ANALYSIS OF RELATIVISTIC NUCLEUS-NUCLEUS COLLISIONS TO
SEARCH FOR PHASE TRANSITION

A. KAMAL

Department of Basic Sciences, Deanship of Preparatory Year,
Umm Al-Qura University, P. O. Box 715,
Makkah-21955, Kingdom of Saudi Arabia
E-mail: akarshad@uqu.edu.sa, arshadhep@gmail.com

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Abstract. We have strived to look at the behavior of Levy stability index, \( \mu \) and fractal structure in the distributions of relativistic charged particles in terms of modified \( G_{mq} \) moments using experimental and model (FRITIOF and HIJING, Heavy Ion Jet INteraction Generator) based simulated 14.5 GeV per nucleon \( ^{28}\text{Si-AgBr} \) collisions.

The dynamical component of the multifractal moments is gleaned using correlation-free Monte Carlo (MC-RAND) simulated events. The results are compared with those obtained earlier using scaled factorial moments, \( F_q \). The investigation reveals that the distribution of relativistic charged particles in pseudorapidity space exhibits fractal behavior, dynamical fluctuations and multifractality in both experimental and generated data. Our findings also reveal manifestation of non-thermal phase transition in the experimental data in the two approaches and the value of Levy index, \( \mu \) for the experimental data is consistent with the Levy law description of density distribution of produced charged particles. Furthermore, the values of \( \mu \) for the experimental data are reproduced by the FRITIOF generated data but the values of \( \mu \) obtained by analysing HIJING data specifically in terms of \( G_{mq} \) and \( F_q \) moments violate the Levy stability condition: \( 0 \leq \mu \leq 2 \).

Key words: Levy index, Quark-gluon plasma, Multifractal moments, Scaled factorial moments and multifractality.

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1. INTRODUCTION

Analysis of the interactions caused by cosmic rays [1] has revealed large fluctuations in the rapidity density distribution; similar features have also been observed in the accelerator experiments [2, 3]. Such fluctuations can be considered a possible signature of phase transition from normal nuclear matter to quark-gluon plasma (qgp), an idea also supported by Quantum Chromodynamics. This has resulted in an unprecedented interest among high energy physicists to study production of systems of multi-particles in such collisions [1]—[4].

Fluctuations in various variables due to low statistics are termed as statistical fluctuations, whereas fluctuations in the observables in high multiplicity events are
believed to arise due to some dynamical causes, such as presence of a fireball, fractal, clusters, jets, etc., just after the collision, are referred to as non-statistical fluctuations. One may, therefore, expect that the fluctuations which originate from thermalization, hydrodynamical expansion or hadronization may contain new physics.

Patterns of non-statistical fluctuations can be explained in terms of short-range correlations, production of jets through self-similar cascade process, qgp formation and its subsequent transition to confined state of hadrons, etc. However, none of these models satisfactorily address the physics issues involved.

To explain the mechanism of multi-particle production in heavy-ion collisions in terms of multifractality, both experimental and theoretical studies have been carried out [5, 6]. Information about existence of non-statistical fluctuations in A-A collisions have been obtained [7, 8] by examining multifractality in these collisions. Study of fractals [9] is important because it can help explain chaos in nonlinear physics. It has been suggested [10] fluctuations in the rapidity distributions should have significant multifractal configuration. It may be stressed that study of scaled factorial moments, \( F_q \), which are defined for only positive order of the moments, whereas study of multifractality uses both positive and negative orders of the moments. Several authors [5, 10, 11] have investigated the nature of fractal structures adopting different approaches.

Multifractality was for the first time investigated by Hwa [5], using multifractal moments, \( G_q \). However, due to small particle multiplicities in fixed target experiments, self-similar pattern can’t be expected to emerge in the case of finite scales of resolutions. Hence, it is difficult to separate the dynamical fluctuations using \( G_q \) moments as these might be suppressed by the statistical fluctuations; statistical fluctuations affect \( G_q \) moments in the cases of small multiplicities. For addressing the issue of (nature and pattern) of non-statistical fluctuations, Hwa and Pan [6] have introduced a step function, ‘\( \theta \)’ in the definition of \( G_q \) moments.

Important characteristics of a multifractal system can be explained in terms of Levy stability index, \( \mu \) [12]. It has been used to examine the presence of multifractality as a self-similar entity in high energy nuclear collisions. It also provides information regarding the nature of fluctuations around the tails of pseudorapidity distributions. It may be noted that the value of \( \mu \) characterises the nature of fractality; i) \( \mu = 0 \) refers to monofractality, ii) a value of \( \mu \) less than unity corresponds to “calm singularity” and iii) \( \mu \) greater than unity represents “wild singularity”. Thus, the value of \( \mu \) measures the strength of multifractality.

Intermittency regimes which result from phase transitions during the cascading procedure can be classified in terms of \( \mu \). Thermal phase transition must satisfy [13]–[16] the condition, \( 0 < \mu < 1 \), whereas for non-thermal phase transition \( \mu \) must be greater than unity [14]–[15],[17]–[20]. Self-similar cascading is linked to thermal phase transitions to take place. The new phase may not be characterized by
thermodynamic behaviour. Such a transition may happen during parton-shower cascade as visualized by Van-Hove [21].

Peschanski [22] views that if the dynamics of intermittency is linked to self-similar cascading; then the corresponding transition will be non-thermal. In the normal phase events are populated by spikes and holes, contrary to the situation in spin-glass phase where events are dominated by a few spikes. As suggested [23],[24] that this might be the possible reason for the appearance of intermittency in multiparticle production in relativistic nuclear collisions. It may be interest to note that the behaviour of Levy index and thermodynamic variables/quantities in the thermal and non-thermal regimes have been critically studied in refs [12], [25].

Significance of studying factorial moments to understand hadronization in nuclear collisions stated in our earlier publications [26],[27]. As already stated \( G_q \) and \( F_q \) moments tend to saturate when \( \delta \eta \rightarrow 0 \) and particle multiplicity in the non-empty bin becomes unity. In order to overcome this difficulty, modified \( G_q \) moments, \( \tilde{G}_q^{m} \), are used in the present analysis. Several workers [26, 27], [28]–[33] have investigated phase transition in relativistic nuclear collisions in terms of \( \mu \). But a few attempts have been made for searching evidence of non-thermal phase transition in terms of \( \mu \) using \( \tilde{G}_q^{m} \) moments. Keeping in view its importance and academic interest, the present study was carried out.

Values of Levy index, \( \mu \) generalized fractal dimensions, \( D_q^{dyn} \) and fractal index, \( t_q^{dyn} \) which are believed to be linked to multifractality are calculated to look at non-thermal phase transition, multifractal structure and dynamical fluctuations using \( \tilde{G}_q^{m} \) moments for the experimental, FRITIOF and HIJING generated data on 14.5 A·GeV/c \(^{28}\text{Si}-\text{AgBr} \) collisions.

For extracting the dynamical component of the multifractal moments, values for the experimental and correlation free Monte Carlo events are compared; these values are also compared with those reported earlier [26]. Analysis reveals presence of multifractal structure and dynamical fluctuations in both experimental and simulated data; there is also indication that non-thermal phase transition takes place.

2. DETAILS OF THE DATA

Data collected by us from ILFORD G5 emulsion exposed to 14.5 A·GeV \(^{28}\text{Si} \) beam from Alternating Gradient Synchrotron at Brookhaven National Laboratory (BNL), USA are used. A random sample, which consists of 605 interactions with \( n_h \geq 8 \), while \( n_h \) denotes the number of charged particles emitted from an interaction with relative velocities \( \beta \leq 0.7 \) are analysed. Events with \( n_h \geq 8 \) are envisaged to be produced exclusively due to interactions with AgBr group of nuclei, whereas those with \( n_h \leq 7 \) are either due to the interactions with H or CNO group of targets or due to the peripheral collisions with AgBr group of nuclei [34],[35]. The method of line
scanning was followed in searching the interactions. Each beam track was picked up at least 3mm from the entrance edge and followed until it interacted or left the pellicle. Only those primary tracks which lie at distances more than 50µm from the air and glass surfaces were selected for line scanning. It may be pointed out that 40X objectives and 10X eyepieces on Japan made NIKON binocular compound microscopes were used for the line scanning. Coordinate method was used for measuring the space angles (θ) of the tracks of secondary charged particles. Apart from this, other important criteria relating to scanning, rule for sorting out events etc., are given in reference [35] and earlier publications [36]−[39] from our laboratory.

All secondary tracks produced by relativistic charged particles were classified following standard emulsion terminology. The relativistic shower track with specific ionization \( g^* \leq 1.4 \) (\( g^* = g/g_0 \), where \( g_0 \) is the minimum ionization of a relativistic singly charged particle and \( g \) is the ionization of the charged secondary) are mainly produced by relativistic charged pions. The total number of such tracks produced in an interaction is denoted by \( n_s \). Tracks with specific ionization \( g^* > 10 \) are taken as black tracks. Black tracks are mostly produced by protons but may contain small percentage of multiply charged fragments. The number of black tracks produced in an interaction is designated by \( n_b \). The tracks with specific ionization \( g^* \) lying in the range: \( 1.4 \leq g^* \leq 10 \) are termed as grey tracks. Most of the particles producing grey tracks are recoil target protons, which carry vital information about intra-nuclear cascading. The number of grey tracks in an interaction is denoted by \( n_g \). The projectile fragments produce tracks which have constant ionizations, long ranges and small emission angles. Their number is represented by \( n_f \).

Black and grey tracks, taken together, are referred to as heavy tracks produced by heavily ionizing particles. The number of heavy tracks in an interaction is denoted by \( n_h = n_b + n_g \).

Pseudorapidity distribution is one of the most fundamental distributions in high energy experiments. Pseudorapidity is defined as \( \eta = -\ln \tan(\frac{\theta}{2}) \), where \( \theta \) is the emission angle with respect to the mean direction of the incident beam. It is considered to be the most interesting experimental variable to investigate the mechanism of multi-particle production in hadronic and nuclear collisions. At a very high energy, it approximates rapidity, \( Y = \frac{1}{2} \ln \frac{(E+p_l)}{(E-p_l)} \), where \( E \) and \( p_l \) denote respectively the total energy and longitudinal momentum of a particle. Experimentally, it is not always possible to measure the energy and momentum of a particle, distribution in rapidity space is, therefore, generally studied in terms of pseudorapidity. Furthermore, in order to compare the experimental results with the corresponding values calculated using certain models, analysis of matching numbers of events with the same description as the experimental ones, generated by FRITIOF and HIJING is carried out.
Fig. 1 – Pseudorapidity distributions of relativistic charged particles for the experimental and simulated data on 14.5 A· GeV/c $^{28}$Si-AgBr collisions. The dashed curves represent Gaussian fits to the distributions.

Distributions of the normalized pseudorapidity, $(\frac{1}{N_{\text{evt}}} \frac{dN}{d\eta})$, of relativistic charged particles produced in 14.5 A· GeV $^{28}$Si-AgBr interactions for the experimental, FRITIOF and HIJING simulated data are displayed in Fig. 1, where $N_{\text{evt}}$ represents the total number of events. It may be noted that the curves in the figure are Gaussian which incidentally reproduce the shapes of the experimental, FRITIOF and HIJING distributions reasonably well; similar distributions have also been observed by other workers [40]–[46].

3. MATHEMATICAL FORMALISM IN TERMS OF MODIFIED ‘$G_2^{\eta}$’ MOMENTS

Let the pseudorapidity range $\Delta \eta$ be divided into M bins of equal width, that is, $\delta \eta = \frac{\Delta \eta}{M}$, where $\Delta \eta = \eta_{\text{max}} - \eta_{\text{min}}$ (max and min, respectively, represent the maximum and minimum rapidity values in an event). Further, let $n_j$ be the number of particles in $j^{th}$ bin, where $j$ labels the bins running from 1 to M, including both empty
(some of the bins may have no particles) and non-empty (constitute the fractal set) bins, total number of particles produced in an event is estimated using the following relation:

\[ n = \sum_{j=1}^{M'} n_j \]  \hspace{1cm} (1)

If \( P_j \) denotes the fraction of particles located in \( j^{th} \) bin then \( P_j \) can be calculated from \( P_j = \frac{n_j}{M} \), where \( P_j \) is a small real number for a small bin width, and may vary from bin to bin. Then a set of \( G_q \) moments for \( n_j \) particle multiplicity in \( j^{th} \) bin are defined [5] as:

\[ G_q = \sum_{j=1}^{M'} P_j^q \]  \hspace{1cm} (2)

where \( M' \) is the total number of non-empty bins which constitute the fractal set, summation is carried out over the non-empty bins, \( M' \) only and \( q \) represents the order of the moments.

It is essential to overcome the issue of statistical part, to understand the process of self-similar cascade mechanism and for distinguishing dynamical fluctuations from statistical ones. Accordingly, following form of modified ‘\( G_q^{m} \)’ moments, proposed by Hwa and Pan [6], are used:

\[ G_q^{m} = \sum_{j=1}^{M'} P_j^q \theta(n_j - q) \]  \hspace{1cm} (3)

where \( \theta(n_j - q) \) represents the usual step function which is added to the earlier form of \( G_q \) moments for filtering out the noise.

\[ \theta(n_j - q) = \begin{cases} 1 & \text{if } n_j \geq q \\ 0 & \text{if } n_j < q \end{cases} \]  \hspace{1cm} (4)

\( P_j \) and \( n_j \) have the same meanings as in the definition of \( G_q \) moments. For large particle multiplicity, \( q << \frac{n}{M} \), where \( n \) and \( M \) denote total multiplicity in an event and number of bins respectively. In such a situation \( G_q \) and \( G_q^{m} \) tend to become essentially the same [47].

If for a particular data sample, averaging is done over all the events, the average value of modified ‘\( G_q^{m} \)’ moments for a given \( q \) is calculated from

\[ < G_q^{m} > = \frac{1}{N_{\text{ext}}} \sum_{1}^{N_{\text{ext}}} G_q^{m} \]  \hspace{1cm} (5)

The concept of multifractality [5] envisages that a self-similar particle emission pro-
cess must show linear behavior of the following type [5, 10]:

\[
\langle G^m_q \rangle \propto M^{-\tau^m_q}
\]

where \( \tau^m_q \) represent modified fractal mass exponents, which is extracted from the linear dependence of \( \ln \langle G^m_q \rangle \) on \( \ln M \); \( \tau^m_q \) is expressed [6] as:

\[
\tau^m_q = \frac{\Delta \ln \langle G^m_q \rangle}{\Delta \ln M}
\]

The dynamical component of \( \langle G^m_q \rangle \), \( G^{dyn}_q \), can be extracted [48] from

\[
\langle G^{dyn}_q \rangle = \frac{\langle G^m_q \rangle}{\langle G^{stat}_q \rangle} \cdot M^{1-q}
\]

where superscripts “ dyn ” and “ stat ” denote the contributions from dynamics and statistics respectively. Moreover, \( \langle G^{stat}_q \rangle \) appearing in Eq. (8) are obtained from Eq. (6) by distributing \( n \) particles randomly in the specified \( \Delta \eta \) interval. If \( \langle G^m_q \rangle \) is purely statistical, then \( \langle G^{dyn}_q \rangle \), according to Eq. (8), is \( M^{1-q} \), which results from trivial dynamics.

It may be pointed out that trivial dynamics without statistical fluctuations envisages \( \frac{dn}{d\eta} \) to be flat for each event, so the probability \( P_j \) that a particle is in bin \( j \) is \( \frac{1}{M} \) for all \( j \), and

\[
\langle G^{dyn}_q \rangle = \sum_j P_j \cdot q = M^{1-q}
\]

For trivial dynamics Eq. (9) implies that \( \tau^q_{dyn} = q - 1 \); any deviation from this would mean that there is dynamical information. If we put all three \( G^q_q \) factors in Eq. (8) exhibiting their respective power-law behaviors as described by Eq. (6), we can get the following relation:

\[
\tau^q_{dyn} = \tau^m_q - \tau^q_{stat} + q - 1
\]

where \( \tau^q_{stat} \) is the slope corresponding to correlation free Monte Carlo randomly generated events. Any deviation of \( \tau^q_{m} \) from \( \tau^q_{stat} \) would lead to a deviation of \( \tau^q_{dyn} \) from \( (q-1) \), implying dynamical contribution to \( \tau^q_{m} \). It is important to mention that this is true whether or not \( \langle G^{stat}_q \rangle \) dominates over \( \langle G^m_q \rangle \). Hence, \( \tau^m_q \) is a sensitive measure of the dynamical fluctuations than \( \langle G^m_q \rangle \).

A comparison of \( F_q \) with \( \langle G^m_q \rangle \) moments would help interpret self-similar property of intermittency phenomenon. Following is the expression for scaled factorial moments, \( F_q \), for \( q^{th} \) order of moment is defined [49], [50] as:

\[
F_q = M^{q-1} \sum_{m=1}^{M} \frac{n_{m}(n_{m-1})\ldots\ldots(n_{m-q+1})}{N(N-1)\ldots\ldots(N-q+1)}
\]
where \( \textbf{M} \) is the number of bins and \( n_m \) is the number of particles in \( m^{th} \) bin for a single event, \( N \) is the total number of charged particle in an event lying in the pseudorapidity range \( \Delta \eta \) and \( q \) is the order of the moments.

A relationship between intermittency indices, \( \phi_q \), and fractal indices, \( t_{q}^{\text{dyn}} \), was established in ref [5, 6]; both indices are linked through the following expression [5, 6]:

\[
\phi_q \simeq q - 1 - t_{q}^{\text{dyn}}
\]  

(12)

Fractal indices and generalized fractal dimensions are related [6] as:

\[
D_{q}^{\text{dyn}} = \frac{t_{q}^{\text{dyn}}}{(q - 1)}
\]  

(13)

Anomalous fractal dimensions, \( d_q \), is the co-dimension described [6] as:

\[
d_q = 1 - D_q
\]  

(14)

that is the difference between the topological and Renyi dimensions. Similarly, we have

\[
d'_q = 1 - D_{q}^{\text{dyn}}
\]  

(15)

Intermittency indices, \( \phi_q \), and anomalous fractal dimensions, \( d_q \), are related [51] in the following way:

\[
\phi_q = d_q(q - 1)
\]  

(16)

Another important parameter linked to the description of intermittency is intermittency exponent, \( \beta_q \), which is expressed [20] as:

\[
\beta_q = \frac{\phi_q}{\phi_2} = (q - 1) \frac{d_q}{d_2}
\]  

(17)

According to Brax and Peschanski [13],[52], Levy index, \( \mu \) and intermittency exponent, \( \beta_q \) is related [13], [52]–[54] as:

\[
\beta_q = \frac{q^\mu - q}{2^\mu - 2}
\]  

(18)

On substituting \( \beta_q \) in Eq. (17), we get the following expression for \( \frac{d_q}{d_2} \):

\[
\frac{d_q}{d_2} = \frac{(q^\mu - q)}{(2^\mu - 2)} \frac{1}{(q - 1)}
\]  

(19)

For \( \mu=2 \), Eq. (19) gives \( \frac{d_q}{d_2} = \frac{q}{2} \), corresponding to self-similar branching process reproducible using the recognized Gaussian distribution while other condition \( \mu=0 \) obviously leads to \( d_q = d_2 \) and multifractality tends to manifest monofractal behavior, which is envisaged to result from 2\( ^{nd} \) order phase transition, from qgp to confined hadrons [55].
4. RESULTS AND DISCUSSION

Figure 2 compares the behaviors of $\ln <G_q^m>$ vs. $\ln M$ plots for the experimental, FRITIOF, HIJING and correlation free Monte Carlo randomly generated (MC-RAND) data sets. It is noticed from the figure that $\ln <G_q^m>$ vs. $\ln M$ plots exhibit linearity of the same kind as expected from Eq. (6), indicating thereby self-similarity. For extracting the statistical contributions to $<G_q^m>$, values of $\ln <G_q^m>$ for the Monte Carlo randomly generated events, invoking independent particle emission hypothesis (IPEH), are calculated and plotted against $\ln M$ in Fig. 2. It is evident that the values of $<G_q^{stat}>$ are smaller in comparison to the values of $<G_q^m>$ for higher values of $q$. This result agrees reasonably well with those reported earlier [6]. The sharp straight lines in the figure are linear fits to the data sets. In Fig. 2 $\ln <G_q^m>$ vs. $\ln M$ plots for the experimental and simulated data nicely match but for $q = 5$ and 6 and smaller values of bin sizes (higher values of $M$), the experimental points are higher than the corresponding fitted values and more scattered in the case of experimental data for higher values of $q$. This can be explained by introducing a step function, which can not only suppress the effect of statistical noise but also can eliminate the contribution of lower multiplicity ($n_j < 5$, for $q = 5$ and $n_j < 6$, for $q = 6$) in a given bin. For $n_j = 1$ and 2 in a given bin, most of the experimental points can be explained by using statistical noise, but for $n_j = 4$ most
of these are not due to the statistical noise. Hence, modified ‘$G_q^m$’ moments, as suggested by Hwa and Pan [5, 6], are not effective in the multifractal moment analysis for higher $q$.

The chi-square values for the experimental data for $q= 5$ and 6 are found to be 0.10 and 0.17 respectively. Furthermore, coefficient of determination ($R^2$) values for the experimental data corresponding to $q= 5$ and 6 are found to be 0.982 and 0.973 respectively. Similar results for the experimental data for higher values of $q$ or $M$ have also been reported by other workers [56]–[57].

The values of modified mass exponents, $t_q^m$, are obtained by least squares fits to the data points; $t_q^{stat}$ are calculated for the Monte Carlo randomly generated data in similar manner. The values of $t_q^{dyn}$ are calculated using Eq. (10). The values of $t_q^m$, $t_q^{stat}$, $t_q^{dyn}$, $D_q^{dyn}$, $d_q$ and $\beta_q$ for $q= 2$-6 for the experimental and models based simulated events are listed in Table 1. Displayed in Fig. 3 are the values of $t_q^{dyn}$ for the three types of data sets. In the figure $t_q^{dyn}$ are observed to increase for all values of $q$ in all the three cases. Furthermore, the values of $t_q^{dyn}$ are found to be comparatively higher, for $q \geq 3$ for the HIJING events in comparison to the values for the experimental and FRITIOF data.

Shown in Fig. 4 are the variations of $D_q^{dyn}$ with $q$ obtained with the help of $t_q^{dyn}$ using modified ‘$G_q^m$’ moments. Also, for comparison we have displayed in Fig.
Values of $t^m_q$, $t^{stat}_q$, $t^{dyn}_q$, $D^{dyn}_q$, $d'_q$ and $\beta_q$ for the experimental, FRITIOF and HIJING generated data on $14.5 \text{ A-GeV} \ ^{28}\text{Si-AgBr}$ interactions.

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<th>$D^{dyn}_q$</th>
<th>$d'_q$</th>
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4, the values of generalized dimensions, $D_q$, estimated in terms of $F_q$ moments from our data [26]. From the figure, $D^{dyn}_q$ and generalized dimensions, $D_q$, are found to decrease with increasing $q$. It is worth noting that a decreasing trend in the value of $D^{dyn}_q$ may indicate existence of multifractality. However, the values of $D_q$ calculated
from $F_q$ moments using our data [26] are relatively lower for the experimental and FRITIOF data sets, excepting the values estimated for HIJING data for $q = 2$ and 3, in comparison to the values of $D_q^{dyn}$ obtained using modified ‘$G_q^m$’ moments. Different trends of $D_q$ variations may be attributed to the related formalism and limitation of the individual method. Also, one can see in Fig. 4, the values of $D_q^{dyn}$ and $D_q$, instead of being unity, are smaller than 1; this may indicate presence of non-trivial dynamical fluctuations.

Values of $D_q$, $D_q^{stat}$, $D_q^{dyn}$ and anomalous fractal dimensions, $d_q$, estimated from modified ‘$G_q^m$’ and $F_q$ moments using our data [26] are presented in Fig. 5. Multifractal structure of self-similarity is clearly discernible from the decreasing trend of $D_q^{dyn}$ with increasing $q$. It may be noted that Fig. 5 reflects complimentary nature of the intermittency phenomenon.

Dependence of intermittency index, $\phi_q$ [26], $(q - 1 - t_q^{stat})$ and $(q - 1 - t_q^{dyn})$ on $q$ are exhibited in Fig. 6. From the figure, $\phi_q$ values are found to be slightly different from $(q - 1 - t_q^{stat})$. This small difference is expected to arise due to different approaches adopted in calculating $F_q$, and modified ‘$G_q^m$’ moments. A noticeable feature from the figure is that the values of $t_q^{dyn}$ are markedly different from $q - 1$, revealing the existence of some fluctuations of dynamical nature.

Anomalous fractal dimensions, $d_q'$, are estimated using Eq. (15) for the three variety of data and their values are given in Table 1. From the table the values of

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**Fig. 4** – Generalized dimensions, $D_q$, vs. $q$ plots. The filled symbols denote $D_q^{dyn}$ values estimated from modified ‘$G_q^m$’ moments and open ones represent $D_q$ obtained from $F_q$ moments.
Fig. 5 – Variations of $D_q, D_q^{\text{stat}}, D_q^{\text{dyn}}$, calculated using modified $G_m^q$, moments, and anomalous fractal dimensions, $d_q$ using ($F_q$ moments) on 14.5 A·GeV with $q$.

$d_q$ are seen to increase with $q$ in each case. The patterns of the variations of $d_q$ with $q$ support the occurrence of self-similar cascade mechanism accompanied by a non-thermal phase transition.

Table 2

Estimated values of $\mu$ for $q = 2 \ldots 6$ obtained for the three types of data.

<table>
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<th>Moments</th>
<th>Data Sample</th>
<th>$\mu$</th>
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<td>Mod. $G_q$ [Present work]</td>
<td>EXPT.</td>
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<td>FRITIOF</td>
<td>1.86±0.12</td>
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<tr>
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<td>HIJING</td>
<td>2.23±0.10</td>
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<td>$F_q$ [26]</td>
<td>EXPT.</td>
<td>1.79±0.05</td>
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<td>FRITIOF</td>
<td>1.28±0.07</td>
</tr>
<tr>
<td></td>
<td>HIJING</td>
<td>2.31±0.17</td>
</tr>
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</table>

Figure 7 exhibits the variations of $\beta_q$ calculated by applying the methodology of modified $G_m^q$ and $F_q$ moments with $q$. It may be of interest to mention that the value of $\beta_2$ shall obviously be unity in each case. Values of $d_q$ and $\beta_q$ are calculated using Eqs. (16) and (17) and taking the values of intermittency indices, $\phi_q$, from reference [26]. On the basis of Fig. 7 it can be emphasized that $\beta_q$ increase with $q$. Besides, it is found in the figure that the values of $\beta_q$, when calculated using...
modified $\phi_{q}(F_{q} \text{ moments})$, $(q - 1 - t_{q}^{m})$ and $(q - 1 - t_{q}^{d y n})$ against $q$.

Eq. (18) is used to evaluate the value of $\mu$. The values of $\beta_{q}$ for the HIJING events are nearly the same, when calculated using both types of moments.

As stated earlier if the value of $\mu$ lies in the interval $(0, 1)$, it would indicate thermal phase transition and if it falls in the range $(1, \infty)$ it will be considered to be linked to non-thermal phase transition. Thus, the values of $\mu > 1$ found for the experimental data in the two approaches are consistent with the Levy law description of density distribution and also reveal a possible manifestation of non-thermal phase transition during particle production. Also, the values of $\mu$ for the experimental data in the two approaches are reasonably consistent with the values of $\mu$ obtained for the FRITIOF generated data. Furthermore, it is seen in Table 2 that the calculated values of Levy index using the two types of moments for the HIJING events are found to be
greater than 2. It may be noted that EHS/NA22 collaboration [58] has reported that the value of \( \mu \), approximately exceeds the upper limit of 2 [20]. Also, Levy index within its restricted stability region: \( 0 < \mu \leq 2 \), fails to reproduce the ratio \( \frac{d_q}{d_2} \).

5. CONCLUSIONS

Investigation of the experimental and simulated data on 14.5 A·GeV \( ^{28}\)Si-AgBr collisions in terms of ‘\( G_q^m \)’ and \( F_q \) moments reveals multifractal structure. The dynamical fluctuations seem to get separated from the statistical ones when the data are analyzed in terms of ‘\( G_q^m \)’ moments. Generalized dimensions, \( D^d_{q, \text{dyn}} \), obtained from the modified ‘\( G_q^m \)’ moments and \( D_q \), estimated using \( F_q \) moments are observed to decrease with increasing \( q \) and are found to be smaller than unity. Values of the anomalous fractal dimensions, \( d_q \), evaluated using \( F_q \) moments increases linearly with \( q \) in each case and the values of \( t_q^\text{dyn} \) are found to be different from \( q - 1 \). Deviation of \( t_q^\text{dyn} \) from \( q - 1 \) supports the presence of multifractal behavior of dynamical origin as well as non-trivial dynamical fluctuations. These observations indicate that multi-particle production may be proceeding through a self-similar cascade mechanism possibly accompanied by a non-thermal phase transition.

Different values of \( \mu \), obtained by analyzing the data in terms of ‘\( G_q^m \)’ and
\( F_q \) moments, explain different degrees of multifractality. This may be happening because of different formalism of \( F_q \) and \( C^{m \to}_q \) moments. Manifestation of non-thermal phase transitions are discernible in the experimental data in the two approaches. The values of \( \mu \) for the experimental and \textsc{fritiof} generated data are consistent with Levy stable law. Moreover, the values of \( \mu \) for the experimental data are nicely reproduced by the \textsc{fritiof} generated data. It is important to mention that the \( \mu \) has been found to be greater than 2 for \textit{HIJING} data in the two cases. Finally, Levy index, \( \mu \), for \textit{HIJING} data fails to satisfy Eq. (19) and also violates Levy stability condition, \( 0 \leq \mu \leq 2 \).

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\textbf{REFERENCES}