FURTHER SOLUTIONS OF THE FALKNER-SKAN EQUATION

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Abstract. In this paper, we propose a reliable treatment for the Falkner-Skan equation, which can be described as the non-dimensional velocity distribution in the laminar boundary layer over a flat plate. We propose an algorithm of two steps that will introduce an exact solution to the equation, followed by a correction to that solution. Three different special cases: Hiemenz flow (\(\beta = 1\)), Homann flow (\(\beta = \frac{1}{2}\)), and Blasius problem (\(\beta = 0\)) have been considered. When the pressure gradient parameter \(\beta\) takes sufficiently large values, we use a transformation of variable that reduces the Falkner-Skan equation into an equivalent boundary value problem in a finite domain, and we solve this problem by a different technique that allows us to find \(f''(0)\). Also, various exact solutions can be obtained in a straightforward manner by using a direct method when \(\beta = -1\). The new technique, as presented in this paper, has been shown to be very efficient for solving the Falkner-Skan equation.

Key words: Falkner-Skan equation; Hiemenz flow; Homann flow; Blasius problem; approximate solution; exact solution.

1. INTRODUCTION

The Falkner-Skan equation describes the class of the so-called similar laminar flows in boundary layer on a permeable wall, and at varying main-stream velocity \([1, 2]\). This well-known nonlinear third-order differential equation is given by

\[ f''' + f'' + \beta \left(1 - (f')^2\right) = 0, \quad 0 < \eta < \infty, \]  

subject to the boundary conditions

\[ f(0) = \gamma, \quad f'(0) = 0 \]  

and

\[ f'(\infty) = 1, \]  

where \(f\) is a stream function, \(\eta\) is a distance from the wall, which is called similarity variable, \(f'\) defines the velocity component in \(\eta\)-direction, \(f''\) defines the shear stress in the boundary layer.

The mass-transfer parameter \(\gamma\) in the boundary condition sets the measure for the Romanian Journal of Physics 63, 102 (2018)
mass flow rate through the wall boundary in either direction. Positive values de-
termine flows with suction, negative with blowing through the wall boundary. The
zero value corresponds to flow along impermeable wall with zero mass transfer. The
numerical parameter $\beta$ (positive or negative) in the Falkner-Skan equation sets a
degree of acceleration or deceleration of main stream [3]. The Falkner-Skan equation
(1), together with (2)-(3), constitute the theoretical basis of convective heat and mass
transfer. The existence and uniqueness question for the problem (1)-(3) was given
by [4] and [5]. Weyl [6] proved that for each value of the parameter $\beta$ there exists
a physical solution with a positive monotonically decreasing, in $[0,1)$, second deriva-
tive that approaches zero as the independent variables goes to infinity. In the case
$\beta > 1$, Craven and Pelietier [7] computed solutions for which $\frac{df}{d\eta} < 0$ for some values
of $\eta$. Coppel [8] presented an elegant proof of the existence and uniqueness for $\beta > 0$
and showed that the wall shear stress $f''(0)$ is an increasing function of $\beta$. In the
works [9, 10], the extension for $\beta < 0$ was discussed and properties of solutions were
investigated further. New theoretical results for the solutions of the Falkner-Skan
equation were given in Refs. [10] and [11]. Different methods have been proposed
to solve this problem: finite difference [9, 12–14], shooting [15–20], finite element
[21], group invariance theory [22, 23], Chebyshev spectral method [24, 25], diffe-
rential transformation method [26], non-iterative transformation method (ITM) [27],
homotopy analysis method [28], and Adomian decomposition method (ADM) [29].
Other direct methods and numerical techniques to study nonlinear differential equa-
tions of physical relevance have been recently reported [30–32].

To our knowledge, a general analytic solution of Eq. (1) satisfying the bound-
ary conditions (2)-(3) does not exist. The purpose of this work is to present an al-
gorithm of two steps that will introduce approximate solutions to the Falkner-Skan
equation followed by a correction to that solution. Furthermore, we will consider
three special cases: Hiemenz flow ($\beta = 1$), Homann flow ($\beta = \frac{1}{2}$) and Blasius pro-
b lem ($\beta = 0$). We use a transformation of variables that reduces (1)-(3) into an
equivalent boundary value problem (BVP) in a finite domain. Finally, we solve this
problem by a different technique that allows us to find $f''(0)$, when the pressure gra-
dient parameter $\beta$ takes sufficiently large values. Also, various exact solutions can
be obtained in a straightforward manner by using a direct method when $\beta = -1$. The
results lead to very accurate approximate solutions. Listed below are some special
cases when Eqs. (1)-(3) are solvable.
2. CASE 1: LARGE VALUES OF $\beta$

The Falkner-Skan can be written in an equivalent form utilizing $m$ as the Falkner-Skan pressure parameter instead of $\beta$:

$$f''' + ff'' + \frac{2m}{m+1} (1 - (f')^2) = 0,$$

(4)

where $0 \leq m \leq \infty$. The authors have considered only the case of $0 \leq \beta \leq 2$. For large values of $m \to \infty$, Eq. (1) becomes

$$f''' + ff'' + 2 (1 - (f')^2) = 0.$$

(5)

This case has been studied in Ref. [5]. One of the main problems appears when $\beta \to \infty$.

In this section, we will present a direct approach to solve this problem when the pressure gradient parameter $\beta$ takes sufficiently large values, i.e. $\beta \to \infty$.

Let us consider the following transformation [33]:

$$z(x) = f''(\eta) \text{ and } x = f'(\eta).$$

(6)

Differentiating Eq. (1) with respect to $\eta$, we get

$$f''' + ff'' + (1 - 2\beta) f' f'' = 0$$

(7)

and from Eq. (6), we have

$$z(x) = \frac{dx}{d\eta},$$

(8)

$$z'(x) = \frac{dz}{dx} = \frac{f''}{f'} = -f - \beta \frac{1 - f'^2}{f''}$$

(9)

and

$$z''(x) = \frac{d^2z}{dx^2} = \frac{f'''}{f'f''} - \frac{f''^2}{f''^3}.$$  

(10)

Substituting these equations into Eq. (7), we obtain

$$z''(x) - \beta (1 - x^2) \frac{z'(x)}{z^2(x)} + (1 - 2\beta) x \frac{1}{z(x)} = 0.$$  

(11)

Since $f'(\infty) = 1$ and $f(0) = 0$, dealing, for example, with a simple case $\gamma = 0$, we have $z(1) = f''(\infty) = 0$ and $z'(0) = -\beta$ and $z(0) = \sigma$, where $f''(0) = \sigma$. Hence, the initial-boundary value problem for the Falkner-Skan equation can be transformed into the new problem

$$
\begin{cases}
    z''(x) - \beta (1 - x^2) \frac{z'(x)}{z^2(x)} + (1 - 2\beta) x \frac{1}{z(x)} = 0, & 0 < x < 1, \\
    z(0) = \sigma, z'(0) = -\frac{\beta}{\sigma}, z(1) = 0.
\end{cases}
$$

(12)
Multiplying both sides of the first-equation of (12) by \( \xi(x) = e^{\int \frac{(1-2\beta)x}{\beta(1-x^2)} \, dx} \) and taking into account that \( \xi'(x) = \frac{(1-2\beta)x}{\beta(1-x^2)} \xi(x) \), we obtain

\[
\xi(x) \frac{d^2 z}{dx^2} + \beta (1-x^2) \left( \frac{1}{z(x)} \right)' = 0. 
\tag{13}
\]

We have

\[
\xi(x) = e^{\int \frac{(1-2\beta)x}{1-x^2} \, dx} = (1-x^2)^{\frac{1}{2\beta}}. 
\tag{14}
\]

Thus

\[
z''(x) + \beta (1-x^2)^{\frac{1}{2\beta}} \left( \frac{1}{z(x)} \right)' = 0. 
\tag{15}
\]

The function \((1-x^2)^{\frac{1}{2\beta}}\) can be approximated by 1 for sufficiently large values of \( \beta \), and then \( \xi(x) \approx 1-x^2 \).

More precisely, we obtain the approximated equation

\[
z''(x) + \beta (1-x^2)^{\frac{1}{2\beta}} \left( \frac{1}{z(x)} \right)' = 0. 
\tag{16}
\]

Integrating Eq. (16) from 0 to \( x \) and taking into account that \( z'(0) = -\beta \) and \( z(0) = 0 \), we get

\[
z'(x) + \beta \xi(x) \frac{1}{z(x)} = 0, 
\tag{17}
\]

which can be written as follows

\[
(z^2(x))' = -2\beta \xi(x). 
\tag{18}
\]

Integrating Eq. (18) from 0 to \( x \) and taking into account that \( z(0) = 0 \), we get

\[
z^2(x) = -2\beta \int_0^x \xi(x) \, dx + \sigma^2. 
\tag{19}
\]

Consequently,

\[
z^2(x) = -2\beta (x - \frac{x^3}{3}) + \sigma^2. 
\tag{20}
\]

Thus, it is required to find \( \sigma \) by the given condition \( z(1) = 0 \). Then we have

\[
\sigma^2 = \frac{4}{3} \beta. 
\tag{21}
\]

Thus

\[
f''(0) = \sigma = \pm 2 \sqrt{\frac{\beta}{3}}. 
\tag{22}
\]

It follows that

\[
z(x) = \pm \sqrt{-2\beta (x - \frac{x^3}{3}) + \frac{4}{3} \beta}. 
\tag{23}
\]
3. CASE 2: HOMANN FLOW ($\beta = \frac{1}{2}$)

For $\beta = \frac{1}{2}$, Eq. (12) becomes

\[
\begin{align*}
\begin{cases}
  z''(x) - \frac{1}{2} (1 - x^2) \frac{z'(x)}{z(x)} &= 0, \\
  z(0) = \sigma, z'(0) = -\frac{\beta}{\sigma}, z(1) &= 0.
\end{cases}
\end{align*}
\]

(24)

We start first by noting that the assumption

\[z(x) = k,\]

(25)

where $k$ is a parameter to be determined, satisfies the first equation of (24). Using the boundary condition $z(0) = \sigma$, we obtain $k = \sigma$. However, this solution does not satisfy both boundary conditions $z'(0) = -\frac{\beta}{\sigma}$ and $z(1) = 0$. In view of this, we will employ the given conditions together with (25) to achieve the second goal of our approach by making a correction to (25). We do this by writing the first equation of (24) in an equivalent form as

\[z''(x) = \frac{1}{2} (1 - x^2) \frac{z'(x)}{z^2(x)}.\]

(26)

Inserting the obtained solution (25) in the right hand side (RHS) of Eq. (26), we obtain

\[z(x) = ax + b,\]

(27)

where $a$ and $b$ are two constants of integration. Taking into account the boundary conditions $z(0) = \sigma$, $z'(0) = -\frac{\beta}{\sigma} = -\frac{1}{2\sigma}$, $z(1) = 0$, we immediately find that

\[a = -\frac{1}{2\sigma}, \quad b = \sigma\]

(28)

with $\sigma = \frac{1}{\sqrt{2}} = 0.7071$. The value of the skin friction for Homann stagnation was determined to be $\sigma = 0.927682$ [15].

Returning to the original transformation, we get

\[f''(\eta) = af'(\eta) + b,\]

(29)

so that

\[f'(\eta) = ce^{a\eta} - \frac{b}{a}\]

(30)

and

\[f(\eta) = \frac{c}{a}e^{a\eta} - \frac{b}{a}\eta + d,\]

(31)

where $c$ and $d$ are two constants of integration. Using the boundary conditions (2)-(3), we obtain $c = -2\sigma^2$ and $d = -4\sigma^3$.

Consequently, the approximate solution of the Homann flow problem is given as

\[f_p(\eta) = 4\sigma^3 e^{-\frac{1}{2\pi} \eta} - 2\sigma^2 \eta - 4\sigma^3, \quad \sigma = 0.7071.\]

(32)
3.1. THE ERROR REMAINDER

An error analysis $ER(\eta)$ for $f_p(\eta)$ is considered in Table 1, where the residual $ER(\eta)$ or error remainder is calculated for different values of $\eta$ by substituting $f_p(\eta)$ for $f(\eta)$ in Eq. (1) when $\beta = \frac{1}{2}$:

$$ER_1(\eta) = -\frac{1}{2} e^{-\frac{1}{2} \eta} + \sigma e^{-\frac{1}{2} \eta} \left( 4\sigma^3 e^{-\frac{1}{2} \eta} + 2\sigma^2 \eta - 4\sigma^3 \right) + \frac{1}{2} \left[ 1 - 4\sigma^4 \left( 1 - e^{-\frac{1}{2} \eta} \right)^2 \right].$$  (33)

This demonstrates that the approximate solution is a good approximation of the actual solution. The complete error analysis is given in Tables 1-4.

4. CASE 3: HIEMENZ STAGNATION FLOW ($\beta = 1$)

We note that the solution of the form

$$f(\eta) = \beta \eta + \lambda,$$  (34)

where $\lambda$ is a parameter to be determined, satisfies Eq. (1). Then we have

$$f'(\eta) = \beta, f''(\eta) = f'''(\eta) = 0.$$  (35)

Inserting this solution into Eq. (1) to get

$$\beta(1 - \beta^2) = 0.$$  (36)

Thus $\beta = -1, 1, 0$. For the Hiemenz flow problem ($\beta = 1$), we have

$$f(\eta) = \eta + \lambda.$$  (37)

Inserting now the boundary condition $f(0) = \gamma$ we get $\lambda = \gamma$.

We write Eq. (1) in an equivalent form as

$$f''' + ff'' = -\left( 1 - (f')^2 \right).$$  (38)

Inserting the obtained solution Eq. (37) in the RHS of Eq. (38), we obtain

$$\frac{f''}{f'} = -f$$  (39)

Integrating Eq. (39) twice, we obtain

$$f'(\eta) = c \sqrt{\pi} e^{\frac{\eta^2}{2}} erf \left( \frac{\eta + \gamma}{\sqrt{2}} \right) + d,$$  (40)

where $erf$ is the error function given by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$  (41)
Consequently,

\[ f_p(\eta) = c \sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} \left[ (\eta + \gamma) er\left(\frac{\eta + \gamma}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} e^{-\frac{(\eta + \gamma)^2}{2}} \right] + d\eta + e, \quad (42) \]

where \(c, d,\) and \(e\) are constants of integration.

Using the boundary conditions \(f'(0) = 0\) and \(f'(\infty) = 1\), we obtain

\[ c \sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} + d = 0 \quad (43) \]

and

\[ c \sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} + d = 1. \quad (44) \]

Solving this system for \(c\) and \(d\), we obtain

\[ c = \frac{1}{\sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} - \sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} er\left(\frac{\eta}{\sqrt{2}}\right)} \quad (45) \]

and

\[ d = \frac{\sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} er\left(\frac{\eta}{\sqrt{2}}\right)}{\sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} er\left(\frac{\eta}{\sqrt{2}}\right) - \sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}}}. \quad (46) \]

The constant \(e\) can be determined from the condition \(f(0) = \gamma\). Thus

\[ e = \gamma - c \sqrt{\frac{\pi}{2} e^{-\frac{\eta^2}{4}}} \left[ \gamma er\left(\frac{\gamma}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} e^{-\frac{\gamma^2}{4}} \right]. \quad (47) \]

Hence the approximate solution of the Hiemenz flow problem \((\beta = 1)\) is given by Eq. (42). Figures 1 and 2 show the approximate solution.

### 4.1. THE ERROR REMAINDER

An error analysis \(ER(\eta)\) for

\[ f_p(\eta) = \eta er\left(\frac{\eta}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} e^{-\frac{\eta^2}{4}} - \sqrt{\frac{2}{\pi}} \quad (48) \]

when \(\gamma = 0\) is considered in Table 3, where the residual \(ER_2(\eta)\) or error remainder is calculated for different values of \(1 \leq \eta < \infty\).

\[ ER_2(\eta) = \sqrt{\frac{2}{\pi}} e^{-\frac{\eta^2}{4}} \left[ -\sqrt{\frac{2}{\pi}} \eta + \eta er\left(\frac{\eta}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} e^{-\frac{\eta^2}{4}} - \sqrt{\frac{2}{\pi}} \right] + 1 - \left( er\left(\frac{\eta}{\sqrt{2}}\right) \right)^2. \quad (49) \]
Fig. 1 – The approximate solution of the Hiemenz flow problem $f_p(\eta)$ when $\gamma = 0$.

Table 1

<table>
<thead>
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<th>$\eta$</th>
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</tr>
<tr>
<td>1</td>
<td>-0.0710278091</td>
</tr>
<tr>
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<td>0.0420318148</td>
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</tr>
<tr>
<td>20</td>
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</tr>
</tbody>
</table>
(a) $\beta = 0$

(b) $\beta = 0.5$

(c) $\beta = 1$

Fig. 2 – The approximate solution compared to numerical solution, $\gamma = 0$. 
(a) $0 \leq \eta \leq 20$

(b) $0 \leq \eta \leq 5$

Fig. 3 – The approximate solution of the Blasius equation $f_p(\eta)$ when $\gamma = 0$ and $\alpha = 0, 0.5, 1, 1.5, 2$. 
(a) $0 \leq \eta \leq 30$

Fig. 4 – The approximate solution of the Blasius equation $f_p(\eta)$ when $\gamma = 0$ and $\alpha = 0, -0.5, -1, -1.5, -2$. 
(a) $0 \leq \eta \leq 30$

![Graph showing $f_p(\eta)$ for different values of $\gamma$ when $\alpha = 0$ and $\gamma = 0, 0.5, 1, 1.5, 2$.](image)

(b) $0 \leq \eta \leq 5$

![Graph showing $f_p(\eta)$ for different values of $\gamma$ when $\alpha = 0$ and $\gamma = 0, 0.5, 1, 1.5, 2$.](image)

Fig. 5 – The approximate solution of the Blasius equation $f_p(\eta)$ when $\alpha = 0$ and $\gamma = 0, 0.5, 1, 1.5, 2$. 
The error remainder \( ER_3(\eta) \) for different value of \( \gamma \).

<table>
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<th>( ER_3(\eta), \gamma=0.5 )</th>
<th>( ER_3(\eta), \gamma=1 )</th>
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The error remainder \( ER_2(\eta) \) for the value of \( \gamma = 0 \).

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<th>( ER_2(\eta) )</th>
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</tr>
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5. CASE 4: BLASIUS FLOW (\( \beta = 0 \))

5.1. \( \gamma \neq 0 \)

For \( \beta = 0 \), we have

\[
    f(\eta) = \gamma.
\]  

(50)

Proceeding as before, inserting the obtained solution Eq. (50) in the RHS of Eq. (39), we obtain

\[
    f(\eta) = \frac{\gamma}{\gamma} e^{\gamma \eta} + d\eta + e, \quad \gamma \neq 0.
\]  

(51)
Table 4

The error remainder $ER_1(\eta)$ for the value of $\gamma = 0$.

<table>
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<td>10</td>
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</tr>
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<td>20</td>
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</tr>
<tr>
<td>50</td>
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</tr>
<tr>
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</tr>
<tr>
<td>200</td>
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</tr>
<tr>
<td>500</td>
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</tr>
<tr>
<td>1000</td>
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</tr>
<tr>
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</tr>
<tr>
<td>5000</td>
<td>0.19179816064 $\times 10^{-4}$</td>
</tr>
</tbody>
</table>

Using the boundary conditions (2)-(3), we obtain

\[ c = \gamma, \quad d = 1, \quad e = \gamma - \frac{1}{\gamma}. \]  \hfill (52)

Consequently, the approximate solution of Eqs. (1)-(3) when $\beta = 0$ is given as

\[ f_p(\eta) = \frac{1}{\gamma} e^{-\gamma \eta} + \eta + \gamma - \frac{1}{\gamma}, \quad \gamma \neq 0. \]  \hfill (53)

If $f'(0) = \alpha$, where $\alpha$ is a constant, then $c = \gamma (1 - \alpha)$. The initial condition $f'(0) = \alpha$ indicates the slip condition at the wall. The case where $\alpha = 0$ represents no-slip.

5.2. $\gamma = 0$

Proceeding as before, we obtain

\[ f(\eta) = \frac{c}{k^2} e^{-k \eta} + d \eta + e. \]  \hfill (54)

Using the boundary conditions (2)-(3), we obtain

\[ c = k, \quad d = 1, \quad e = -\frac{1}{k}. \]  \hfill (55)

Consequently, the approximate solution of Eqs. (1)-(3) when $\beta = 0$ is given as

\[ f_p(\eta) = \frac{1}{k} e^{-k \eta} + \eta - \frac{1}{k}. \]  \hfill (56)
To find the value of $k$, we use the transformation [33]:

$$
\begin{align*}
z(\eta) z''(\eta) + \eta &= 0, \\
z'(0) &= 0, z(1) = 0
\end{align*}
$$

(57)

with $z(0) = \sigma$.

Proceeding as in [34], we define the solution $z(\eta)$ by an infinite series in the form

$$
z(\eta) = \sum_{n=0}^{\infty} C_n \eta^n.
$$

(58)

Thus

$$
z'(\eta) = \sum_{n=0}^{\infty} (n+1)C_{n+1} \eta^n,
$$

(59)

$$
z^2(\eta) = \sum_{n=0}^{\infty} \eta^n \sum_{k=0}^{n} C_k C_{n-k}
$$

(60)

and

$$
z'^2(\eta) = \sum_{n=0}^{\infty} \eta^n \sum_{k=0}^{n} (k+1)(n+1-k)C_{k+1} C_{n+1-k}.
$$

(61)

The substitution into Eq. (57) yields

$$
\sum_{n=0}^{\infty} \eta^n \sum_{k=0}^{n} C_k C_{n-k} = \sigma^2 - \frac{1}{3} \eta^3
$$

$$
-2 \sum_{n=2}^{\infty} \frac{\eta^n}{n(n-1)} \sum_{k=0}^{n-2} (k+1)(n-1-k)C_{k+1} C_{n-1-k}.
$$

(62)
We equate the coefficients of the like powers of $\eta$ on the left side and on the right side to arrive at recurrence relations for the coefficients. Thus

\[
\begin{align*}
C_0^2 &= \sigma^2, \\
\sum_{k=0}^{1} C_k C_{1-k} &= 0, \\
\sum_{k=0}^{2} C_k C_{2-k} &= -C_1^2, \\
\sum_{k=0}^{3} C_k C_{3-k} &= -\frac{1}{3} + \frac{1}{3} \sum_{k=0}^{1} (k+1)(2-k)C_{k+1}C_{2-k}, \\
\sum_{k=0}^{4} C_k C_{4-k} &= +\frac{1}{3} \sum_{k=0}^{2} (k+1)(3-k)C_{k+1}C_{3-k}, \\
\sum_{k=0}^{5} C_k C_{5-k} &= +\frac{1}{6} \sum_{k=0}^{3} (k+1)(4-k)C_{k+1}C_{4-k}, \\
\sum_{k=0}^{6} C_k C_{6-k} &= +\frac{1}{6} \sum_{k=0}^{4} (k+1)(5-k)C_{k+1}C_{5-k}, \\
&\vdots
\end{align*}
\]

The coefficients $C_0, C_1, \ldots$ are

\[
\begin{align*}
C_0 &= \sigma, \\
C_1 &= C_2 = 0, \\
C_3 &= -\frac{1}{6\sigma}, \\
C_4 &= C_5 = 0, \\
C_6 &= -\frac{1}{180\sigma^3}, \\
&\vdots
\end{align*}
\]

We can now compute the truncated decomposition series $z(\eta) = z_0(\eta) + z_1(\eta) + z_2(\eta) + z_3(\eta) + z_4(\eta) + z_5(\eta) + z_6(\eta)$. Consequently, the solution is given by

\[
z(\eta) = \sigma - \frac{\eta^3}{6\sigma} - \frac{\eta^6}{180\sigma^3}.
\]

Thus, it is required to find $\sigma$ by the given condition $z(1) = 0$; then we get

\[
\sigma - \frac{1}{6\sigma} - \frac{1}{180\sigma^3} = 0.
\]
Then, we get $\sigma = 0.4417$, that is $f''_p(0) = 0.44174$. The numerical value of $f''(0)$ by the Runge-Kutta method is given as $f''(0) = 0.46960$. Finally, substituting this value into Eq. (56), we obtain $k = 0.44174$. Consequently, the approximate solution of the initial-boundary value problem for the Blasius equation when $\gamma = 0$, which is characterized by $k = 0.44174$, is given by Eq. (56).

5.3. THE ERROR REMAINDER

The error remainder $ER(\eta)$ is calculated for different values of $\eta$:

5.3.1. Case 1: $\gamma \neq 0$

$$ER_3(\eta) = e^{-2\gamma \eta} + e^{-\gamma}(\eta - 1).$$

(67)

5.3.2. Case 2: $\gamma = 0$

$$ER_4(\eta) = e^{-2k\eta} + e^{-k\eta}(k\eta - 1 - k^2), k = 0.44174.$$

(68)

where the residuals $ER_i(x)$, $i = 3, 4$ or errors remainders are calculated for small and large values of $\eta$, which show that the present solutions are highly accurate. The figures 3-5 have been drawn to show the approximate solutions of the Blasius equation.

6. EXACT SOLUTIONS OF THE FALKNER-SKAN EQUATION WHEN $\beta = -1$

The solutions for the value of $\beta < 0$ represent decelerating flows. We integrate Eq. (1) from 0 to $\eta$, using the fact that

$$ff'' = (ff')' - f'^2.$$  

(69)

The substitution of Eq. (69) into Eq. (1) leads to

$$f'' + (ff')' = 1.$$  

(70)

Integrating Eq. (70) from 0 to $\eta$ and taking into account that $f(0) = \gamma$ and $f'(0) = 0$, we obtain

$$f'' + ff' = \eta + f''(0).$$  

(71)

Consequently Eq. (71) can also be integrated analytically, and we obtain the nonlinear Riccati equation

$$f' + \frac{1}{2}f^2 = \frac{1}{2}\eta^2 + f''(0)\eta + \frac{\gamma^2}{2}.$$  

(72)
The first result in the analysis of Riccati equation (72) is: if one particular solution to Eq. (72) can be chosen as follows $f_1 = \eta + f''(0)$, then $f''(0)$ must satisfy
\[ f''(0)^2 + 2 = \gamma^2, \]  
that is
\[ f''(0) = \gamma^2 - 2, \quad \gamma^2 - 2 \geq 0 \]  
and the general solution is obtained as
\[ f(\eta) = f_1(\eta) + \frac{1}{w(\eta)}, \]  
where $w$ satisfies the corresponding first-order linear equation
\[ w' - f_1 w = \frac{1}{2}, \]  
and we can obtain its closed form solution
\[ w(\eta) = \frac{\sqrt{2} e^{\frac{f''(0)}{2}} \text{erf} \left( \frac{\eta + f''(0)}{\sqrt{2}} \right) + 2C}{2e^{-\frac{\eta^2}{2}} - f''(0)\eta}, \]  
where $\text{erf}$ is the error function and $C$ is a constant of integration.

A set of solutions to the Riccati equation is then given by
\[ f(\eta) = \eta + f''(0) + \frac{2e^{-\frac{\eta^2}{2}} - f''(0)\eta}{\sqrt{2} e^{\frac{f''(0)}{2}} \text{erf} \left( \frac{\eta + f''(0)}{\sqrt{2}} \right) + 2C}, \]  
and therefore $f$ satisfies the boundary condition (3), that is $f'(\infty) = 1$.

Inserting now the boundary condition $f(0) = \gamma$ into Eq. (78) to get
\[ f''(0) + \frac{2}{\sqrt{2} e^{\frac{f''(0)}{2}} \text{erf} \left( \frac{f''(0)}{\sqrt{2}} \right) + 2C} = \gamma. \]  
Consequently, this leads to
\[ C = \frac{1}{\gamma - f''(0)} - \frac{1}{2} \sqrt{\frac{\pi}{2} e^{\frac{f''(0)}{2}} \text{erf} \left( \frac{f''(0)}{\sqrt{2}} \right)}. \]  
Thus we have proved.

**Lemma 1**

The solution of the Falkner-Skan equation when $\beta = -1$ subject to the boundary conditions (2)-(3) can be exactly obtained by Eq. (80), where $C$ is determined by Eq. (79) and $\gamma^2 - 2 \geq 0$. 
7. CONCLUSION

In this work, we have considered the Falkner-Skan equation, which can be described as the non-dimensional velocity distribution in the laminar boundary layer over a flat plate. Very good approximate analytic solutions are obtained by a direct method when $\beta$ takes large values. Also, three special cases: Hiemenz flow ($\beta = 1$), Homann flow ($\beta = \frac{1}{2}$), and Blasius problem ($\beta = 0$) have been considered. We have also demonstrated that the exact solution can be obtained in a straightforward manner by using a direct method when the pressure gradient parameter takes the value $\beta = -1$.

REFERENCES