi'^* - \frac{p^*}{p + p^*} + a_1^* - 1) + \frac{pp^*}{(p + p^*)^2}],
\]

(16)
where $p = 1$, and $a_1$ is a freely complex constant. After a shift of time and space coordinates, $a_1$ can be eliminated and the fundamental rogue wave can be written as

$$u = e^{2i\gamma}[1 - \frac{16i\gamma + 4}{(4x + 2y + 4\gamma + 1)^2 + 16\gamma^2 + 1}].$$  \hspace{1cm} (17)$$

It follows that $|u|$ has three extreme lines given by

\begin{align*}
L_1 : y &= -2x - 2\gamma + \frac{1}{2}, \\
L_2 : y &= -2x - 2\gamma + \frac{1}{2} + \frac{1}{2}\sqrt{48\gamma^2 + 3}, \\
L_3 : y &= -2x - 2\gamma + \frac{1}{2} - \frac{1}{2}\sqrt{48\gamma^2 + 3}. 
\end{align*}  \hspace{1cm} (18)$$

Based on the analysis of critical values for rational solution $|u|$ (17), the maximum amplitude of $|u|$ is given by

$$|u|_{\text{max}} = |u|_{L_1} = \sqrt{1 + \frac{8}{16\gamma^2 + 1}}$$

and the minimum amplitude of $|u|$ is given by

$$|u|_{\text{min}} = |u|_{L_2} = |u|_{L_3} = \sqrt{1 - \frac{1}{16\gamma^2 + 1}}.$$

We define $L_{\text{width}}$ as a distance between $L_2$ and $L_3$, thus

$$L_{\text{width}} = \sqrt{\frac{48\gamma^2}{5}}.$$

It is not easy to find that $|u|_{\text{max}} \to |u|_{\text{min}} = 1$ and $L_{\text{width}} \to \infty$ when $|\gamma| \to \infty$, which means that the rational solution $u$ defined in (17) approaches to a constant background as $|\gamma|$ goes to $\infty$, and has a large amplitude for only a short period of time. Thus solution $u$ defined in (17) is a line rogue wave. This solution is shown in Fig. 1. As can be seen, this solution possesses a line profile with a varying height, which is different from the moving line solitons of the multi-dimensional soliton equations. Line solitons maintain a perfect profile without any decay during their propagation in the $(x, y)$-plane, but the solution $|u|$ approaches the constant background when $|\gamma| \to 0$, whereas at intermediate times it reaches a much higher amplitude. Note that $|u|$ reaches the maximum amplitude $3$ (i.e., three times the background amplitude) along the line $L_1$, and the minimum amplitude $0$ along lines $L_2$ and $L_3$ at $\gamma = 0$.

The above discussion focused on the fundamental rogue wave solutions of the $(2 + 1)$-dimensional NLS equation. Below we will discuss the high-order rogue wave solutions corresponding to those given in Theorem 1, for $N \geq 2$. 
3.2. HIGH-ORDER ROGUE WAVE SOLUTIONS

For an arbitrary given value of $N$, the $N$th-order rogue waves for the $(2+1)$-dimensional NLS equation (1) can be generated by using the results of Theorem 1. These rogue wave solutions have different dynamics in the $(x,y)$-plane and in the $(x,t)$-plane or $(y,t)$-plane. In the $(x,y)$-plane, the $N$th-order rogue waves consist of $N$ individual fundamental line rogue waves, and these line rogue waves are parallel lines. That is different from nonfundamental rogue waves in the Davey-Stewartson (DS) equations [65, 66] and Fokas systems [70], since the latter are no longer line waves. In the $(x,t)$-plane or $(y,t)$-plane, the $N$th-order rogue waves are made up of $\frac{N(N+1)}{2}$ localized waves, and these localized waves feature as $(1+1)$-dimensional rogue waves.

To demonstrate high-order rogue waves, we first consider the case of $N = 2$. 

Fig. 1 – (Color online) A fundamental line rogue wave $|u|$ given by (17), for the $(2+1)$-dimensional NLS equation (1), plotted in the $(x,y)$-plane. Here the parameter $a_1 = 0$. 

$$t = -5$$ $$t = -1$$ $$t = 0$$ $$t = 5$$
In this case, the explicit form of the second-order rogue wave solutions is

$$u = e^{2it} \begin{vmatrix} m^{(1)}_{11} & m^{(1)}_{13} \\ m^{(0)}_{11} & m^{(0)}_{13} \\ m^{(1)}_{31} & m^{(1)}_{33} \\ m^{(0)}_{31} & m^{(0)}_{33} \end{vmatrix},$$

(19)

where $m^{(n)}_{ij}$ are given by (5). With parameter choices

$$a_0 = 1, a_1 = 0, a_2 = 0, a_3 = -\frac{1}{12},$$

(20)

the final expression of this solution is given as

$$u = e^{2it} (1 - \frac{\phi_2}{f_2}),$$

(21)

where

$$f_2 = 4096t^6 + 12288t^5x + 6144t^4y + 18432t^4xy + 4608t^4y^2 + 16384t^3x^3 + 24576t^3x^2y + 12288t^3xy^2 + 2048t^3y^3 + 9216t^2x^4 + 18432t^2x^3y + 13824t^2x^2y^2 + 4608t^2xy^3 + 576t^2y^4 + 3072tx^5 + 7680tx^4y + 7680tx^3y^2 + 3840tx^2y^3 + 960txy^4 + 96ty^5 + 512x^6 + 1536x^5y + 1920x^4y^2 + 1280x^3y^3 + 480x^2y^4 + 96xy^5 + 8y^6 - 3072t^6 - 9216t^5x - 4608t^4y - 12288t^3x^2 - 3072t^3y^2 - 9216t^2x^3 - 13824t^2x^2y - 9012t^2xy^2 - 1152t^2y^3 - 3840tx^4 - 7680tx^3y - 5760tx^2y^2 - 1920txy^3 - 240tx^4y - 768x^5 - 1920x^4y - 1920x^3y^2 - 960x^2y^3 - 240xy^4 - 24y^5 + 1536t^4 + 2304tx^3 + 1152t^3y + 3456t^2x^2 + 3456t^2xy + 864t^2y^2 + 2304tx^3 + 3456tx^2y + 1728txy^2 + 288ty^3 + 576x^4 + 1152x^3y + 864x^2y^2 + 288xy^3 + 36y^4 - 64t^3 - 576t^2x - 288t^2y - 768tx^2 - 768txy - 192ty^2 - 256x^3 - 384x^2y - 192y^2x - 32y^3 + 288t^2 + 240tx + 120y + 120x^2 + 120xy + 30y^2 - 36tx - 36x - 18y + 5,$$

$$\phi_2 = -6 + 96x + 96t - 24y - 384x^4 + 48y - 72x^2 - 288x^3 - 864t^2 + 48y^3 + 384x^3 + 1536t^3 - 4608t^2 + 288t^2y^2 + 288txy - 288y - 3072t^3y - 768x^3y - 576tx + 1152tx^2 - 1536tx^3 - 6144t^3x - 4608t^2x^2 + 2304tx^2 + 1152t^3y - 576x^2y - 1152tx^2 - 576x^2y^2 - 192ty^3 + 192txy^2 - 1152txy^2 - 4608t^2xy - 2304tx^2y - 12288txy^3 - 9216tx^2y^3 + 3072tx^2y^2 - 768txy^3 + 4608t^3xy^2 + 2304t^2xy^2 + 1152t^3xy^2 - 12288t^2y^3 - 6144t^2y^3 - 3072t^3y^2 - 6144t^2y^3 - 768ty^4 - 1536tx^4 + 96ty^4 + 6144t^3x + 3072t^3y + 4608t^2x^2 + 1152t^3y^2 + 1536tx^3 + 192tx^3 - 96ty^4 - 6144t^3x + 3072t^3y + 4608t^2x^2 + 1152t^3y^2 + 1536tx^3 + 192tx^3 - 9216ty + 6144t^3x + 3072t^3y - 384t^2 - 192t^2 - 120t.$$

This solution is shown in Fig. 2. It is seen that when two parallel line rogue waves arise from the constant background in the $(x, y)$-plane, the region of their intersection acquires higher amplitude first (see the panel at $t = -2$). At the intermediate time, the superposition of two parallel line rogue waves generates one main
peak posing maximum amplitude and several lower peaks (see the panel at $t = 0$). At larger time, the two parallel line rogue waves decay back to the constant background (see the panel at $t = 5$). It is noticed that the maximum of $|u|$ is equal to 5 ($i.e.$, five times the height of the background), which is much higher than the second-order rogue waves in the DS systems [65, 66], since the latter does not exceed four times the constant background for all the time.
The second-order rogue wave solution in the $(x,t)$ plane are shown in Fig. 3. Visually, this solution consists of three fundamental rogue waves as second-order rogue waves in $(1+1)$-dimensional systems. The interaction of these three rogue waves can also generate some common wave patterns, such as the fundamental pattern (see panel (a) of Fig. 3), and the triangular pattern (see panel (b) of Fig. 3).

Next, we proceed to consider more complicated cases of Theorem 1. For instance, with $N = 3$, and parameter choices

$$a_0 = 1, a_1 = 0, a_2 = 0, a_3 = -\frac{1}{12}, a_4 = 0, a_5 = -\frac{1}{240},$$

the corresponding solutions are shown in Figs. 4 and 5. Comparing to the second-order rogue wave shown in Figs. 2 and 3, wave patterns of the third-order rogue wave are more complicated. In the $(x,y)$-plane, the third-order rogue wave consist of three individual fundamental line rogue wave, which arise from the constant background and then decay back to the constant background at larger time, see Fig. 4. It is noticed that in this whole process, these three fundamental line wave keep parallel, and the maximum value of solution $|u|$ is equal to $7$ (i.e., seven times the height of the background). Thus the interaction between these three line rogue waves can generate very high peaks. In the $(x,t)$-plane, the third-order rogue wave for the $(2+1)$-dimensional NLS equation are composed by 6 fundamental localized waves as third-order rogue waves in $(1+1)$-dimensional soliton equations, see Fig. 5. Three basic patterns, namely the fundamental pattern, the triangular pattern, and the circular (ring) pattern, are also exhibited (see panels (a), (b), and (c) of Fig. 5, respectively).

For larger $N$, these high-order waves have qualitatively similar behaviors, besides that more line rogue waves will arise and interact with each other, and the transitional profiles of these solutions become much more complicated. Motivated
Fig. 4 – (Color online) The third-order rogue wave \(|u|\) of the \((2+1)\)-dimensional NLS equation (1), with the parameters \(N = 3, a_1 = 0, a_2 = 0, a_3 = -\frac{1}{12}, a_4 = 0, a_5 = -\frac{1}{2007}\), plotted in the \((x,y)\)-plane.

by the key features of rogue waves up to third-order, which are shown in Figs. 1-5, higher-order rogue waves \(|u|\) given in Theorem support the following conjectures:

1) In the \((x,y)\)-plane, the \(N\)th-order parallel line rogue waves consist of \(N\) individual fundamental line rogue waves. The maximum values of \(|u|\) is \(2N + 1\) (i.e., \(2N + 1\) times the height of the background). For the fundamental pattern, the \(N\)th-order rogue wave has \(n(n+1) - 1\) peaks, and the central peak is surrounded by
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Fig. 5 – (Color online) Three patterns of the third-order rogue wave solution $|u|$ of the (2 + 1)-dimensional NLS equation (1), plotted in the $(x,t)$-plane. (a) The fundamental pattern with parameters $N = 3, a_1 = 0, a_2 = 0, a_3 = -\frac{1}{12}, a_4 = 0, a_5 = -\frac{1}{240}, y = 0$. (b) The triangular pattern with parameters $N = 3, a_1 = 0, a_2 = 0, a_3 = 20, a_4 = 0, a_5 = 0, y = 0$. (c) The ring pattern with parameters $N = 3, a_1 = 0, a_2 = 0, a_4 = 0, a_5 = 10i, y = 0$.

$N(N+1) - 2$ gradually decreasing peaks.

(2) In the $(x,t)$-plane or $(y,t)$-plane, the $N$th-order rogue waves are made up of $\frac{N(N+1)}{2}$ rogue waves as in $(1 + 1)$-dimensional soliton equations. They also possess some common patterns, namely fundamental pattern, triangular, and circular patterns. For the triangular and circular patterns, there are $N(N+1)/2$ uniform peaks. For the circular pattern, an order-$N$ rogue wave displays a ring structure, the outer ring has $2N - 1$ uniform peaks, and the inner structure is an order-$(N-2)$ rogue wave.

4. SUMMARY AND DISCUSSION

In summary, we have derived a general formula for the $N$-th order rogue wave solutions of the (2 + 1)-dimensional NLS equation (1) by employing the bilinear transformation method in Theorem 1, in which solutions are expressed explicitly
in terms of determinants. The first-order rogue wave solution is given explicitly in Eq. (17). The asymptotic behaviors of this solution are analyzed, and the typical evolution dynamics is shown in Fig. 1. It is seen that the fundamental rogue wave is a line rogue wave that arises from the constant background and then disappears into the constant background. The explicit form of the second-order rogue wave solution is given in Eq. (21). Figures 2 and 3 display the dynamical features of the second-order rogue waves, which consist of three parallel line rogue waves. The third-order rogue waves are composed of three parallel line rogue waves in the \((x, y)\)-plane, and these line waves just exist on the constant background for a short period of time, see Fig. 4. The dynamics of the third-order rogue waves in the \((x, t)\)-plane is shown in Fig. 5. The main features of higher-order rogue waves are summarized as follows:

(1) In the \((x, y)\)-plane, the \(N\)th-order rogue waves consist of \(N\) parallel line rogue waves.

(2) In the \((x, t)\)-plane or the \((y, t)\)-plane, the \(N\)th-order rogue waves are composed of \(\frac{N(N+1)}{2}\) localized waves as in \((1+1)\)-dimensional soliton equations.

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