NONLINEAR WAVES IN OPTICAL AND MATTER-WAVE MEDIA:
A TOPICAL SURVEY OF RECENT THEORETICAL AND EXPERIMENTAL
RESULTS

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Abstract. We present a survey of some recent theoretical and experimental studies on nonlinear waves that form and propagate in a broad class of optical and matter-wave media. The article is structured as a resource paper that outlines a large series of results for spatiotemporal optical solitons, ultrashort (few-cycle) optical pulses, nonlinear modes in parity-time-symmetric systems, rogue waves in scalar, vectorial, and multidimensional nonlinear systems, and matter-wave solitons and vortices.

Key words: nonlinear waves, spatiotemporal optical solitons, few-cycle optical pulses, parity-time-symmetric nonlinear waves, rogue waves, matter-wave solitons.

1. INTRODUCTION

The formation, propagation, and stability of nonlinear waves in optical and matter-wave media is a research field of broad interest in the context of both fundamental and applied studies, as can be seen from books and review articles on the topic [1–17] and a series of influential works published in the course of the past three decades – see, in particular, Refs. [18–37].

In this topical-resource paper we focus on some recent theoretical and experimental studies dedicated to nonlinear waves in a variety of physical settings involving optical and matter-wave media. The paper is structured in six Sections, the first one being an introduction, and the last one presenting a conclusion. Other Sections are dedicated to nonlinear waves in dispersive and/or diffractive optical media (Sec. 2), nonlinear localized structures in parity-time ($\mathcal{PT}$)-symmetric systems (Sec. 3), rogue waves (RWs) in various settings (Sec. 4), and localized matter-wave modes (Sec. 5). The chapters are, largely, mutually independent and can be used as resource references in the corresponding research fields.
Among the plethora of nonlinear-wave patterns, two- and three-dimensional (2D and 3D) ones stand out due to their fundamentally important role in many physical settings; see a recent viewpoint [9] on multidimensional solitons and their legacy in contemporary atomic, molecular, and optical physics and the review articles [11] and [16] on well-established results and novel findings in the area of multidimensional solitons in nonlinear optics and atomic Bose-Einstein condensates (BECs).

The phenomena of the formation of multidimensional solitons in optical media and their robust propagation over distances that are much longer than the underlying diffraction/dispersion lengths, were, in particular, investigated in detail in connection to the fascinating possibility of using spatiotemporal optical solitons (alias “light bullets” [18]) as data bits in all-optical processing devices [4, 20]. Other potential applications of temporal, spatial, and spatiotemporal solitons were elaborated in the context of the use of dissipative solitons in mode-locked lasers [38], supercontinuum generation in photonic crystal fibers [39, 40] and the use of ultrashort (few-cycle) solitons for the same purpose [41, 42], and the possibility of using temporal cavity solitons as data carriers in all-optical buffers [43]. A very important application that has drawn a great deal of current interest is the use of dissipative temporal solitons circulating in microresonators for the generation of frequency combs [44, 45]; for a review on dissipative Kerr solitons in optical microresonators, see a recent paper by Kippenberg et al. [46]. Applications of 3D matter-wave solitons are also envisaged – in particular, in highly precise interferometry [47–49].

The paper is organized as follows. In Sec. 2 we overview the recent research activity in the area of multidimensional solitons in optical media, namely: i) theoretical and experimental studies of linear and nonlinear light bullets; ii) theoretical studies of 2D and 3D solitons forming in carbon nanotubes, and iii) recent theoretical and experimental activities in the area of ultrashort (few-cycle) optical pulses. Section 3 overviews recent results on nonlinear localized modes in \( \mathcal{PT} \)-symmetric optical systems. In Sec. 4 we briefly review recent work on RWs in scalar, vectorial, and multidimensional physical settings. Multidimensional matter-wave localized states, including recently predicted and experimentally created “quantum droplets”, are addressed in Sec. 5. Section 6 concludes the article.

2. NONLINEAR WAVES IN DISPERSIVE AND/OR DIFRACTIVE OPTICAL MEDIA

It is well known that in nonlinear dispersive and/or diffractive optical media the multidimensional solitons (2D and 3D ones) are self-trapped as a result of the balance between linear and nonlinear effects. The combined effects of diffraction and group-velocity dispersion tend to broaden the optical wave packet in both spatial and temporal dimensions. The nonlinear effects tend to induce self-focusing or defocusing of the optical waveform. In particular, temporal solitons form and propagate
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stably over long distances in monomode optical fibers. The fiber solitons are one-dimensional (1D) waveforms and can be stable if certain conditions are fulfilled, see, for example, the book by Kivshar and Agrawal [1]. However, the multidimensional optical solitons (2D and 3D ones) can be stable only in specially selected physical settings. The self-focusing cubic (Kerr-type) effect leads to the unavoidable collapse of the optical wave, see a review paper by Bergé [50] on the wave collapse with specific applications to optics and plasmas, and a book by Fibich [51].

While fundamental (zero-vorticity) solitons are destabilized by the wave collapse in 2D and 2D settings, bright solitons with embedded vorticity are destroyed by instability that breaks the axial symmetry of these topological modes, leading to their splitting into several fragments (typically, these are fundamental solitons) [16, 17]. The stabilization of fundamental and vortex solitons in 2D and 3D settings is a profoundly important issue from both the theoretical and experimental point of view, as well as for potential applications. To stabilize zero-vorticity solitons, non-Kerr nonlinearities are widely used, of quadratic or saturable types, in which the collapse is arrested in 2D and 3D geometries. However, vortex solitons cannot be stabilized with the help of these nonlinearities [52–55]. Another method uses effective potentials induced by photonic lattices, which readily stabilize both 2D and 3D fundamental and vortex solitons [16, 17]. In media with competing optical nonlinearities (e.g., self-focusing cubic terms in combination with self-defocusing quintic ones), vortex solitons may be stable in some parts of their existence domains; we mention here the experimental results demonstrating robust trapping of fundamental [56] and vortex (2+1)D solitons [57] in such media. In particular, transient stabilization of the vortex soliton in the latter case was supported by nonlinear losses. In this connection, it is relevant to mention a new technique for the stabilization of weakly localized vortex beams (strictly speaking, ones with an infinite total energy) in Kerr media with self-focusing nonlinearity by means of higher-order dissipative terms representing multiphoton absorption [58]. Kartashov et al. [59] have recently investigated the existence and stability domains for rotating vortex clusters in optical media with inhomogeneous defocusing cubic nonlinearity growing toward the periphery. A series of stable vortex clusters nested in a common localized envelope have been shown to exist in this setting.

Zeng et al. [60] have recently introduced 1D and 2D models of optical and matter-wave media with self-repulsive cubic nonlinearity, whose local strength is subject to spatial modulation. Flat-top solitons of various types, including fundamental ones, 1D multipoles, and 2D vortices have been produced analytically and numerically in that model, and their stability domains have been identified by means of the linear-stability analysis and direct numerical simulations. The problem of spontaneous vortex nucleation in nonlocal nonlinear media has been recently addressed by Biloshytkskyi et al. [61]. The formation and stability of vortex rings and vortex lines,
emerging in the course of propagation of self-trapped wave beams in nonlocal nonlinear media, have been investigated numerically. These types of vortex complexes can be seen as building blocks for spontaneous or engineered creation of complex topological patterns representing structured light, see Ref. [61] for further details. Liang [62] has studied the revolving and spinning of optical patterns by two coaxial spiraling elliptic beams in nonlocal nonlinear optical media, both analytically, using a variational method, and numerically. The self-induced periodic interfering behavior of dual Airy beams in strongly nonlocal media has been studied by Zang et al. [63]. The analytical and numerical results show that these dual Airy beams in strongly nonlocal media exhibit a periodic focusing and defocusing behavior, and form interference fringes between the corresponding focusing and defocusing positions, see Ref. [63] for details.

A plethora of management techniques can also be used to stabilize multidimensional fundamental solitons; see article [9] for a discussion of key physical mechanisms that may be used for producing robust multidimensional optical solitons.

2.1. LINEAR AND NONLINEAR LIGHT BULLETS: RECENT THEORETICAL AND EXPERIMENTAL STUDIES

In what follows we briefly overview some recent theoretical and experimental studies of light-bullet formation and stability in both linear and nonlinear physical settings. Recently, a series of theoretical works dealing with spatiotemporal localized modes in the free space have been published. In particular, Peng et al. [64] have considered controllable Airy-Airy vortex light bullets (2D vortex solitons of the Airy-Airy type in a system with quadratic nonlinearity were investigated too [65]). Generation, control, and manipulation of self-accelerating Airy-Laguerre-Gaussian light bullets in strongly nonlocal 2D nonlinear media have been investigated, both analytically and numerically, by Wu et al. [66]. Salamin [67] has provided approximate solutions for ultrashort tightly-focused radially polarized laser pulses in plasma, in the form of Bessel-Bessel light bullets.

Self-accelerating Airy-Ince-Gaussian and Airy-helical-Ince-Gaussian spatiotemporal wave packets in strongly nonlocal nonlinear media have been studied analytically and numerically by Peng et al. [68]. Zhong et al. [69] have investigated 3D Airy light bullets in self-defocusing nonlinear media. The 3D localized Airy-Cartesian and Airy-helical-Cartesian wavepackets in free space have been analytically investigated in a recent work by Huang and Deng [70]. Salamin [71] has recently reported analytic expressions for wave fields of ultrashort tightly-focused Bessel pulses with an arbitrary order (i.e., Bessel light bullets) in plasma. Self-trapped pulsed beams with finite power in nonlinear Kerr media excited by time-diffracting, spatiotemporal beams have been studied by Porras [72]. We also refer here to recent theoretical and experimental studies of spatiotemporal wave packets,
which are pulsed optical beams that propagate diffraction- and dispersion-free, as a consequence of tight correlations between their spatial and temporal spectra, see Refs. [73, 74] for details. In the framework of a graded-index model, the coexistence of collapse and stable spatiotemporal solitons in multimode optical fibers has been investigated by Shtyrina et al. [75], using a variational approach. The existence of stable dipole-mode spatiotemporal solitons has been also predicted in Ref. [75].

Finite-energy spatiotemporally localized Airy wave packets have been discussed in a recent work by Besieris and Shaarawi [76], which was motivated by an earlier study of Porras [77] on Gaussian beams diffracting in time. In a recent work, Sakaguchi [78] has put forward new models for multidimensional stable solitons in both optical and matter-wave media. By means of both analytical and numerical techniques, the existence of robust 1D and 2D tail-free self-accelerating solitons and vortices has been shown by Qin et al. [79], in models for either two-component BECs coupled by a microwave field, or optical media with thermal nonlinearities. The robust propagation of multidimensional dark-soliton wave packets featuring Bloch oscillations (BOs) has been recently studied numerically by Driben et al. [80]. The interplay of the self-focusing nonlinearity, refractive-index gradient, and normal temporal dispersion is able to support robust hybrid waves, which feature BOs of dark solitons in multidimensional photonic settings. The robust evolution of multidimensional light bullets of a mixed dark-bright type can also feature BOs, see Ref. [80] for details.

Next we overview a few recent theoretical and experimental works in the area of spatiotemporal dissipative structures in various physical settings. These unique structures were theoretically predicted long time ago in two pioneering works by Turing [81] and by Prigogine and Lefever [82]. A plethora of dissipative structures have been observed in nonlinear systems far from equilibrium in diverse settings in biological, chemical, and optical systems; see an overview paper by Rozanov [83], a review by Tlidi et al. [84], and a recent comprehensive overview by Tlidi and Panajotov [85]. We also refer here to an Introduction by Tlidi, Clerc, and Panajotov [86] to a topical issue on dissipative structures in matter out of equilibrium, which was dedicated to the memory of Ilya Prigogine. It is worth mentioning that the generic dynamical model for describing the unique features of temporal, spatial, and spatiotemporal dissipative structures is the complex cubic-quintic Ginzburg-Landau (CQGL) equation in one, two, and three dimensions.

Veretenov, Fedorov, and Rosanov [87] have recently predicted a new wide class of 3D topological dissipative optical solitons (“hula-hoop solitons”) in homogeneous laser media with fast saturable absorption, namely topological vortex and knotted dissipative 3D optical solitons that are generated by 2D vortex solitons. In a recent work, Akhmediev et al. [88] have numerically investigated dissipative solitons with extreme spikes in either normal or anomalous dispersion regimes. It was found that
such types of solitons exist in large domains of the parameter space of the complex CQGL equation, the variation of any of the five parameters of the model resulting in a rich structure of bifurcations. Grishin et al. [89] have reported experimental observation of self-generation of chaotic dissipative multisoliton complexes supported by competing nonlinear spin-wave interactions. Descalzi and Brand [90] have numerically investigated interaction scenarios for stationary and oscillatory dissipative solitons in the framework of two coupled complex CQGL equations for counter-propagating waves. Two main interaction scenarios have been identified: (i) the occurrence of bound states and compound states of exploding dissipative solitons, and (ii) the onset of spatiotemporal disorder due to the creation, interaction, and annihilation of dissipative solitons, see Ref. [90]. Milián et al. [91] have introduced a new class of self-sustained states, which may exist as single solitons, or build multisoliton clusters, in driven passive cylindrical microcavities. These multidimensional cavity solitons are stabilized by the radiation which they emit and exhibit pronounced polychromatic conical tails [91]. Kochetov and Tuz [92] have considered the effect of replication of fundamental dissipative solitons and vortices modeled by a 2D complex CQGL equation in the presence of a locally applied potential. It was found that the replication of fundamental dissipative solitons remains persistent in the presence of strong fluctuations of the potential, while the replication of dissipative vortices is only possible under very weak fluctuations [92].

In a series of works [93–95], the generation of stable vortex clusters, ring-shaped vortices, and multivortex ring beams has been investigated numerically in dissipative media with effective inhomogeneous diffusion. In a recent paper, Fedorov, Veretenov, and Rosanov [96] have predicted an irreversible hysteresis of internal structure of 3D tangle dissipative optical solitons. Using the generic complex Ginzburg-Landau model for dissipative localized structures, Kochetov [97] has investigated controllable transitions between stationary and moving dissipative solitons, viz., transitions between solitons of the plain, composite, and moving types. We also mention here a recent summary [98] of analytical and numerical studies of the existence and stability of dissipative light bullets and diverse light-bullet complexes that form in a slew of physical settings, and a recent work by Djoko et al. [99] on robust propagation of optical vortex beams, necklace-ring solitons, soliton clusters, and uniform-ring beams that are described by a generalized (3+1)D complex Ginzburg-Landau equation including higher-order effects.

Next we briefly overview some relevant recent experimental results reported in the area of spatiotemporal dynamics of optical solitons in different physical settings. Numerous experimental studies of nondiffracting optical wavepackets, e.g., Airy beams, Bessel beams, Airy-Bessel beams, etc., have been reported in the course of past few years; see, for example, Refs. [100–104]. We also refer here to an overview paper by Efremidis et al. [105] on recent advances in the area of Airy
beams and accelerating optical waves in diverse physical settings.

Recently, Liu et al. [106] have experimentally demonstrated the buildup of soliton molecules in a mode-locked laser. External perturbations, such as a change of the light polarization state and the fluctuation of pump power, can strongly influence the buildup process and the dynamics of soliton molecules, see Ref. [106]. Recently, Salhi et al. [107] have experimentally demonstrated the generation of a broadly tunable dissipative soliton resonance in dual-amplifier figure-of-eight fiber lasers. Asymmetric and single-side splitting of dissipative solitons described by complex CQGL equations with asymmetric wedge-shaped potentials have been reported by Liao et al. [108]. “Molecular complexes” of optical solitons in passively mode-locked fiber lasers have been recently observed by Wang et al. [109]. Ultrashort optical pulses propagating in dissipative nonlinear systems can also interact and form optical soliton molecules. Wang et al. [109] have demonstrated that two soliton-pair “molecules” can bind to form a stable complex. The formation of femtosecond dissipative solitons in an enhancement cavity with Kerr nonlinearity and a spectrally tailored finesse has been experimentally demonstrated in a recent work by Lilienfein et al. [110]. The vast potential for diverse applications of these dissipative optical solitons includes spatiotemporal filtering, compression of ultrashort (femtosecond) laser pulses, and cavity-enhanced nonlinear frequency conversion, see Ref. [110] for details.

The work by Li et al. [111] has reported experimental realization of 3D versatile linear vortex light bullets, which combine a high-order Bessel beam and an Airy pulse. These spatiotemporal wave packets propagate without distortion while carrying a nonzero orbital angular momentum. In a recent work [112], the creation of one-dimensional pulsed light sheets, which propagate self-similarly in free space in the absence of nonlinearity or dispersion, has been demonstrated experimentally. These diffraction-free space-time light sheets may be useful in nonlinear spectroscopy and microscopy, see Ref. [112].

Lahav et al. [113] have recently reported experimental observation of 3D spatiotemporal pulse-train solitons in a photorefractive strontium barium niobate crystal, firing 800-nm pulses at it. For this wavelength, the crystal has a normal dispersion, and, as a consequence, the temporal profile of the waveform amounts to a dark pulse although the produced beam was a bright one, see also a brief discussion by Wise [114] of this important experimental achievement. The experimental demonstration of the fact that the electric field can produce 3D solitary waves representing “bullets” of oscillating molecular director in nematic liquid crystals, self-trapped along all three spatial dimensions, has been recently reported by Li et al. [115]. These bullets preserve their spatially confined shapes and survive collisions. In a recent work, Shumakova et al. [116] have observed chirp-controlled filamentation in ambient air and formation of ultrashort light bullets in the mid-infrared spectral region.
2.2. SOLITONS IN CARBON NANOTUBES

The unique dynamics of ultrashort optical pulses and the formation of both 2D and 3D solitons in carbon nanotubes (CNTs) have been investigated in a series of recent works. A theoretical study of the interaction of a 2D electromagnetic pulse with electron inhomogeneities in arrays of CNTs in the presence of field inhomogeneity has been reported by Zhukov et al. [117] and the propagation dynamics of 2D extremely short electromagnetic pulses in Bragg media containing immersed arrays of CNTs has been reported too [118]. Belonenko et al. [119] have studied the formation of discrete solitons in a Bragg environment containing CNTs. The formation of 2D light bullets in Bragg media with harmonically modulated refractive indices, which contain arrays of CNTs, has been investigated by Nevzorova et al. [120]. Special opto-acoustic effects in arrays of CNTs have been studied by Zhukov et al. [121]. A comprehensive study of collisions of 3D bipolar optical solitons in arrays of CNTs has been reported by Zhukov et al. [122]. Three-dimensional ultrashort optical Airy beams in inhomogeneous media containing CNTs have been also investigated [123]. Such types of “light bullets” exhibit stable propagation, and, by varying the density-modulation period of CNTs, one can control the velocity of these 3D optical modes, see Ref. [123]. The stable propagation of 2D light bullets in media with inhomogeneous density of CNTs has been studied in a recent work by Dvuzhilov and Belonenko [124]. The propagation of 3D bipolar ultrashort electromagnetic pulses in inhomogeneous arrays of semiconductor CNTs has been investigated by Fedorov et al. [125]. The heterogeneity of the propagating medium is represented by a planar region with an increased concentration of conduction electrons. Numerical simulations have demonstrated that, after interacting with the higher electron concentration area, the ultrashort pulse can propagate steadily, without significant spreading [125].

2.3. ULTRASHORT (FEW-CYCLE) OPTICAL WAVEFORMS

Many experimental and theoretical studies of optical pulses with widths ranging from tens of nanoseconds to only a few tens of femtoseconds have been reported in the course of the last three decades; see, in particular, the review papers on the fast growing research area of ultrashort high-power optical pulses [126–136]. The possibility to generate high-intensity ultrashort optical pulses relies on a revolutionary experimental achievement, namely the so-called chirped pulse amplification (CPA) technique, which was introduced in 1985 by Strickland and Mourou [137]. In a subsequent work published in 1987, Mourou, Strickland and their collaborators had put forward an idea of using large-scale Nd:glass amplifiers to extend the CPA technique to the amplification of pulses with the duration of hundreds fs to the hundred-Joule level, leading to petawatt-power pulses and to intensities $\sim 10^{23}$ W/cm$^2$ [138].

Ursescu et al. [135] have recently described a high-power laser system and
experiments planned at the Extreme Light Infrastructure–Nuclear Physics (ELI-NP) facility to be built at Magurele Physics Campus, in outskirts of Bucharest. At this unique experimental facility, ultrashort 10 PW laser pulses with intensities reaching $10^{23}$ W/cm$^2$ are planned to be delivered; see the recent paper [139] describing physical experiments to be carried out with such 10 PW ultra-intense lasers and with 20 MeV high-brilliance gamma beams, and the status of completing the ELI-NP facility. Further, a series of high-field physics and quantum electrodynamics experiments to be performed at the ELI-NP facility have been surveyed in a comprehensive paper by Turcu et al. [136]. We also refer here to a paper by Jirka et al. [140] on the possibility to efficiently generate electron-positron pairs with 10 PW-class laser facilities, by means of a standing wave created by two tightly focused counterpropagating laser pulses, to a recent work by Miron and Apostol [141] on the motion of electric charges in high-intensity electromagnetic fields, and to a recent paper by Yu et al. [142] on a novel approach to the creation of electron-positron pairs in photon-photon collisions driven by 10 PW laser pulses.

In the course of the past few years ultrahigh intensity lasers with peak powers $\sim 1$ PW have been built at different facilities: (a) at the Lawrence Berkeley National Laboratory, USA (BELLA high repetition rate PW class laser) [143], (b) at the Apollon facility in France it is planned to generate 10 PW peak power pulses with duration of 15 fs at a repetition rate of 1 shot/minute [144], (c) at the laser facility in South Korea the generation of 4.2 PW laser pulses with duration of 20 fs at a repetition rate of 0.1 Hz has been demonstrated, using a CPA Ti:sapphire laser system [145], (d) at the laser facility in China the generation of 5.4 PW laser pulses with duration of 24 fs from a CPA Ti:sapphire laser system [146] has been reported, and (e) at the Extreme Light Infrastructure - Beamlines facility in the Czech Republic [147].

The state-of-the-art surface plasma attosource beamlines at Extreme Light Infrastructure Attosecond Light Pulse Source (ELI-ALPS) are currently being designed and developed in Szeged, Hungary, to enable new directions in plasma-based attoscience research, see a recent paper by Mondal et al. [148]. In a recent work, Jahn et al. [149] have investigated the challenging problem of generation of intense isolated attosecond pulses in the extreme ultraviolet and X-ray spectral ranges, from relativistic surface high harmonics. The measurements and particle-in-cell numerical simulations to determine the optimum values for the key physical parameters have been reported in Ref. [149].

In a recent paper, Turcu et al. [150] have produced an overview of quantum-electrodynamics experiments with colliding PW laser pulses, which are planned to be performed at the new superhigh-power laser facilities, which are capable to reach intensities of $10^{23}$ W/cm$^2$ in the laser focus. Ciappina et al. [151] have discussed recent results for atomic diagnostics of ultrahigh laser intensities. A method for
direct and unambiguous measurement of ultrahigh laser intensities in the range of $10^{20} - 10^{24}$ W/cm$^2$ has been put forward and tested numerically, see Ref. [151]. Wilson et al. [152] have investigated theoretically and numerically the dynamics of 3D compression of ultrashort intense laser pulses in plasma. Using the variational method, and starting from the slowly-varying envelope model, equations describing the spatiotemporal evolution of short laser pulses towards collapse have been derived. 3D particle-in-cell numerical simulations have been also carried out to study physical processes both preceding and following the pulse collapse, see Ref. [152] for details. Nonlinear Compton scattering of electrons in the presence of an ultraintense few-cycle laser pulse travelling through a plasma has been investigated by Mackenroth et al. [153] in the framework of quantum electrodynamics, and properties of the quantum vacuum Cherenkov radiation in strong laser pulses have been recently studied by Macleod et al. [154].

Next we briefly overview recent theoretical results for ultrashort optical waveforms. Several research groups [155–162] have investigated a series of topics ranging from the important issue of interplay of diffraction and nonlinear effects in the propagation of very short light pulses [155] to the potential use of focused ultrashort pulses for far-field light nanoscopy [162]. Arkhipov et al. [163] have investigated the possibility of the generation of unipolar pulses in Raman-active media, excited by a series of few-cycle optical pulses, and Pakhomov et al. [164] have put forward a general framework to produce unipolar half-cycle pulses of controllable form in resonant media with nonlinear field coupling. The possibility of generation of unipolar half-cycle pulses via unusual reflection of a single-cycle pulse from thin metallic or dielectric layers has been put forward by Arkhipov et al. [165]. Terniche et al. [166] have investigated the propagation of few-cycle solitons in two parallel optical waveguides, in the presence of linear nondispersive coupling, and it was shown that few-cycle vector solitons in such coupled waveguides are, in fact, optical breathers.

The propagation dynamics of Gaussian spatiotemporal wavepackets in arrays of parallel optical waveguides, assuming linear nondispersive coupling between adjacent guides, has been studied by Leblond et al. [167]. The conditions for formation of few-cycle spatiotemporal optical solitons in such waveguide arrays, were also investigated in Ref. [167], including the identification of their time duration, spatial widths, and energies. A recent work by Leblond and Mihalache [168] has addressed the propagation of Gaussian spatiotemporal inputs in arrays of parallel optical waveguides, assuming both linear and nonlinear non-dispersive coupling between adjacent guides. Depending on the numerical values of the key parameters of the spatiotemporal Gaussian input, the combined effect of linear and nonlinear couplings leads to either spreading or formation of different types of (2+1)D ultrashort spatiotemporal solitons, see Ref. [168] for more details. Note that the formation of relatively broad semi-discrete spatiotemporal solitons in coupled arrays of nonlinear optical
waveguides was analyzed in Ref. [169].

The problem of wavelength-scaled laser filamentation in solids and plasma-assisted generation of subcycle light bullets in the long-wavelength infrared region has been recently addressed by Grynko et al. [170]. The method put forward in Ref. [170] for the generation of ultrashort (subcycle) light bullets in solids at long wavelengths may help to open new possibilities in ultrafast nonlinear optics, including the generation of high-order harmonics and attosecond pulses. Recently, Sazonov and Ustinov [171] have investigated the propagation of few-cycle pulses in nonlinear media modeled by a set of four-level atoms. In this context, an integrable generalization of the sine-Gordon equation has been reported in Ref. [171] without the use of the slowly-varying envelope approximation. Also, a review of the propagation dynamics of both resonant and quasi-resonant envelope optical solitons, as well as few-cycle optical solitons with temporal durations from nanoseconds to femtoseconds, has been recently published by Sazonov [172]. Gao et al. [173] have investigated analytically and numerically the stability and interaction scenarios for few-cycle pulses in Kerr media, while Konobeeva and Belonenko [174] have studied the propagation of 3D few-cycle optical pulses in topological-insulator thin films. Self-compression of laser pulses in an active system of weakly-coupled waveguides has been investigated analytically by Balakin et al. [175], using the variational method based on a spatiotemporal Gaussian ansatz. The obtained results agree well with direct numerical simulations of the corresponding discrete Ginzburg-Landau dynamical model.

The generation of ultrashort (few-cycle) pulses in the terahertz (THz) range is a research topic of great interest during the past decade. Recently, Pakhomov et al. [176] have proposed a new method for generation of subcycle THz pulses with unusual waveshapes such as rectangular or triangular ones. The method is based on control of the phase delay between different parts of the THz wavefront using linear diffractive optical elements, see Ref. [176]. Porras [177] has analyzed theoretically the effects of the coupling between the orbital angular momentum (OAM) and temporal degrees of freedom on few-cycle and sub-cycle optical pulse shapes, in the geometry used in current experiments involving diffraction and focusing. In another recent work, Porras [178] has found the upper bound for the OAM carried by ultra-short (few-cycle) pulses; note that the single-cycle pulse can carry up to 27 units of OAM, see Ref. [178] for details.

In the experimental arena of strong laser pulses and their interaction with matter we refer here to a recent work by Tsatrafyllis et al. [179]. The quantum optical signatures in a strong laser pulse after interaction with semiconductors have been studied [179] and it has been demonstrated that strongly laser-driven semiconductors lead to the generation of nonclassical states of light having subcycle electric field fluctuations that carry the information of the subcycle dynamics of the interaction, see Ref. [179] for details.
Next, we briefly overview some recent relevant experimental results in the active research area of high-power ultrashort (femtosecond) lasers and associated spatiotemporal phenomena. Spatiotemporal mode-locking in multimode fiber lasers has been addressed by Wright et al. [180]. It was demonstrated in Ref. [180] that modal and chromatic dispersions in fiber lasers can be compensated by strong spatial and spectral filtering, allowing locking of transverse and longitudinal modes for the creation of ultrashort laser pulses with a large variety of spatiotemporal profiles. Potential applications of this technique include the realization of super high-power lasers and efficient control of the emittance of the beam, with the objective to create specific spatiotemporal patterns. In a recent work, Zheltikov [181] has reviewed the current status of multioctave supercontinuum generation and the creation of subcycle pulses both in air and in solid materials. Applications of these studies range from the frequency-comb metrology, coherence tomography, nonlinear spectroscopy, and chirped-pulse amplifiers to bioimaging [181]. In a recent experimental work, Gonsalves et al. [182] have demonstrated guiding of relativistically intense laser pulses with peak power of 0.85 PW over 15 diffraction lengths, which allows the electron beam acceleration to 8 GeV in a laser-heated capillary discharge waveguide.

3. NONLINEAR WAVES IN PARITY-TIME-SYMMETRIC SYSTEMS

In the course of the last two decades, the concept of the space-time reflectional symmetry, alias parity-time ($\mathcal{PT}$) symmetry, first proposed in the quantum theory by Bender and Boettcher [183], has been thoroughly explored in diverse areas of fundamental and applied physics, such as optics and photonics, atomic physics and Bose-Einstein condensates, superconductivity, electronic circuits, acoustics, etc. Unique features of linear or nonlinear waves in $\mathcal{PT}$-symmetric physical systems have been investigated in detail. $\mathcal{PT}$-symmetric quantum mechanics is a natural extension of conventional quantum theory, dealing with non-Hermitian Hamiltonians characterized by complex-valued external potentials obeying the $\mathcal{PT}$-symmetry: the spatially odd (antisymmetric) imaginary part of the complex potential represents symmetrically placed and mutually balanced gain and loss elements. The respective non-Hermitian Hamiltonians exhibit all-real, i.e., physically relevant, energy spectra, provided that the strength of gain-loss terms does not exceed a certain critical values, above which the $\mathcal{PT}$ symmetry breaks down. In combination with the usual intrinsic nonlinearity of $\mathcal{PT}$-symmetric media, non-Hermitian Hamiltonians give rise to a plethora of new phenomena that have no counterparts in common dissipative physical systems, see two comprehensive reviews published by Konotop et al. [13] and Suchkov et al. [14]. In particular, unlike normal dissipative systems, in which stable solitons may exist as attractors, separated from all other solutions, $\mathcal{PT}$-symmetric models give rise to continuous families of solitons, sharing the latter property with
conservative models. In optics the $\mathcal{P}\mathcal{T}$ symmetry can be relatively easily implemented, e.g. in planar waveguides [184] and in coupled optical structures [185, 186]; see two relevant experimental works [187, 188].

Theoretical studies of dynamics of spatial solitons in $\mathcal{P}\mathcal{T}$-symmetric optical lattices, reported in the course of the past several years, have been recently surveyed by He and Malomed [189] and by He et al. [190]. $\mathcal{P}\mathcal{T}$-symmetric optical couplers with competing cubic self-focusing and quintic self-defocusing nonlinearities have been theoretically investigated by Burlak et al. [191]. Also, a $\mathcal{P}\mathcal{T}$-symmetric dual-core system with the sine-Gordon nonlinearity and derivative coupling has been studied by Cuevas-Maraver et al. [192] as a relevant extension of the class of nonlinear $\mathcal{P}\mathcal{T}$-symmetric physical models. Barashenkov et al. [193] have studied the problem of integrability and $\mathcal{P}\mathcal{T}$-symmetry restoration in nonlinear Schrödinger (NLS) dimers with gain and loss. The problem of nonlinear parity-time-symmetry breaking in optical waveguides with complex-valued Gaussian-type potentials has been addressed in Ref. [194], and the existence and stability of both symmetric and asymmetric solitons that form in self-focusing optical waveguides with $\mathcal{P}\mathcal{T}$-symmetric double-hump Scarf-II potentials have been analyzed in Ref. [195]. Three-dimensional solitons supported by different types of $\mathcal{P}\mathcal{T}$-symmetric complex-valued external potentials have been also investigated numerically in recent papers [37, 196, 197]. Kartashov et al. [37] have reported a first example, to the best of our knowledge, of continuous families of stable 3D propagating solitons supported by $\mathcal{P}\mathcal{T}$-symmetric complex-valued external potentials.

Continuous families of one- and two-dimensional stable solitons in $\mathcal{P}\mathcal{T}$-symmetric NLS equations with position-dependent effective masses have been investigated by Chen et al. [198]. A combination of the $\mathcal{P}\mathcal{T}$-symmetric gain-loss structure with inhomogeneous repulsive nonlinearity, growing fast enough from the center to periphery, makes it possible to predict both 1D [199] and 2D [200] solitons (including vortex solitons, in the 2D case), with unbreakable $\mathcal{P}\mathcal{T}$ symmetry, which extends to arbitrarily large values of the strength of the gain-loss terms. The effects of the third-order dispersion on constant-amplitude solutions of the NLS equation with complex-valued potentials have been considered by Liu et al. [201]. Lombard and Mezhoud [202] have performed a detailed analytical and numerical study of the spectrum of the $\mathcal{P}\mathcal{T}$-symmetric complex-valued linear potential, and in a subsequent work, Lombard et al. [203] have investigated the relationship between complex potentials with real eigenvalues and the inverse scattering problem.

Li et al. [204] have studied the properties of optical solitons in models of waveguides with focusing or defocusing saturable nonlinearity and a $\mathcal{P}\mathcal{T}$-symmetric complex-valued external potential of the Scarf-II type. Fundamental and multipole solitons for both focusing and defocusing signs of the saturable nonlinearity in such $\mathcal{P}\mathcal{T}$-symmetric waveguides have been studied by Li et al. [204]. The exis-
tence, stability, and dynamics of optical solitons that form in media with competing cubic-quintic nonlinearity and $PT$-symmetric complex-valued external potentials have been investigated in Ref. [205]. The effects of periodically-modulated third-order dispersion on periodic solutions of the NLS equation with complex-valued potentials have been studied by Liu et al. [206]. Wang et al. [207] have investigated numerically the evolution dynamics of continuous families of dipole solitons in media with self-defocusing Kerr nonlinearity and partial $PT$-symmetric optical potentials. This type of complex-valued external potentials can support one-parameter families of diagonal and adjacent dipole solitons with two humps, which feature either the out-of-phase or in-phase structure. Both diagonal out-of-phase and diagonal in-phase dipole solitons can propagate stably at moderate values of their power. It was pointed out in Ref. [207] that only adjacent in-phase dipole solitons can be stable in a certain region of their existence domain.

Analytical and numerical studies of the optical-beam dynamics in single active waveguides and in a dimer of active waveguides with quadratic nonlinearities have been recently reported by Tsoy et al. [208]. Stationary solutions for a dimer with identical waveguides and for a $PT$-symmetric dimer were obtained, and the respective stability analysis has demonstrated that stable optical beams exist in a wide range of parameters of the model. It is worth mentioning that the active monomer can be used as an amplitude filter or as an optically controllable switch, see Ref. [208]. Li et al. [209] have investigated numerically the stability, dynamics, and bifurcation scenarios of both symmetric and asymmetric solitons supported by saturable nonlinear media in the presence of a $PT$-symmetric external potential. In a recent work, Scheel and Szameit [210] have studied the impact of gain and loss on the evolution of photonic quantum states, concluding that $PT$ symmetric states do not exist in the framework of the quantum field theory.

Kartashov and Vysloukh [211] have recently addressed the formation of edge and bulk dissipative solitons in modulated $PT$-symmetric continuous waveguide arrays composed of waveguides with amplifying and absorbing sections, whose density gradually increases either towards the center of the array or towards its edges. It was found that in such modulated $PT$-symmetric waveguide arrays, the symmetry breaking phenomenon may occur via collision of eigenvalues of compact edge or bulk modes, which are localized in the domains where the waveguides have the smallest separation, see Ref. [211] for more details. Wang et al. [212] have investigated analytically and numerically the families of solitons pinned to a defect described by a $PT$-symmetric potential given by a $\delta$-functional dipole, which is embedded in a 1D uniform medium with a general self-focusing nonlinearity. A physical scheme for creation of combined linear and nonlinear optical potentials with $PT$-symmetry, with the aim to address scattering of optical solitons by such types of potentials in coherent atomic gases was put forward by Qin et al. [213].
Yang [214] has recently reported a new type of symmetry-breaking bifurcation of solitons in optical media with cubic-quintic or saturable nonlinearities and $\mathcal{PT}$-symmetric complex-valued external potentials. The two bifurcated branches of asymmetric solitons exhibit opposite stability, which contrasts all previous symmetry-breaking bifurcations in either conservative or dissipative physical systems, see Ref. [214] for details.

The results briefly outlined in this Section pave the way for exploiting and better understanding the unique properties of non-Hermitian systems in many physical settings.

4. ROGUE WAVES

In this Section we discuss recent theoretical and experimental advances in the area of rogue waves (RWs), alias “freak waves”. These special wave structures emerge in a broad class of physical systems ranging from optics to Bose-Einstein condensates, see the comprehensive reviews [215–218]. In the course of the past years, theoretical and experimental studies of such high-amplitude waves have mainly focused on optical and hydrodynamic settings; see, for example, Refs. [219–232]. The plethora of theoretical and experimental papers published in the past decade in the area of RWs were strongly influenced by the seminal works of Peregrine [233], Kuznetsov and Ma [234, 235], and Akhmediev et al. [236], where exact rational solutions of the integrable NLS equation were reported; see also the papers published in 1993 by Mihalache and Panoiu [237] and by Mihalache et al. [238] where the above-mentioned exact rational solutions of the NLS equation and other types of “first-order” exact solutions of this nonlinear equation were obtained in the context of the pulse propagation in optical fibers, for both the anomalous- and the normal-dispersion regime. The multiparameter families of first-order exact solutions of the NLS equation in the normal-dispersion regime have been independently reported in 1993 by Akhmediev and Ankiewicz [239] and by Gagnon [240]. There are also other integrable nonlinear partial differential equations admitting Peregrine-type solitons, as well as Akhmediev and Kuznetsov-Ma breathers [234, 235]; see, for example, the recent review [218].

In what follows we will overview theoretical and experimental works performed in the past two years in this vast research area. Ankiewicz and Akhmediev [241] reported a plethora of RW solutions of the integrable NLS hierarchy with an infinite number of terms, namely higher-order dispersion and nonlinear terms. Some mechanisms of generation of fundamental RW spatiotemporal structures in $N$-component coupled systems have been considered by Ling et al. [242]. Chowdury and Krolikowski [243] have presented analytical one- and two-breather solutions of the fourth-order NLS equation in the degenerate, soliton, and RW limits. The pos-
sibility of the generation of higher-order RWs in optical fibers from multibreathers by dint of a double-degeneracy procedure has been investigated recently by Wang et al. [244]. Baronio [245] has investigated the formation of Akhmediev breathers and Peregrine solitary waves in quadratic nonlinear media in the regime of cascading second-harmonic generation. In a subsequent work, Baronio et al. [246] have reported the formation and key properties of quadratic Peregrine solitary waves, in the presence of significant group-velocity mismatch between the waves (or beam walk-off, in terms of the Poynting vector).

Ankiewicz and Akhmediev [247] have studied the relation between the number of “elementary units” of multi-rogue waves and the triangular numbers. It is well known that multi-rogue-wave solutions of integrable nonlinear evolution equations have a specific number of elementary components within their structures. Ankiewicz and Akhmediev [247] have revealed that, for the $n$-th-order rogue-wave solutions, these numbers are given by triangular numbers.

Chan and Chow [248] have performed a numerical investigation of the generation and dynamics of “hot spots” as models of dissipative RWs. Effects of gain or loss on the dynamics of RWs have been investigated in the framework of the generic complex Ginzburg-Landau equation and the robustness of the Peregrine soliton in such dissipative systems has been studied by means of numerical methods, see Ref. [248]. The effects of a seed pulse on RW formation for mid-infrared supercontinuum generation in chalcogenide photonic-crystal fibers have been studied numerically by Zhao et al. [249]. Yang et al. [250] have studied, analytically and numerically, possibilities of obtaining controllable optical RWs via nonlinearity management. Bright-dark solitons and semi-rational RW solutions for the coupled Sasa-Satsuma equations have been obtained by Liu et al. [251]. Zhang and Yan [252] have systematically investigated, with the aid of the Darboux-transformation method, the dark-bright mixed higher-order semi-rational solutions of $n$-component nonlinear Schrödinger equations. Peregrine solitons beyond the threefold-amplitude magnification limit have been recently studied by Chen et al. [253]. Within the framework of coupled Fokas-Lenells model, the Peregrine solitons can reach peak amplitudes as high as five times the background level, see Ref. [253]. El-Tantawy and Wazwaz [254] have studied, analytically and numerically, the formation of RWs and dark-soliton collisions in dusty plasmas.

A theoretical study of extreme events occurring in phononic lattices has been reported by Charalampidis et al. [255]. This work illustrates the potential of dynamical lattices towards the experimental observation of phononic (acoustic) RWs. We also mention here the work of Randoux et al. [256] on nonlinear spectral analysis of data recorded in optics and in hydrodynamic experiments that reported the pioneering observation of nonlinear waves with spatiotemporal localization similar to Peregrine solitons. Related results were reported by Chen et al. [257], in the form of some
special types of exact two- and three-soliton solutions of the Kadomtsev-Petviashvili II equation, written in terms of hyperbolic cosines. The obtained exact solutions exhibit rich intriguing interaction patterns on a finite background, see Ref. [257]. A comprehensive theoretical study of RWs in ultracold bosonic gases has been recently reported by Charalampidis et al. [258]. We also point out that the analytic forms of loop RW solutions for the Wadati-Konno-Ichikawa equation have been reported by Zhang et al. [259] using the Darboux transformation and inverse hodograph transformation. RWs and hybrid solutions of the Davey-Stewartson I equation have been obtained recently by Liu et al. [260] using the Hirota bilinear method. Zhang et al. [261] have recently studied, by means of analytical and numerical methods, the modulational instability and higher-order vector RW and multi-dark soliton structures in the framework of the general coupled Hirota equations, which describe the propagation of two ultrashort pulses in an optical fiber.

Different types of first-order and second-order rational solutions and periodic solutions of the Sasa-Satsuma equation have been obtained by Guo et al. [262] using the Darboux-transform method. Unique dynamics of dust-acoustic and dust-cyclotron freak waves in a magnetized dusty plasma has been recently studied analytically and numerically by Akhtar et al. [263]. Families of semi-rational solutions to the Fokas system, named lump-soliton modes, have been reported by Rao et al. [264], using the Hirota bilinear method. First-order and second-order rogue waves on a periodic background for the Kaup-Newell equation have been obtained analytically by Liu et al. [265], also using the Darboux transform. Chen and Yan [266] have recently investigated the matrix Riemann-Hilbert problem for the Hirota equation with non-zero boundary conditions. Using the \(n\)-fold Darboux transform, higher-order rogue waves for the Hirota equation have been obtained in a closed form, see Ref. [266]. The complex dynamics of vector rogue waves on a double-plane wave background has been investigated by Zhao et al. [267]. Degasperis et al. [268] have explored a connection between the instability properties of the plane-wave states and the existence and key features of rogue-wave solutions of integrable versions of a system of two coupled NLS equations. Using the Darboux transform, rogue waves in the nonlocal PT-symmetric NLS model have been investigated recently by Yang and Yang [269]. The obtained exact solutions for these types of rogue waves exhibit rich patterns, most of which having no counterparts in the local NLS model. Recently, Ye et al. [270] have obtained the general rogue wave solutions of the coupled Fokas-Lenells equations using a non-recursive Darboux transformation technique.

Chen et al. [271] have recently investigated both analytically and numerically \textit{superchirped rogue waves} in optical fibers within the framework of a generalized nonlinear Schrödinger equation including the group-velocity dispersion, cubic and quintic nonlinearities, and self-steepening effect. These super-rogue waves involve a frequency chirp, localized in both time and space, and are robust against white-noise
perturbations. The possibility of generating them in a turbulent field suggests their accessibility to experimental observation, see Ref. [271].

It is necessary to stress that all RWs are unstable states, as they are supported by a continuous-wave background that is subject to the modulational instability. In this connection, it is relevant to mention a scenario for stabilization of these states by means of dispersion and nonlinearity management [272].

As concerns experiments, we mention the work by Klein et al. [273] on the observation of ultrafast RW patterns in the form of single-peak, twin-peak, and triple-peak states in fiber laser systems. In this connection, it is worth mentioning that the fast dynamics of fiber lasers makes them very convenient tools for studying rare, extreme events, such as RWs [273]. In a recent work, Coulibaly et al. [274] have investigated, both experimentally and theoretically, turbulence-induced rogue waves in Kerr resonators. The existence of rogue waves as extreme events characterized by long-tail statistics has been unveiled, see Ref. [274] for details.

5. MATTER-WAVE SOLITONS, VORTICES, AND QUANTUM DROPLETS

In this Section we briefly overview recent theoretical and experimental studies of physical properties of matter-wave fundamental solitons and vortices in atomic BECs; see also a viewpoint on multidimensional soliton structures and their legacy in contemporary atomic, molecular, and optical physics [9], and a recent brief review [275] of the creation of matter-wave solitons by means of spin-orbit coupling, and a review [276] on 2D vortex solitons in spin-orbit-coupled dipolar BECs.

In a recent work, Adhikari [277] has studied, by means of a variational method and numerical simulations, the possibility of creating self-bound stable 3D matter-wave spherical boson-fermion quantum balls, in the presence of an attractive boson-fermion interaction and weak repulsive three-boson interaction. Stable and metastable bright vortex solitons in 3D spin-orbit-coupled three-component BECs with spin $F = 1$ have been studied by Gautam and Adhikari [278], using the numerical solution of mean-field equations and a Gaussian variational approximation for bright vortex solitons. Li et al. [279] have obtained 3D stable solitons in binary atomic BECs with spin-orbit coupling and out-of-phase linear and nonlinear Bessel optical lattices, employing a variational method and direct numerical simulations of a coupled system of Gross-Pitaevskii (GP) equations. Stable solitons with the semivortex and mixed-mode structures have been found, depending on the strength of the intra- and intercomponent spatial modulation of the nonlinearity and spin-orbit coupling. Solitary-vortex evolution in 2D harmonically trapped BECs has been investigated by Wang and Zhou [280], using a variational method based on a Gaussian ansatz and taking asymmetric perturbation effects into regard. Wang and Yang [281] have investigated, analytically and numerically, the dynamics of 2D vor-
tex solitons described by the GP model that incorporates a harmonic trapping potential and higher-order nonlinear interactions. The stability of a trapless dipolar BEC with temporal modulation of the short-range contact interaction has been studied by Sabari and Dey [282]. Analytical and numerical considerations have shown that the oscillatory contact interaction prevents the collapse of the trapless dipolar BEC, see Ref. [282]. A spectral and dynamical analysis of single vortex rings in anisotropic harmonically trapped 3D BECs has been performed by Ticknor et al. [283]. In a recent work, Salerno and Baizakov [284] have studied normal-mode oscillations of nonlocal composite matter-wave solitons by means of the variational approach and numerical simulations. The dynamics of traveling dark-bright solitons in spin-orbit-coupled BECs has been investigated by D’Ambroise et al. [285]. Sakaguchi et al. [286] have reviewed the properties of 2D matter-wave solitons, governed by the spinor system of GP equations with cubic nonlinearity, including the spin-orbit coupling and the Zeeman splitting.

Recently, Sakaguchi [287] has investigated, both analytically (using a variational method) and numerically, soliton lattices in a physical setting described by the GP equation with nonlocal and repulsive coupling. Also, effects of the rotating harmonic potential and spin-orbit coupling have been studied numerically in the same work. A numerical study of collision of two vortex dipoles propagating in opposite directions along parallel lines in 2D BECs has been reported by Yang et al. [288]. Three generic scenarios of the interaction of vortex dipoles in homogeneous BECs have been identified by varying the impact parameter: (i) the vortex recombination mode, (ii) the encircling mode, and (iii) the flyby mode. The axial-symmetry-breaking instability of leapfrogging vortex rings in BECs has been investigated numerically by Ikuta et al. [289], using both the GP equation and the vortex-filament model. It has been revealed that three coaxial quantized vortex rings exhibit aperiodic leapfrogging dynamics, see Ref. [289] for details. Kartashov and Zezyulin [290] have numerically identified families of stable multiring and rotating solitons in 2D spin-orbit-coupled BECs with radially periodic potentials, by solving a system of Gross-Pitaevskii equations with the Rashba spin-orbit coupling, see Ref. [290] for a detailed discussion of the physical model. A theoretical study of Bose-Einstein condensation of light via laser cooling of an ensemble of two-level atoms inside a Fabry-Perot cavity has been recently reported by Wang et al. [291].

A true breakthrough in the creation of stable multidimensional soliton-like states, in the form of “quantum droplets”, which are stabilized by Lee-Huang-Yang corrections to the mean-field dynamics, has been reported recently. These corrections take effects of quantum fluctuations into regard [292]. As demonstrated by Petrov [35] for two-component BECs with contact repulsive intra-component and attractive inter-component interactions, the accordingly modified GP equation for the condensate’s wave function, $\psi$, in 3D acquires an additional repulsive (hence, stabi-
lizing) quartic term $\sim |\psi|^4\psi$ (an effect of the Rabi coupling between the components was additionally considered in Ref. [293]).

In the 2D setting, the effective nonlinearity is different, $\sim \ln \left( |\psi|^2/\psi_0^2 \right) |\psi|^2\psi$, with some density constant, $\psi_0^2$ [294]. In the 1D limit, in which the quantum-fluctuation correction is represented by an attractive term $\sim |\psi|^2\psi$, the structure and dynamics of localized states was studied in Ref. [295]. Two different types of 1D droplets were identified, namely small ones with an approximately Gaussian shape and broad flat-top states. Three-dimensional droplets with embedded vorticity have been studied by Kartashov et al. [296]. By means of numerical and approximate analytical methods, families of self-trapped vortex tori, carrying the topological charges $m_1 = m_2 = 1$ or $m_1 = m_2 = 2$ in their components, were constructed and their stability was investigated numerically. The stability regions of the vortex droplets are broad for $m_{1,2} = 1$ and are narrower for $m_{1,2} = 2$, whereas the droplets with hidden vorticity (that is, when $m_1 = -m_2 = 1$ in the two components) are completely unstable in the relevant parameter domain. In this connection, Gautam and Adhikari [297] have recently investigated, by means of a Gaussian variational approximation, the formation of stable self-trapped spherical quantum balls in the model of binary BECs with the above-mentioned quartic terms and additional repulsive quintic ones. Collisions between such spherical balls have been also investigated by numerical methods in Ref. [297].

In the 2D geometry, vortex droplets were predicted in Ref. [298], which may be stable up to $m_1 = m_2 = 5$. Furthermore, 2D hidden-vorticity states have their stability region too. Ring-shaped clusters of 2D droplets were very recently predicted in Ref. [299]. In the presence of spin-orbit coupling, 2D droplets of the semi-vortex type were considered in Ref. [300]). The problem of spontaneous symmetry breaking (SSB) of QDs in a dual-core trap has been recently investigated by Liu et al. [301]. One-dimensional QDs in a binary bosonic gas loaded in a symmetric double-core cigar-shaped potential have been addressed in Ref. [301]. We note that unlike the usual SSB mechanism for matter-wave solitons, which is induced by mean-field interactions, the SSB of QDs in this physical system is driven by the interplay of the mean-field and Lee-Huang-Yang quadratic attractive terms in the linearly coupled GP equations. Also, collisions between moving two-components QDs have been considered by Liu et al. [301], which demonstrate inelastic interactions, leading to the merger of QDs into breathers.

In the experiment, nearly two-dimensional [302, 303] and fully three-dimensional [304, 305] quantum droplets have been created in a condensate of $^{39}\text{K}$ atoms, with two components represented by different hyperfine states. Another prediction and realization of stable 3D quantum droplets was demonstrated in single-component BEC with long-range dipole-dipole interactions between atoms carrying permanent magnetic dipole moments [306–308]. However, unlike the above-mentioned droplets,
supported by the contact interactions, with embedded vorticity, vortex droplets in dipolar condensates are completely unstable [309].

Recently, Kartashov et al. [310] have introduced an exactly integrable nonlinear model describing the dynamics of spinor solitons in space-dependent matrix gauge potentials of rather general types. As an example of this dynamical model, a self-attractive BEC with random spin-orbit coupling has been considered in Ref. [310]. If the Zeeman splitting is also included in the model, the nonlinear system becomes nonintegrable. However, for a strong Zeeman splitting, the integrability is restored, see Ref. [310].

The results surveyed in this Section are largely based on the broad arsenal of analytical and numerical frameworks currently available for solving the GP equations and its siblings in nonlinear optics settings. Among relevant models that are currently available in this context, we mention 1D and 2D equations used to approximately describe 3D systems [311–317] (in addition to the above-mentioned dimensional reduction for quantum droplets), and a large family of numerical solvers for nonlinear partial differential equations, such as the time-dependent GP equation and its generalizations [318–322].

On the experimental side, we refer here to a work by Danaila et al. [323] in which an experimental evidence for the existence, stability, and dynamics of dark-antidark solitary waves in two-component BECs, confined in elongated traps, has been reported. Theoretical and numerical investigation of vector dark-antidark states in the 2D setting have been also reported in Ref. [323]. Experimental demonstration of the existence of robust three-component soliton states in spinor $F = 1$ BECs, of the dark-bright-bright and dark-dark-bright types, has been recently reported by Bersano et al. [324]. These multicomponent states have lifetimes on the order of hundreds of milliseconds. Kang et al. [325] have recently reported the observation of spin domain walls bounded by half-quantum vortices in spin-1 BEC with antiferromagnetic interactions. In a recent work, Nguyen et al. [326] have investigated, both experimentally and theoretically, the response of an elongated BEC to direct modulations of the interaction, induced by the Feshbach resonance. The parametric excitation of BEC leads to two distinct modulation regimes: formation of Faraday patterns, and the formation of granular states featuring irregular fluctuating spatial patterns.

6. CONCLUSION

The objective of this article has been to provide a brief overview of recently obtained results in theoretical and experimental studies of nonlinear-wave phenomenology in the areas of nonlinear optics and matter waves in BEC. The article has been structured, essentially, as a resource text, which provides a large number of
references to relevant publications reporting recent advances in the research field of nonlinear waves in optical and matter-wave media. In particular, included are new findings concerning few-cycle (ultra-narrow) optical pulses, parity-time-symmetric nonlinear modes, rogue waves in various media, quantum droplets recently predicted and experimentally created in bosonic condensates, multidimensional (2D and 3D) and multicomponent (spinor and vector) settings, and some other topics.

The survey of the currently available results clearly demonstrates that there remains a vast room for further theoretical and experimental studies of solitons and related self-trapped modes in photonics, BEC, and other quickly developing areas of physics.

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