INTEGRABLE DISCRETIZATION OF COUPLED ABLOWITZ-LADIK EQUATIONS WITH BRANCHED DISPERSION

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Abstract. We construct an integrable discretization of differential-difference multicomponent Ablowitz-Ladik system with branched dispersion relation, using the Hirota bilinear formalism. The multisoliton solutions are also discussed, first for the fully discrete version of Ablowitz-Ladik equation, second for the coupled Ablowitz-Ladik system with 2 and N equations.

Key words: Hirota bilinear formalism, discretization, soliton solutions.

1. INTRODUCTION

Constructing integrable discretizations for partial differential or differential-difference equations is a challenging task in the topic of integrable systems. One of the most powerful methods that can be used in constructing integrable discretizations is the bilinear method proposed by Hirota [1–4], which we detail in the first section of the article. Usually, the integrability for a given partial differential or a differential-difference equation can be proven by the existence of an infinite number of independent integrals in involution which can be computed from the Lax pairs [5]. However, in the Hirota formulation the integrability relies on the existence of a general multisoliton solution. By the word general we mean the solution describing multiple collisions of an arbitrary number of solitons with arbitrary parameters and phases and for all branches of dispersion relations.

In this paper, using the very efficient discretization method of Hirota, we build an integrable fully discrete version of the multicomponent Ablowitz-Ladik system with branched dispersion. The starting point is the integrable semidiscrete version of Ablowitz-Ladik system [6]. The completely integrable variant of semidiscrete nonlinear Schrödinger equation, known as the Ablowitz-Ladik equation was introduced by J. Ablowitz and J. Ladik more than forty years ago [7, 8].

The nonlinear Schrödinger (NLS) equation has been intensively studied since the seventies. The equation first appeared in the biophysical context [9], then in nonlinear optics, Bose-Einstein condensates, etc. The semidiscrete NLS equation has applications in various subjects, such as nonlinear lattices in condensed matter physics, phase plane patterns, the dynamics of discrete curves and surfaces. Curiously enough, the full discretization of nonlinear Schrödinger equation has not been studied so far. In this article we are making a few steps forward in the topic.

Also, the study of integrable coupled semi-discrete and discrete equations is still not well developed. Several authors [11–14] have proposed coupled soliton equations. The Hirota bilinear formalism turns out to be very effective in obtaining integrable discretization of such systems [2].

An important fact, which can be observed also in this article, is that the multisoliton solution for fully discrete equations has exactly the same phase factors and interaction terms as in the differential-difference case. What differs is the dispersion relation. This fact has first been observed by Hirota and Tsujimoto in [1, 3, 15] for many examples including lattice mKdV, lattice NLS, lattice coupled mKdV.

2. THE HIROTA DISCRETIZATION METHOD

The method is very efficient for deriving integrable discretizations of bilinear systems. This discretization method can be applied if we have a semidiscrete bilinear form [1–4] of our equation/system. Starting from here we can prove its complete integrability using three steps:

1. The first step consists in the direct discretization of the Hirota operator

$$D_t f \cdot g \equiv \frac{df}{dt} g - f \frac{dg}{dt}$$

   by replacing ordinary derivatives with finite differences

$$\Delta_m f \cdot g \equiv [\Delta_t f] g - f [\Delta_t g] \ ,$$

   where $\Delta_t f = [f(t+\delta) - f(t)]/\delta$ and δ is the discretization step on the time axis. In the rest of the equation we replace t with $m\delta$.

2. The second step consists in imposing to the bilinear equations obtained at previous step, $T(\Delta_m, \Delta_n, ...) f \cdot g = 0$, the gauge invariance:

$$f \to f e^{pn+qm} \ , \quad g \to g e^{pn+qm},$$

   where p and q are constants.

3. The third step consists in computing the multi-soliton solution and recovering the nonlinear form of our equation.

3. TIME DISCRETIZATION OF THE GENERAL SYSTEM OF COUPLED ABLOWITZ-LADIK

The general system of semidiscrete coupled Ablowitz-Ladik with N coupled equations is:

$$\begin{aligned}
i\dot{q}_1 &= (1+|q_1|^2)(\overline{q_2}+\underline{q_N})\\
i\dot{q}_2 &= (1+|q_2|^2)(\overline{q_3}+\underline{q_1})\\
i\dot{q}_3 &= (1+|q_3|^2)(\overline{q_4}+\underline{q_2})\\
&\ldots\ldots\\
i\dot{q}_{N-1} &= (1+|q_{N-1}|^2)(\overline{q_N}+\underline{q_{N-2}})\\
i\dot{q}_N &= (1+|q_N|^2)(\overline{q_1}+\underline{q_{N-1}}).
\end{aligned}$$

As we have shown in [6], using the nonlinear substitutions: $q_\mu = G_\mu/F_\mu$, the system can be cast in the Hirota bilinear form:

$$i\mathbf{D_t}G_\mu \cdot F_\mu = \overline{G_{\mu+1}}\underline{F_{\mu-1}} + \underline{G_{\mu-1}}\overline{F_{\mu+1}} \tag{1a}$$

$$F_\mu^2 + |G_\mu|^2 = \overline{F_{\mu+1}}\underline{F_{\mu-1}}, \quad \mu = 1,...,N \tag{1b}$$

where $F_0 = F_N$, $F_{N+1} = F_1$, $G_0 = G_N$, $G_{N+1} = G_1$. Here F_N are real functions, while G_N and complex valued functions.

We are going to perform fully integrable discretization of system (1) (t discretization) using the Hirota bilinear formalism [4], presented in the previous section. Replacing the continuous bilinear operator with a discrete one

$$\mathbf{D_t}G_\mu \cdot F_\mu = (G_\mu)_t F_\mu - G_\mu(F_\mu)_t = \frac{\tilde{G}_\mu - G_\mu}{\delta}F_\mu - G_\mu\frac{\tilde{F}_\mu - F_\mu}{\delta} = \frac{1}{\delta}(\tilde{G}_\mu F_\mu - G_\mu\tilde{F}_\mu)$$

(where $\tilde{G}_\mu = G_\mu((m+1)\delta,n)$, $\tilde{F}_\mu = F_\mu((m+1)\delta,n)$, $t \to m\delta$) and imposing the gauge-invariance (*i.e.* invariance with respect to the multiplication with exponential of linears) [4], from the first bilinear equation (1a) we get:

$$i(\widetilde{G}_\mu F_\mu - G_\mu\widetilde{F}_\mu) = \delta(\widetilde{\overline{G_{\mu+1}}}\underline{F_{\mu-1}} + \underline{G_{\mu-1}}\widetilde{\overline{F_{\mu+1}}}),$$

where $F_\mu = F_\mu(m\delta,n)$, $G_\mu = G_\mu(m\delta,n)$, $\widetilde{\overline{G_\mu}} = G_\mu((m+1)\delta,n+1)$ and $\widetilde{\overline{F_\mu}} = F_\mu((m+1)\delta,n+1)$.

Using Hirota-Tsujimoto approach [4, 16] we are not changing the second bilinear equation (otherwise the bilinear system will not have two-soliton solution). Our fully discrete bilinear system will be:

$$i(\widetilde{G}_\mu F_\mu - G_\mu\widetilde{F}_\mu) = \delta(\widetilde{\overline{G_{\mu+1}}}\underline{F_{\mu-1}} + \underline{G_{\mu-1}}\widetilde{\overline{F_{\mu+1}}}) \tag{2a}$$

$$F_\mu^2 + |G_\mu|^2 = \overline{F_{\mu+1}}\underline{F_{\mu-1}}, \quad \mu = 1,..,N. \tag{2b}$$

4. MULTISOLITON SOLUTION

In order to prove integrability of the bilinear system (2) we construct the multisoliton solution. For simplicity and more clarity of the procedure, before solving the system with N equations, we first consider the cases of discrete coupled Ablowitz-Ladik for $N=1$ and $N=2$.

4.1. MULTISOLITON SOLUTION FOR N=1

For $N=1$, the system (2) turns into the bilinear form of fully discret NLS:

$$i(\widetilde{G}_1 F_1 - G_1 \widetilde{F}_1) = \delta(\widetilde{\overline{G_1}}\underline{F_1} + \underline{G_1}\widetilde{\overline{F_1}}) \tag{3a}$$

$$F_1^2 + |G_1|^2 = \overline{F_1}\underline{F_1}\ , \tag{3b}$$

where F_1 is a real function, while G_1 is a complex valued function.

Considering the following ansatz for the one soliton solution:

$$G_1 = \alpha_1 e^{\eta_1}, \quad F_1 = 1 + e^{\eta_1+\eta_1^*+\phi_1}, \quad \eta_1 = k_1 n + \omega_1 \delta m + \eta_1^{(0)},$$

the first bilinear equation (3a) gives the dispersion relation:

$$\exp\delta\,\omega_1(k_1) = \frac{i+\delta e^{-k_1}}{i-\delta e^{k_1}},$$

while the second equation (3b) gives:

$$e^{\phi_1} = \frac{\alpha_1\alpha_1^*}{2[\cosh(k_1+k_1^*)-1]}\ ,$$

where α_1 can be any complex number. For simplicity we fix $\alpha_1 = 1$.

In constructing the 2-soliton solution we have to take into account that all types of solitons must be considered (in this case both directions of propagation). Straightforward calculation gives a 2-soliton solution that describes interaction of two solitons for any propagation direction:

$$\begin{aligned} G_1 =& e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+\eta_2^*+\phi_{12}+\phi_{14}+\phi_{24}} + e^{\eta_1+\eta_2+\eta_1^*+\phi_{12}+\phi_{13}+\phi_{23}} \\ F_1 =& 1 + e^{\eta_1+\eta_1^*+\phi_{13}} + e^{\eta_2+\eta_2^*+\phi_{24}} + e^{\eta_1+\eta_2^*+\phi_{14}} + e^{\eta_2+\eta_1^*+\phi_{23}} + \\ & + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\sum\limits_{1\le i<j}^{4}\phi_{ij}}\ , \end{aligned}$$

where:

$$e^{\phi_{ij}} = \begin{cases} \frac{1}{2}\left[\cosh(k_i+k_j^*)-1\right]^{-1}, & \text{if } i=1,2 \text{ and } j=3,4; \\ 2\left[\cosh(k_i-k_j)-1\right]\ , & \text{if } i=1,2 \text{ and } j=1,2 \\ & \text{or } i=3,4 \text{ and } j=3,4; \end{cases}$$

with

$$\eta_j = k_j n + \omega_j \delta m + \eta_j^{(0)}, \quad k_{j+2} = k_j^* \ ,$$

and the dispersion relation is given by:

$$\exp \delta\, \omega_j(k_j) = \frac{i + \delta e^{-k_j}}{i - \delta e^{k_j}} \ , \quad j = 1,2 \ .$$

Although the existence of 3-soliton solution in Hirota form for NLS is a strong indicator for the complete integrability [17], we give here the N-soliton solution:

$$G_1 = \sum_{\mu=0,1} D_2(\underline{\mu}) \exp\left(\sum_{i=1}^{2N} \mu_i \eta_i + \sum_{1 \le i < j}^{2N} \mu_i \mu_j \phi_{ij} \right)$$
$$F_1 = \sum_{\mu=0,1} D_1(\underline{\mu}) \exp\left(\sum_{i=1}^{2N} \mu_i \eta_i + \sum_{1 \le i < j}^{2N} \mu_i \mu_j \phi_{ij} \right)$$

where:

$$e^{\phi_{ij}} = \begin{cases} \frac{1}{2}\left[\cosh(k_i + k_j^*) - 1\right]^{-1}, & \text{if } i = 1, ...N \text{ and } j = N+1, ..., 2N\,; \\ 2\left[\cosh(k_i - k_j) - 1\right] \ , & \text{if } i = 1, ...N \text{ and } i = 1, ..., N \\ & \text{or } i = N+1, ..., 2N \text{ and } j = N+1, ..., 2N\,; \end{cases}$$

with

$$\eta_j = k_j n + \omega_j t + \eta_j^{(0)} \ , \quad \eta_{j+N} = \eta_j^* \ , \quad k_{j+N} = k_j^* \ ,$$
$$\exp \delta\, \omega_j(k_j) = \frac{i + \delta e^{-k_j}}{i - \delta e^{k_j}} \ , \quad \omega_{j+N} = \omega_j^* \ , \quad j = 1, ..., N$$

and

$$D_1(\underline{\mu}) = \begin{cases} 1 \ , & \text{when } \sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \mu_{i+N}\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

$$D_2(\underline{\mu}) = \begin{cases} 1 \ , & \text{when } \sum_{i=1}^{N} \mu_i = 1 + \sum_{i=1}^{N} \mu_{i+N}\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

4.2. MULTISOLITON SOLUTION FOR N=2

We consider the bilinear form of coupled Ablowitz-Ladik with two equations:

$$i(\widetilde{G}_1 F_1 - G_1 \widetilde{F}_1) = \delta(\widetilde{\widetilde{\overline{G_2}}}\underline{F_2} + \underline{G_2}\widetilde{\widetilde{\overline{F_2}}}) \tag{4a}$$

$$i(\widetilde{G}_2 F_2 - G_2 \widetilde{F}_2) = \delta(\widetilde{\widetilde{\overline{G_1}}}\underline{F_1} + \underline{G_1}\widetilde{\widetilde{\overline{F_1}}}) \tag{4b}$$

$$F_1^2 + |G_1|^2 = \overline{F_2}\underline{F_2} \tag{4c}$$

$$F_2^2 + |G_2|^2 = \overline{F_1}\underline{F_1}, \tag{4d}$$

where F_1, F_2 are real functions, while G_1, G_2 are complex valued functions. In order to solve this bilinear system we take for the one soliton solution the following ansatz:

$$G_1 = \alpha_1 e^{\eta_1}, \quad G_2 = \alpha_2 e^{\eta_1}, \quad F_1 = 1 + e^{\eta_1+\eta_1^*+\phi_1}, \quad F_2 = 1 + e^{\eta_1+\eta_1^*+\phi_2},$$

where amplitudes are given by the components of the "polarization vector" (α_1, α_2). From (4a)-(4b) we get an homogeneous algebraic system with the unknowns α_1, α_2. Its compatibility condition gives the dispersion relation:

$$\exp\delta\,\omega_1(k_1) = \frac{i + \delta\epsilon^{-1}e^{-k_1}}{i - \delta\epsilon e^{k_1}}\,, \quad \epsilon \in \{+1, -1\}.$$

Next, the bilinear equations (4c)-(4d) give two possible relations: $e^{\phi_1} = e^{\phi_2}$ or $e^{\phi_1} = -e^{\phi_2}$. The last one gives $\alpha_1 = \alpha_2 = 0$ so it remains only $\phi_1 = \phi_2 \equiv \phi_{12}$. Also, from (4a)-(4b) we get $\epsilon\alpha_1 = \alpha_2$. Now fixing $\alpha_1 = 1$ we can write down the one soliton solution in the simplest form:

$$G_1 = e^{\eta_1}, \quad G_2 = \epsilon e^{\eta_1}, \quad F_1 = F_2 = 1 + e^{\eta_1+\eta_1^*+\phi_{12}}$$

where

$$e^{\phi_{12}} = \frac{1}{2[\cosh(k_1+k_1^*) - 1]}\,, \quad \eta_1 = k_1 n + \omega_1 \delta m + \eta_1^{(0)}\,.$$

Again, we have to take into account that all types of solitons must be considered (in this case both directions of propagation). The 2-soliton solution that describes interaction of two solitons for any propagation direction has the form:

$$G_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+\eta_2^*+\phi_{12}+\phi_{14}+\phi_{24}} + e^{\eta_1+\eta_2+\eta_1^*+\phi_{12}+\phi_{13}+\phi_{23}}$$

$$G_2 = \epsilon_1 e^{\eta_1} + \epsilon_2 e^{\eta_2} + \epsilon_1 e^{\eta_1+\eta_2+\eta_2^*+\phi_{12}+\phi_{14}+\phi_{24}} + \epsilon_2 e^{\eta_1+\eta_2+\eta_1^*+\phi_{12}+\phi_{13}+\phi_{23}}$$

$$F_1 = 1 + e^{\eta_1+\eta_1^*+\phi_{13}} + e^{\eta_2+\eta_2^*+\phi_{24}} + e^{\eta_1+\eta_2^*+\phi_{14}} + e^{\eta_2+\eta_1^*+\phi_{23}} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\sum_{1\le i<j}^{4}\phi_{ij}}$$

$$F_2 = 1 + e^{\eta_1+\eta_1^*+\phi_{13}} + e^{\eta_2+\eta_2^*+\phi_{24}} + \frac{\epsilon_1}{\epsilon_2}e^{\eta_1+\eta_2^*+\phi_{14}} + \frac{\epsilon_2}{\epsilon_1}e^{\eta_2+\eta_1^*+\phi_{23}} + \\ + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\sum_{1\le i<j}^{4}\phi_{ij}}$$

where:

$$e^{\phi_{ij}} = \begin{cases} \frac{1}{2}\left[\epsilon_i\epsilon_j \cosh(k_i+k_j^*)-1\right]^{-1}, & \text{if } i=1,2 \text{ and } j=3,4\,; \\ 2\left[\epsilon_i\epsilon_j\cosh(k_i-k_j)-1\right]\,, & \text{if } i=1,2 \text{ and } j=1,2 \\ & \text{or } i=3,4 \text{ and } j=3,4\,; \end{cases}$$

$$\eta_j = k_j n + \omega_j \delta m + \eta_j^{(0)}, \quad k_{j+2}=k_j^*, \quad \epsilon_{j+2}=\epsilon_j^* \tag{5}$$
$$\exp\delta\,\omega_j(k_j) = \frac{i+\delta\epsilon_j^{-1}e^{-k_j}}{i-\delta\epsilon_j e^{k_j}}\,, \quad \epsilon_j\in\{-1,1\}\,, \quad j=1,2\,.$$

As the system is completely integrable we can write the N-soliton solution:

$$G_1 = \sum_{\mu=0,1} D_2(\underline{\mu})\exp\left(\sum_{i=1}^{2N}\mu_i\eta_i + \sum_{1\le i<j}^{2N}\mu_i\mu_j\phi_{ij}\right)$$
$$G_2 = \sum_{\mu=0,1} D_2(\underline{\mu})\exp\left(\sum_{i=1}^{2N}\mu_i[\eta_i+\log(\epsilon_i)] + \sum_{1\le i<j}^{2N}\mu_i\mu_j\phi_{ij}\right)$$
$$F_1 = \sum_{\mu=0,1} D_1(\underline{\mu})\exp\left(\sum_{i=1}^{2N}\mu_i\eta_i + \sum_{1\le i<j}^{2N}\mu_i\mu_j\phi_{ij}\right)$$
$$F_2 = \sum_{\mu=0,1} D_1(\underline{\mu})\exp\left(\sum_{i=1}^{2N}\mu_i[\eta_i+\log(\epsilon_i)] + \sum_{1\le i<j}^{2N}\mu_i\mu_j\phi_{ij}\right)$$

where:

$$e^{\phi_{ij}} = \begin{cases} \frac{1}{2}\left[\epsilon_i\epsilon_j \cosh(k_i+k_j^*)-1\right]^{-1}, & \text{if } i=1,...N \text{ and } j=N+1,...,2N\,; \\ 2\left[\epsilon_i\epsilon_j\cosh(k_i-k_j)-1\right]\,, & \text{if } i=1,...N \text{ and } i=1,...,N \\ & \text{or } i=N+1,...,2N \text{ and } j=N+1,...,2N\,; \end{cases}$$

$$\eta_j = k_j n + \omega_j t + \eta_j^{(0)}\,, \quad \eta_{j+N}=\eta_j^*\,, \quad k_{j+N}=k_j^*\,,$$
$$\exp\delta\,\omega_j(k_j) = \frac{i+\delta\epsilon_j^{-1}e^{-k_j}}{i-\delta\epsilon_j e^{k_j}}\,, \quad \omega_{j+N}=\omega_j^*\,, \quad j=1,...,N \tag{6}$$
$$\log(\epsilon_{j+N}) = \log(\epsilon_j)^*\,, \quad \log(\epsilon_j)\in\{\pi i, 2\pi i\}\,,$$

and

$$D_1(\underline{\mu}) = \begin{cases} 1\,, & \text{when } \sum_{i=1}^{N}\mu_i = \sum_{i=1}^{N}\mu_{i+N}\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

$$D_2(\underline{\mu}) = \begin{cases} 1\,, & \text{when } \sum_{i=1}^{N} \mu_i = 1 + \sum_{i=1}^{N} \mu_{i+N}\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

4.3. MULTISOLITON SOLUTION FOR THE GENERAL CASE

The N-soliton solution for multicomponent Ablowitz-Ladik system with branched dispersion relation has the form:

$$G_\mu = \sum_{\nu=0,1} D_2(\underline{\nu}) \exp\left(\sum_{i=1}^{2N} \nu_i [\eta_i + (\mu-1)\log(\epsilon_i)] + \sum_{1\le i<j}^{2N} \nu_i \nu_j \phi_{ij} \right)$$

$$F_\mu = \sum_{\nu=0,1} D_1(\underline{\nu}) \exp\left(\sum_{i=1}^{2N} \nu_i [\eta_i + (\mu-1)\log(\epsilon_i)] + \sum_{1\le i<j}^{2N} \nu_i \nu_j \phi_{ij} \right)$$

where:

$$e^{\phi_{ij}} = \begin{cases} \frac{1}{2}\left(\frac{\epsilon_i^2+\epsilon_j^2}{2\epsilon_i\epsilon_j} \cosh(k_i+k_j^*) + \frac{\epsilon_i^2-\epsilon_j^2}{2\epsilon_i\epsilon_j} \sinh(k_i+k_j^*) - 1 \right)^{-1}, \\ \qquad \text{if } i = 1,...N \text{ and } j = N+1,...2N\,; \\ 2\left(\frac{\epsilon_i^2+\epsilon_j^2}{2\epsilon_i\epsilon_j} \cosh(k_i-k_j) + \frac{\epsilon_i^2-\epsilon_j^2}{2\epsilon_i\epsilon_j} \sinh(k_i-k_j) - 1 \right), \\ \qquad \text{if } i = 1,...N \text{ and } i = 1,...N \\ \qquad \text{or } i = N+1,...2N \text{ and } j = N+1,...2N\,; \end{cases}$$

$$\eta_j = k_j n + \omega_j \delta m + \eta_j^{(0)}, \quad \eta_{j+N} = \eta_j^*\,, \quad k_{j+N} = k_j^*\,, \quad j = 1,...N$$

$$\log(\epsilon_{j+N}) = \log(\epsilon_j)^*\,, \quad \omega_{j+N} = \omega_j^*\,, \quad \log(\epsilon_j) \in \{l\frac{2\pi i}{N}\}\,, \quad l = 1,...N\,.$$

$$D_1(\underline{\mu}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \mu_{i+N}; \\ 0 & \text{otherwise}; \end{cases}$$

$$D_2(\underline{\mu}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \mu_i = 1 + \sum_{i=1}^{N} \mu_{i+N}; \\ 0 & \text{otherwise}. \end{cases}$$

Each of the N solitons can have any of the following branches of dispersion:

$$\exp\delta\,\omega_j(k_j) = \frac{i + \delta\epsilon_j^{-1} e^{-k_j}}{i - \delta\epsilon_j e^{k_j}}\,, \quad \epsilon_j \in \left\{ e^{l\frac{2\pi i}{N}} \right\}, \;\; l = 1,...,N\,, \;\; j = 1,...,N$$

where k_j is the wave number and j is the index of the soliton. Because we have N soliton solution we expect the system to be completely integrable.

5. RECOVERING THE NONLINEAR FORM

Dividing (2a) by $F_\mu \widetilde{F}_\mu$ and putting $q_\mu = G_\mu / F_\mu$ and $\Gamma_\mu = \widetilde{\overline{F_{\mu+1}}}\,\underline{F_{\mu-1}} / F_\mu \widetilde{F}_\mu$ we get:

$$i(\widetilde{q}_\mu - q_\mu) = \delta(\widetilde{\overline{q_{\mu+1}}} + \underline{q_{\mu-1}})\Gamma_\mu \tag{7a}$$

$$1 + |q_\mu|^2 = \frac{\overline{F_{\mu+1}}\,\underline{F_{\mu-1}}}{F_\mu^2}\,. \tag{7b}$$

But, using (7b) we obtain:

$$\frac{\overline{\Gamma_{\mu+1}}}{\Gamma_\mu} = \frac{\widetilde{\overline{\overline{F_{\mu+2}}}}\,F_\mu}{\overline{F_{\mu+1}}\,\widetilde{\overline{F_{\mu+1}}}}\;\frac{F_\mu\,\widetilde{F}_\mu}{\widetilde{\overline{F_{\mu+1}}}\,\underline{F_{\mu-1}}} = \left(\frac{\widetilde{\overline{\overline{F_{\mu+2}}}}\,\widetilde{F}_\mu}{\widetilde{\overline{F_{\mu+1}}}^{\,2}}\right)\left(\frac{F_\mu^2}{\overline{F_{\mu+1}}\,\underline{F_{\mu-1}}}\right) = \frac{1+|\widetilde{\overline{q_{\mu+1}}}|^2}{1+|q_\mu|^2}\,.$$

Finally, our fully discrete Ablowitz-Ladik is the following system:

$$i(\widetilde{q}_\mu - q_\mu) = \delta(\widetilde{\overline{q_{\mu+1}}} + \underline{q_{\mu-1}})\Gamma_\mu$$

$$\frac{\overline{\Gamma_{\mu+1}}}{\Gamma_\mu} = \frac{1+|\widetilde{\overline{q_{\mu+1}}}|^2}{1+|q_\mu|^2}\,, \quad \mu = 1, ..., N\,.$$

Now, eliminating Γ_μ, we will find the following (higher order) fully discrete Ablowitz-Ladik system:

$$\left(\frac{\widetilde{\overline{q_{\mu+1}}} - \overline{q_{\mu+1}}}{\widetilde{\overline{\overline{q_{\mu+2}}}} + q_\mu}\right)\left(\frac{\widetilde{\overline{q_{\mu+1}}} + \underline{q_{\mu-1}}}{\widetilde{q}_\mu - q_\mu}\right)\left(\frac{1+|q_\mu|^2}{1+|\widetilde{\overline{q_{\mu+1}}}|^2}\right) = 1\,, \quad \mu = 1, ..N\,. \tag{8}$$

6. CONCLUSIONS

In this paper we have presented an integrable discretization of a differential-difference multicomponent Ablowitz-Ladik system with branched dispersion relation. The main procedure was discretizing the differential Hirota bilinear operator and then recovering the nonlinear form with the aid of some auxiliary functions. This approach may lead to higher order nonlinear equations. The Hirota bilinear formalism was also used for proving integrability.

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