ENTANGLEMENT QUANTIFICATION USING VARIOUS INSEPARABILITY CRITERIA FOR CORRELATED PHOTONS

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Abstract. In this paper, a detailed comparison among the exhibited nature of entanglement of the cavity radiation of the non-degenerate three-level cascade laser with a coherently driven parametric amplifier and coupled to a two-mode thermal reservoir by applying different inseparability criteria is presented. Although the achievable degree of entanglement is generally found to vary with the applied inseparability criteria, there are cases for which more than three of the applied criteria lead to a significant degree of entanglement for certain parameters. In particular, Duan-Giedke-Cirac-Zoller (DGCZ) criterion, logarithmic negativity, Hillery-Zubairy, and Cauchy-Schwartz inequality inseparability criteria predict a similar pattern of entanglement except when the atoms are initially prepared in a maximum atomic coherent superposition. The presence of the parametric oscillator leads to an increase in the degree of squeezing and entanglement. Moreover, the degree of entanglement for the cavity radiation is significantly enhanced with the linear gain coefficient.

Key words: Atomic coherence, parametric amplifier, entanglement, photon number correlations.

1. INTRODUCTION

Three-level cascade lasers have a great deal of interest over the years in connection with its potential as a source of the strongly correlated two-mode cavity light that in turn leads to two-photon entanglement which is one of the various interesting non classical properties [1–6]. One of the possible mechanism of producing this strong correlation is linked to atomic coherence that induced by preparing the atoms initially in a coherent superposition of the top and bottom levels [7]. In this regard, we can define three-level laser as a two-photon quantum optical device capable of producing strongly correlated light with a number of non-classical features.

Quantum entanglement has been considered as the non-locality aspect of quantum correlations with no classical counterpart. This surprising property was investigated in the seminal paper of EPR [8]. Later, Bell recognized that entanglement leads to experimentally testable deviations of quantum mechanics from classical physics [9]. Finally, with the advent of quantum information theory, entanglement was recognized as a resource, enabling tasks like quantum cryptography [10], quantum computation and communication [11], quantum dense coding [12], quantum teleportation.

entanglement swapping [14], sensitive measurements [15], and quantum telecloning [16]. It turns out that the correlation induced by the initial preparation of the atomic superposition in a three-level laser leads to a generation of a strong continuous variable entanglement [17, 18]. Efforts have been made to propose theoretically ways to produce a strong entangled light from three-level laser using different techniques [19–33]. Authors in [22] and [24] have used parametric amplifier to study its effect on the quantum properties of light generated by the three-level laser. Abebechew [22] found the parametric amplifier in the laser cavity increasing the degree of entanglement.

On the other hand, some authors have studied the effect of thermal reservoir on quantum properties of light [34–38]. Moreover, Ghiu et al. [34] study the polarization of a quantum radiation field under a de-Gaussification process. Specifically, they consider the addition of photons to a two-mode thermal state to get mixed non-Gaussian and nonclassical states which are still diagonal in the Fock basis. In addition, Isar [37] investigates the possibility to generate quantum steering in a system of two coupled bosonic modes interacting with a common thermal reservoir. The initial state of the subsystem is taken of Gaussian form and the evolution under the quantum dynamical semigroup assures the preservation in time of the Gaussian form of the state. Furthermore, Zubarev et al. [38] describe the time evolution of the fidelity of teleportation and logarithmic negativity in a system composed of two coupled bosonic modes in contact with a thermal bath, in the framework of the theory of open systems based on completely positive quantum dynamical semigroups.

In this paper, the entanglement properties of the light produced by a non-degenerate three-level laser with non-degenerate parametric amplifier and coupled to a two-mode thermal reservoir is studied at or below the threshold condition. Different criteria of entanglement quantification are used to detect the entanglement of the two-mode light generated by the two-photon optical device under consideration. We consider a non-degenerate three-level laser in which the pump mode emerging from the parametric amplifier does not couple the top and bottom levels of the injected atoms. This could be realized by putting on the right-side of the nonlinear crystal a screen which absorbs the pump mode. We carry out our analysis applying the pertinent master equation describing the dynamics of the optical device. Using the resulting solutions, the degree of entanglement for the cavity radiation using the Duan-Giedke-Cirac-Zoller, logarithmic negativity, Hillery-Zubairy, and Cauchy-Schwartz inequality inseparability criteria are measured.

2. HAMILTONIAN AND MASTER EQUATION

We represent the top, intermediate, and bottom levels of a three-level atom in a cascade configuration by \( |3\rangle, |2\rangle, \) and \( |1\rangle \), respectively, as shown in Fig. 1. In addi-
tion, we assume the two modes $a_1$ and $a_2$ to be at resonance with the two transitions $|3\rangle \rightarrow |2\rangle$ and $|2\rangle \rightarrow |1\rangle$ dipole allowed respectively, and direct transition between level $|3\rangle$ and level $|1\rangle$ to be dipole forbidden. The interaction of a non-degenerate three-level atom with the cavity modes can be described by the Hamiltonian:

$$\hat{H}_I = ig \left[ |3\rangle \langle 2| \hat{a}_1 - \hat{a}^\dagger_1 |2\rangle \langle 3| + |2\rangle \langle 1| \hat{a}_2 - \hat{a}^\dagger_2 |1\rangle \langle 2| \right],$$

(1)

where $g$ is a coupling constant, which is taken to be the same for both transitions, and $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the two cavity modes. In this paper, we take the initial state of a three-level atom to be $|\psi_A(0)\rangle = C_3(0)|3\rangle + C_1(0)|1\rangle$ and hence the initial density operator for a single atom has the form:

$$\hat{\rho}_A(0) = \rho^{(0)}_{33} |3\rangle \langle 3| + \rho^{(0)}_{31} |3\rangle \langle 1| + \rho^{(0)}_{13} |1\rangle \langle 3| + \rho^{(0)}_{11} |1\rangle \langle 1|,$$

(2)

where $\rho^{(0)}_{33} = |C_3|^2$ and $\rho^{(0)}_{11} = |C_1|^2$ are, respectively, the probabilities for the atom to be initially in the upper and lower levels, and $\rho^{(0)}_{31} = C_3C_1^*$ and $\rho^{(0)}_{13} = C_1C_3^*$. Actually, this assumption corresponds to a situation in which the three-level atom is initially prepared in a coherent superposition of the top and bottom levels.

In addition, we seek to consider when such atoms are injected into a cavity at constant rate $r_n$ and removed after sometime $\tau$, which is long enough for the atoms to decay spontaneously to levels other than the middle or the lower level. The spontaneous decay rate $\gamma$ is taken to be the same for the two upper levels. In the good cavity limit, $\gamma \gg \kappa$, where $\kappa$ is the cavity damping constant, the cavity mode variables change slowly compared with the atomic variables. Hence the atomic variables will reach steady state in relatively short time. The time derivative of such variables

Fig. 1 – (Color online). Schematic representation of a two-mode three-level cascade laser coupled with a two-mode thermal reservoir.
amplifier can be described in the interaction picture by the quantum Hamiltonian:

\[ H = i\varepsilon_1 [\hat{a}_1^\dagger - \hat{a}_1 + \hat{a}_2^\dagger - \hat{a}_2] + i\varepsilon_2 [\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2], \]  

where \( \varepsilon_1 \) is proportional to the amplitude of the driving light modes and \( \varepsilon \) is proportional to the amplitude of the pump mode that drives the NLC (nonlinear crystal). Taking into account equations (3) and (5) along with (6), the master equation of the

\[ \dot{\rho}_1(t) = \frac{A\rho_{33}^{(0)}}{2} \begin{bmatrix} 2\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 \rho - \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 \end{bmatrix} + \frac{A\rho_{11}^{(0)}}{2} \begin{bmatrix} 2\hat{a}_2 \rho \hat{a}_2^\dagger - \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2 \rho - \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2 \end{bmatrix} \]

\[ - \frac{A\rho_{33}^{(0)}}{2} \begin{bmatrix} 2\hat{a}_2 \rho \hat{a}_1 - \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1 \rho - \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1 \end{bmatrix} - \frac{A\rho_{11}^{(0)}}{2} \begin{bmatrix} 2\hat{a}_1 \rho \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rho - \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \end{bmatrix}, \]

where \( A = \frac{2g^3n}{\pi\gamma} \) is linear gain coefficient and we have set \( \rho_{31}^{(0)} = \rho_{31}^{(0)*} \). To this end, following the procedure described by Tesfa [26] and using the density operator of the thermal reservoir:

\[ \rho' = \sum_{n=0}^{\infty} \frac{\bar{n}_{th}^n}{(1 + \bar{n}_{th})^{n+1}} |n\rangle \langle n|, \]  

one can easily obtain the equation of evolution of the density operator for the interaction of the cavity with thermal reservoir:

\[ \dot{\rho}_2(t) = -i[H_S, \rho(t)] + \frac{\kappa}{2}(\bar{n}_{th} + 1)[2\hat{a}_1 \rho \hat{a}_1^\dagger - \hat{a}_1^\dagger \hat{a}_1 \rho - \hat{a}_1^\dagger \hat{a}_1] \]

\[ + \frac{\kappa}{2}\bar{n}_{th}[2\hat{a}^\dagger_1 \rho \hat{a}_1 - \hat{a}_1 \hat{a}_1^\dagger \rho - \hat{a}_1 \hat{a}_1^\dagger \rho + 2\hat{a}_1 \rho \hat{a}_2 - \hat{a}_1 \rho \hat{a}_2] - \rho \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_2 \hat{a}_2^\dagger \rho \]

\[ - \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \rho \]
system turns out to be:

\[
\hat{\rho}(t) = \varepsilon_1 \left[ \hat{\rho} \hat{a}_1 - \hat{\alpha}_1 \hat{\rho} + \hat{\rho} \hat{a}_1^\dagger + \hat{\rho}_2 \hat{\rho}_2^\dagger - \hat{\rho} \hat{a}_2 \right] + \frac{\varepsilon_1}{2} \left[ \hat{\rho} \hat{a}_1 \hat{a}_2 \right]
\]

\[
- \frac{1}{2} (A \rho_{aa}^{(0)} + \kappa n_{th}) [2 \hat{a}_1^\dagger \hat{\rho} \hat{a}_1 - \hat{a}_1^\dagger \hat{\rho} \hat{a}_1^\dagger \hat{\rho} \hat{a}_1^\dagger \hat{\rho} \hat{a}_1^\dagger + \frac{1}{2} (A \rho_{11}^{(0)} + \kappa (n_{th} + 1) + \frac{1}{2} (A \rho_{12}^{(0)} + \kappa n_{th}) [2 \hat{a}_2^\dagger \hat{\rho} \hat{a}_2^\dagger - \hat{a}_2^\dagger \hat{\rho} \hat{a}_2^\dagger + \frac{1}{2} (A \rho_{22}^{(0)} + \kappa (n_{th} + 1)] + \frac{A_{\rho 31}^{(0)}}{2} [2 \hat{a}_2^\dagger \hat{\rho} \hat{a}_1 - \hat{\rho} \hat{a}_1^\dagger \hat{\rho} \hat{a}_2^\dagger + \frac{A_{\rho 32}^{(0)}}{2} [2 \hat{a}_1^\dagger \hat{\rho} \hat{a}_2 - \hat{\rho} \hat{a}_2^\dagger \hat{\rho} \hat{a}_1^\dagger + \hat{\rho} \hat{a}_2^\dagger \hat{\rho} \hat{a}_1^\dagger - \hat{\rho} \hat{a}_2^\dagger \hat{\rho} \hat{a}_1^\dagger].
\]

Employing this master equation, the evolution of the two-mode cavity radiation in terms of e-number variables associated with the normal ordering \(\alpha_1(t)\) and \(\alpha_2(t)\) can be expressed in the form:

\[
\alpha_1(t) = -\gamma_{\alpha} \alpha_1(t) + \nu_{-} \alpha_2(t) + \varepsilon_1 + f_1(t), \tag{8a}
\]

\[
\alpha_2(t) = -\gamma_{\alpha} \alpha_2(t) + \nu_{+} \alpha_1(t) + \varepsilon_1 + f_2(t), \tag{8b}
\]

where \(\gamma_{\alpha} = \frac{1}{2} (\kappa - A \rho_{aa}^{(0)})\), \(\gamma_{\alpha} = \frac{1}{2} (\kappa + A \rho_{cc}^{(0)})\), \(\nu_{\pm} = \varepsilon \pm \frac{1}{2} A \rho_{ac}^{(0)}\) and \(f_1(t)\) and \(f_2(t)\) are the pertinent noise forces, the properties of which remain to be determined. Following the straightforward procedure outlined in [1, 22], it is possible to obtain:

\[
\alpha_1(t) = \Gamma_+(t) \alpha_1(0) + \chi_+(t) \alpha_2^*(0) + G_1(t) + \zeta_+ + \alpha_2(t) = \Gamma_-(t) \alpha_2(0) + \chi_-(t) \alpha_1^*(0) + G_2(t) + \zeta_-,
\]

in which

\[
G_1(t) = \int_0^t \left[ \Gamma_+(t-t') f_1(t') + \chi_+(t-t') f_2(t') \right] dt', \tag{10}
\]

\[
G_2(t) = \int_0^t \left[ \Gamma_-(t-t') f_2(t') + \chi_-(t-t') f_1(t') \right] dt', \tag{11}
\]

\[
\Gamma_\pm(t) = \frac{A_{\pm}}{2 \lambda} e^{-\frac{1}{2} \lambda_{\mp} t} - \frac{A_{\pm}}{2 \lambda} e^{-\frac{1}{2} \lambda_{\mp} t}, \tag{12}
\]

\[
\chi_\pm(t) = \frac{2 \nu_{\mp}}{2 \lambda} e^{-\frac{1}{2} \lambda_{\mp} - \frac{1}{2} \lambda_{\mp} t} - \frac{2 \nu_{\mp}}{2 \lambda} e^{-\frac{1}{2} \lambda_{\mp} - \frac{1}{2} \lambda_{\mp} t}, \tag{13}
\]

\[
\zeta_\pm = \frac{\varepsilon_1}{\lambda} \left[ \frac{A_{\pm} \pm 2 \nu_{\mp}}{\lambda_{\mp}} (1 - e^{-\frac{1}{2} \lambda_{\mp} t}) - \frac{A_{\mp} \pm 2 \nu_{\mp}}{\lambda_{\mp}} (1 - e^{-\frac{1}{2} \lambda_{\mp} t}) \right]. \tag{14}
\]
where the noise forces satisfy the correlations:

\[
\begin{align*}
\langle f_1(t) \rangle &= \langle f_2(t) \rangle = \langle f_1(t') f_1(t) \rangle = 0, \\
\langle f_2(t') f_2(t) \rangle &= \langle f_2(t') f_2(t) \rangle = \langle f_1(t') f_1(t) \rangle = 0, \\
\langle f_1(t) f_1(t') \rangle &= (A\rho^{(0)}_{aa} + \kappa n_{th}) \delta(t - t'), \\
\langle f_2(t) f_2(t') \rangle &= \kappa \bar{n}_{th} \delta(t - t'), \\
\langle f_1(t') f_2(t) \rangle &= \frac{1}{2} \nu_{+} \delta(t - t').
\end{align*}
\]

(15) \quad (16) \quad (17) \quad (18) \quad (19)

It proves to be useful to introduce a new parameter which relates the probabilities of the atom to be in the upper and lower levels. We define the parameter \( \eta \) such that \( \rho^{(0)}_{aa} = 1 - \eta^2 \) with \(-1 < \eta < 1\). For three-level atoms initially in a coherent superposition of the top and bottom levels, one obtains: \( \rho^{(0)}_{cc} = 1 + \eta^2 \) and in view of the relation \( |\rho^{(0)}_{ac}|^2 = \rho^{(0)}_{aa} \rho^{(0)}_{cc} \), one easily finds \( \rho^{(0)}_{ac} = \frac{1}{2} \sqrt{1 - \eta^2} \).

3. ENTANGLEMENT QUANTIFICATION

Here we proceed to study the entanglement condition of the two modes in the cavity. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents.

3.1. DUAN-GIEDKE-CIRAC-ZOLLER (DGCZ) CRITERION

According to the criteria set by Duan et al. [19], a quantum state of the system is entangled provided that the sum of the variances of the two EPR-type operators \( \hat{u} \) and \( \hat{v} \) satisfies the condition [8]:

\[
\Delta u^2 + \Delta v^2 < 2,
\]

(20)

where

\[
\begin{align*}
\hat{u} &= \hat{x}_1 - \hat{x}_2, \\
\hat{v} &= \hat{p}_1 + \hat{p}_2,
\end{align*}
\]

(21) \quad (22)

with \( \hat{x}_1 = \frac{1}{\sqrt{2}} (\hat{a}_1 + \hat{a}_1^\dagger) \), \( \hat{x}_2 = \frac{1}{\sqrt{2}} (\hat{a}_2 + \hat{a}_2^\dagger) \), \( \hat{p}_1 = \frac{i}{\sqrt{2}} (\hat{a}_1^\dagger - \hat{a}_1) \), \( \hat{p}_2 = \frac{i}{\sqrt{2}} (\hat{a}_2^\dagger - \hat{a}_2) \) being the quadrature operators for modes \( \hat{a}_1 \) and \( \hat{a}_2 \). Thus, the sum of the variances of \( \hat{u} \) and \( \hat{v} \) is easily found to be:

\[
\Delta u^2 + \Delta v^2 = 2 \left[ 1 + \langle \alpha_1^*(t) \alpha_1(t) \rangle + \langle \alpha_2^*(t) \alpha_2(t) \rangle - 2\langle \alpha_1(t) \alpha_2(t) \rangle \right].
\]

(23)
In view of equations (9a) and (9b) with (10)-(19), equation (23) turns out to be:

\[
\Delta u^2 + \Delta v^2 = 2 + \frac{\kappa A(1 - \eta)(4\kappa + 3A\eta + A) + 16\varepsilon^2(\kappa + A\eta)}{2[\kappa(\kappa + A\eta) - 4\varepsilon^2](2\kappa + A\eta)} \\
+ \frac{\kappa(4\varepsilon + A\sqrt{1 - \eta^2}) + [4\kappa^2(2\kappa + 2A\eta) + A^2]2\kappa\bar{n}_{th}}{2[\kappa(\kappa + A\eta) - 4\varepsilon^2](2\kappa + A\eta)} \\
- \frac{\kappa(4\varepsilon + A\sqrt{1 - \eta^2})(2\kappa + A\eta + A)}{2[\kappa(\kappa + A\eta) - 4\varepsilon^2](2\kappa + A\eta)} \\
+ \frac{[4\varepsilon(2\kappa + A\eta) + A^2\sqrt{1 - \eta^2}2\kappa\bar{n}_{th}]}{2[\kappa(\kappa + A\eta) - 4\varepsilon^2](2\kappa + A\eta)}
\] (24)

It is not difficult to see from equation (24) that the sum of the variances of the two

Fig. 2 – (Color online). Plots of $\Delta u^2 + \Delta v^2$ [Equation (24)] of the cavity radiation versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon = 0.399$, and for different values of $\bar{n}_{th}$.

Fig. 3 – (Color online). Plots of $\Delta u^2 + \Delta v^2$ [Equation (24)] of the cavity radiation versus $\eta$ for $\kappa = 0.8$, $\bar{n}_{th} = 0.5$, $\varepsilon = 0.399$ and for different values of the linear gain coefficient, $A$.

EPR-type operators are independent of the parameter $\varepsilon_1$ which represents the cavity driving coherent light. This shows that the cavity driving coherent light do not have any effect on the degree of entanglement of the two-mode light. This is due to the fact that the external coherent lights do not introduce additional coherence to the system which is believed to be the source of entanglement in three-level cascade lasers [6, 7, 24, 26].
It is clearly indicated on Figure 2 the cavity radiation is found to be entangled for all parameters under consideration. It can be observed that the degree of entanglement increases for smaller values of the initial preparation of atoms $\eta$, but decreases for the larger values. The maximum possible degree of entanglement in this case is found to be 84% for $A = 1000$ and occurs at $\eta = 0.08$. The introduction of thermal light is observed to improve the degree of entanglement for the minimum atomic coherence. Moreover, it can be seen from Figure 3 that $\Delta u^2 + \Delta v^2$ is less than 2 for all values of $\eta$ except for $\eta = 1$, hence the entanglement criterion (20) is satisfied. In addition, the degree entanglement of the cavity radiation is significantly enhanced with the linear gain coefficient. The maximum possible degree of entanglement in this case is found to be 84% for $A = 100$, $\kappa = 0.8$, $\bar{n}_{th} = 0.5$ and occurs at $\eta = 0.04$.

On the other hand, Figure 4 shows that the degree of entanglement of the cavity radiation versus the parametric amplifier, $\varepsilon$, and the injected atomic coherence, $\eta$, for $\kappa = 0.8$, $\bar{n}_{th} = 0.5$ and $A = 100$. As we see from the plot the degree of entanglement increases with the parametric amplifier and for small values of injected atomic coherence. The effect of the presence of the parametric oscillator is to increase the intra-cavity degree of entanglement for small values of $\eta$.

3.2. LOGARITHMIC NEGATIVITY

Another relevant method is the logarithmic negativity which depicts the presence of entanglement for a two-mode continuous variables based on the negativity of the partial transposition [26]. The logarithmic negativity is combined with negative partial transpose in another case where $V_S$ represents the smallest eigenvalue of the simplistic matrix [26]:

$$V_S = \sqrt{\sigma - \sqrt{(\sigma^2 - 4 \det \Gamma)}}$$

where the invariant and covariance matrices are respectively denoted as:
Fig. 5 – (Color online). A plot of the smallest eigenvalue [Equation (25)] of the two-mode cavity radiation versus \(\eta\) for \(\bar{n}_h = 0.5, \kappa = 0.8, \varepsilon = 0.399\) and for different values of linear gain coefficient.

\[
\sigma = \det \Sigma_1 + \det \Sigma_2 - 2\det \Sigma_{12}, \quad (26)
\]

\[
\Gamma = \begin{pmatrix}
\Sigma_1 & \Sigma_{12} \\
\Sigma_{12}^T & \Sigma_2
\end{pmatrix}, \quad (27)
\]

in which \(\Sigma_1\) and \(\Sigma_2\) are the covariance matrices describing each mode separately while \(\Sigma_{12}\) are the inter-modal correlations. The elements of the matrix in equation (27) are given by:

\[
\Gamma_{ij} = \frac{1}{2}\langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle - \langle \hat{X}_i \rangle \langle \hat{X}_j \rangle, \quad (28)
\]

in which \(i,j = 1,2,3,4\). The quadrature operators are defined as \(\hat{X}_1 = \hat{a}_1 + \hat{a}_1^\dagger\), \(\hat{X}_2 = i(\hat{a}_1^\dagger - \hat{a}_1)\), \(\hat{X}_3 = \hat{a}_2 + \hat{a}_2^\dagger\), and \(\hat{X}_4 = i(\hat{a}_2^\dagger - \hat{a}_2)\). With this introduction the extended covariance matrix goes over into:

\[
\Gamma = \begin{pmatrix}
\Lambda & 0 & \chi & 0 \\
0 & \Lambda & 0 & -\chi \\
\chi & 0 & \Delta & 0 \\
0 & -\chi & 0 & \Delta
\end{pmatrix}, \quad (29)
\]

where \(\Lambda = 2\langle \alpha_1 \alpha_1^* \rangle + 1\), \(\chi = 2\langle \alpha_1 \alpha_2 \rangle\), \(\Delta = 2\langle \alpha_2 \alpha_2 \rangle + 1\) are c-number variables associated with the normal ordering. The logarithmic negativity is defined as:

\[
E_N = \max[0, -\log_2 V_S], \quad (30)
\]

The entanglement is achieved when \(E_N\) is positive within the region of the lowest eigenvalue of covariance matrix \(V_S < 1\).

Next on account of equation (28) along with (29), one can readily show that:

\[
\det \Sigma_1 = [2\langle \alpha_1^* \alpha_1 \rangle + 1]^2, \quad (31a)
\]

\[
\det \Sigma_2 = [2\langle \alpha_2^* \alpha_2 \rangle + 1]^2, \quad (31b)
\]

\[
\det \Sigma_{12} = -4\langle \alpha_1 \alpha_2 \rangle^2. \quad (31c)
\]
It is also possible to establish that:

$$\text{det} \Gamma = \left[ \sqrt{\text{det} \Sigma_1 \text{det} \Sigma_2} - \sqrt{\text{det} \Sigma_1^{T} \text{det} \Sigma_2} \right]^2.$$  

(32)

As it can be noted on Figure 5, the degree of entanglement increases for smaller values of the initial preparation of atoms. It can also be seen that larger values of the linear gain coefficient produces a robust entangled light. The maximum achievable degree of entangled light in this case is 96%, and it occurs for $A = 1000$ and $\eta = 0.02$. This criterion also predicts the absence of entanglement for $\eta = 1$ no matter how we manipulate the rate of atomic injection in the absence of parametric amplifier. It is not difficult to see from Figure 6 that the parametric amplifier produces a considerable degree of entangled light for very small value of the linear gain coefficient regardless of how atoms are initially prepared. The maximum achievable degree of entangled light in this case is 84.5%, and it occurs for $\tilde{n}_{th} = 0$, $A = 100$ and $\eta = 0.08$. Exactly the same feature of DGCZ criterion is observed in this figure except on the maximum achievable degree of entanglement.

On the other hand, Figure 7 shows that the effect the parametric amplifier for large value of the linear gain coefficient does not produce considerable changes to the degree of entanglement. Hence the maximum achievable degree of entangled
light in this case is 82\%, and it occurs for $A = 100$ and $\eta = 0.13$. The behaviors of a non-degenerate three-level laser with non-degenerate parametric amplifier and non-degenerate three-level laser without the parametric amplifier appear to be the same for very large value of the linear gain coefficient (rate of atomic injection) with exception of the existence of the entanglement at the minimum atomic coherence which represents absence of photons in the cavity. Furthermore, it is clearly shown on Figure 8 that for large values of the linear gain coefficient and the parametric amplifier, the enhancement of maximum possible degree of entanglement is occurred.

3.3. PHOTON ANTI-BUNCHING

Photon anti-bunching phenomena happens when the statistics of photons is scattered in the time evolution. The correlation of scattered photons is studied using the second-order correlation function of photo-detection with respect to time. Based on a photo-detection experiment, for a coherent state, $g^{(2)}(\tau) = 1$ represents the highly correlated state. In this state, the probability of joint detection coincides with the probability of independent detection.

On the other hand, $g^{(2)}(\tau) = 0$ when the time delay approaches infinity, $\tau \to \infty$ which is to mean the joint probability of detecting the second photon decreases with time delay. Thus, the situation $g^{(2)}(\tau) < g^{(2)}(0)$ is identified as photon bunching which is to mean two photons tend to be detected simultaneously or after a short time delay [30]. Contrary to this, if $g^{(2)}(\tau) > g^{(2)}(0)$, the joint probability of detecting the second photon increases with time delay. This is known as photon anti-bunching. A field is said to be entangled if the inequality, $g^{(2)}(\tau) > g^{(2)}(0)$ is satisfied. For the coherent state, $g^{(2)}(\tau) = 1$ represents a classical state. However, for a non-classical field state, we have $g^{(2)}(\tau) < 1$ a violation of the classical result. Therefore, the photon anti-bunching phenomena occur when $g^{(2)}(\tau) < 1$ and $g^{(2)}(\tau) > g^{(2)}(0)$, implying the presence of entanglement. Second order equal time correlation function also serve as a helpful tool for Hillery and Zubairy, Cauchy-Stewart inequality inseparability criteria, and Cauchy-Stewart inequality criterion.
3.4. HILLERY-ZUBAIRY (HZ) CRITERION

According to the criterion introduced by Hillery-Zubairy, for two modes of light with \( \hat{a} \) and \( \hat{b} \) operators, the composite state is said to be entangled if condition:

\[
\langle \hat{a}_1 \hat{a}_2 \rangle > \sqrt{\langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle}
\]  

(33)
is satisfied [26]. \( \hat{n}_1 = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \) and \( \hat{n}_2 = \langle \hat{a}_2^\dagger \hat{a}_2 \rangle \) are the pertinent photon numbers corresponding to the involved modes whereas \( \langle \hat{a}_1 \hat{a}_2 \rangle \) is correlated mean photon number of the two-mode light. On the other hand, the photon number correlation for the two-mode cavity light can be expressed in terms of second order correlation function as:

\[
g^{(2)}(t) = 1 + \frac{\langle \alpha_1(t) \alpha_2(t) \rangle^2}{\langle \alpha_1^*(t) \alpha_1(t) \rangle \langle \alpha_2^*(t) \alpha_2(t) \rangle}.
\]  

(34)

Taking into account Hillery and Zubairy criterion, we can put the above equation in the form:

\[
g^{(2)}(t) > 2.
\]  

(35)

It has been shown that the Hillery-Zubairy criterion is another equivalent entanglement criterion with Cauchy-Stewart inequality when the inter-atomic interaction is not taken into account [26].

3.5. VIOLATION OF CAUCHY-SCHWARTZ INEQUALITY (VCSI)

We may also use the second-order correlation to determine the entanglement of a system of two modes cavity radiation [25]. A system of two mode cavity radiation is said to be entangled if it satisfies the Cauchy-Schwartz inequality which can be put in the form:

\[
\langle \hat{a}_1^2 \hat{a}_2^2 \rangle \geq \langle \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \rangle^2.
\]  

(36)

In this principle it is possible to study the non-classical photon number correlation at equal time using the following parameter takes the form:

\[
C_{12} = \left| \frac{\langle \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \rangle}{\langle \hat{a}_1^2 \hat{a}_2^2 \rangle} \right|^2.
\]  

(37)

It can be verified that this expression can be rewritten in terms of equal time correlation function as:

\[
C_{12} = \frac{1}{4} \left( g^{(2)}(t) \right)^2.
\]  

(38)

Taking into account equation (35), the above equation turns out to be:

\[
C_{12} > 1.
\]  

(39)

It is clearly shown in Figure 9 that the photon number correlation, \( C_{12} \geq 1 \), for all values of parameters under consideration. This indicates that the non-degenerate
Fig. 9 – A plot of the steady state photon number correlation function [Equation (38)] of the two-mode cavity radiation versus the initial preparation of atoms and amplitude of parametric amplifier for $n_{th} = 0.25, \kappa = 0.8, \varepsilon = 0.399$.

Fig. 10 – (Color online). Plots of the steady state photon number correlation function [Equation (38)] of the two-mode cavity radiation for $\kappa = 0.8, \varepsilon = 0.399, A = 100$ and for different values of $n_{th}$.

three-level cascade laser with parametric amplifier is a source of entangled light, according to the Cauchy-Schwartz inequality and HZ criteria. It is also observed that the criterion does not include the case for which entanglement is weak when the procedure following from the logarithmic negativity and DGCZ criteria are applied.

Moreover, it can be observed from Figure 10 that $C_{12}$ increases with the decreasing rate of atomic injection. The same situation has been reported in terms of second order correlation function that the degree of entanglement quantification approach is not necessarily directly proportional to the extent to which this criterion is satisfied [25]. It is possible to realize that the Cauchy-Schwartz inequality criterion can be important in predicting the presence of entanglement especially when the atomic coherence is close to maximum.

4. CONCLUSION

In this paper, different inseparability criteria have been used to detect and quantify the entanglement of the two-mode cavity radiation with a coherently driven cavity and whose cavity contains a non-degenerate parametric amplifier coupled to a two-mode thermal reservoir. The degree of entanglement by logarithmic negativity
and DGCZ criteria is greatly enhanced by decreasing the rate of atomic injection when the linear gain coefficient is closer to maximum. In both cases, the weak entangled light is generated when all atoms are initially prepared in the lower energy state and a large number atoms are constantly injected into the cavity regardless of the amplitude of the parametric amplifier. In this situation, the two-mode light is found to be entangled even for the minimum and maximum atomic coherence which respectively correspond to absence and availability of more photons in the cavity.

Moreover, in DGCZ and logarithmic negativity criteria increment in the linear gain coefficient compensates for the degraded degree of entanglement in ways of initial preparation the three-level atoms. Even though Cauchy-Schwartz inequality detects the entanglement of the cavity radiation, it does not account for the effect of a large rate of atomic injection when the initial preparation of atoms is equal or close to the maximum atomic coherence. In this regard, the Cauchy-Schwartz inequality exhibits the same pattern of entanglement quantification for a considerable rate of atomic injection. On the other hand, the inequality does not include the case for which the entangled light is weak when the procedure following from the logarithmic negativity and DGCZ criteria are applied. Contrary to this, Cauchy-Schwartz inequality is found to be an encouraging approach specially when the rate of atomic injection is larger and more photons are available in the cavity.

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