

NOTE ON THE SOLVING THE LAPLACE TIDAL EQUATION WITH LINEAR DISSIPATION

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Abstract. In this paper, we present a new solving procedure for *Laplace tidal equations* (LTEs) with linear dissipation: the analytic algorithm is implemented here for solving momentum equation of LTEs, where the dissipation term with linear dependence on velocity field of fluid flow has been additionally taken into consideration (which is supposed to approximate the decreasing of momentum for the Ocean's flows due to the viscous friction between Ocean's layers if we consider heat fluxes during the lost of energy inside the Ocean).

As a main result of this work, a new ansatz is suggested here for solving LTEs with linear dissipation: solving momentum equation is reduced to solving a system of three linear ordinary differential equations of first order with regard to three components of the flow velocity (depending on time t), along with mandatory using the continuity equation that determines the spatial part of solution.

In our derivation, the main motivation is the proper transformation of the previously presented system of equations to a convenient form, in which the minimum of numerical calculations are required to obtain the final solutions. Preferably, it should be the analytical solutions; we have presented the solution as a *linear combination* of linearly independent fundamental solutions (of real and complex values). We pointed out also the elegant case of partial solution for velocity field of real value. Nevertheless, we should use the continuity equation for identifying the spatial components of velocity field in the case of nonzero fluid pressure in the Ocean, along with nonzero total gravitational potential and the centrifugal potential (due to planetary rotation). It means that the system of Laplace tidal equations with additional *linear dissipation* term (in momentum equation) could not be solved analytically.

Key words: Laplace tidal equations, tidal dissipation, planetary rotation.

1. INTRODUCTION, EQUATIONS OF MOTION

The Laplace tidal equations [1] (LTEs, including continuity equation) describe the dynamics of fluid's velocity under the action of potential forces (including gravity) inside the upper layer of Ocean, which is supposed to be located relatively close to the boundary between the Ocean and the atmosphere of Earth.

In recent research [2], we have presented the proper analytical procedure for solving the Laplace tidal equations of the aforementioned type [1] (including continuity equation). It is worth to note that *Pierre-Simon Laplace*, one of the famous scientists in the field of mechanics, has been spending not less than 30 years of his life exclusively for creating a self-consistent theory of terrestrial tides in its practical application as the dynamics of tidal waves in ocean basins, and for solving the Laplace tidal equations (LTEs, named after him).

Despite the fact that the momentum equation of LTEs presents the appropriate linearization of Navier-Stokes momentum equation (it has been presented in the earlier works of *Pierre-Simon Laplace*), the aforesaid equation was still too hard to solve for him. To understand the importance of brain-storming this problem, let us recall the citation from one of the pioneers among famous scientists who have been involved in solving LTEs, S.S. Hough in his prominent paper published in the year 1898 [3]: “The difficulties experienced by Laplace in his attempts to integrate the equation in question were so great that he abandoned all efforts to obtain a general solution, and confined his discussion to a few of the special cases which present the greatest interest from a practical point of view; even in these simple cases however his original attempts to express the solutions by means of the coefficients associated with his name were discarded in favor of series proceeding according to powers of a certain variable used to define the position of a point on the Earth’s surface”.

Also, we should note that there is a large amount of previous and recent works concerning analytical treatments with respect to these equations, which should be mentioned accordingly [4–9].

The appropriate procedure for solving LTEs has been presented in [2], albeit not achieving the universal analytical formulae for velocity field of motion of the ocean that covers the surface of the planet, but presenting the proper theoretical ansatz to solve the LTEs, depending on given initial conditions (the Cauchy problem in the whole space has been considered in [2] for the system of LTEs).

Nevertheless, there is a generalization of the aforementioned problem (presenting great interest from practical point of view), which has not been considered properly till the year 2011 (see the comprehensive work of Tyler [10]). Namely, the LTEs with linear dissipation in momentum equation is such a generalization of LTEs, where the dissipation term with linear dependence on velocity field of fluids flow has been additionally taken into account.

In the current research, we present the general solution of this problem, using the results of the theory of linear differential equations of the first kind for solving systems of such type.

In the co-rotating frame of a Cartesian coordinate system $\vec{r}=\{x, y, z\}$, the linearized (in \vec{v}) equations of motion of the ocean (*including Rayleigh dissipation term* $\alpha \vec{v}$) that covers the surface of the planet are [10], at given initial or boundary conditions:

$$\nabla \cdot \vec{v} = 0, \quad (1)$$

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + 2\vec{\Omega} \times \vec{v} &= -\nabla \left(\frac{p}{\rho} + \Phi + \chi \right) + \alpha \vec{v}, \\ -\nabla \left(\frac{p}{\rho} + \Phi + \chi \right) &\equiv \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \end{aligned} \quad (2)$$

where \vec{v} is the fluid's velocity, $\vec{v} = \{v_i\}$ ($i = 1, 2, 3$); $\vec{\Omega}$ is the *angular velocity pseudovector*, which determines the actual rate (and direction) of Earth's rotation, $\vec{\Omega} = \{\Omega_i\}$, let us note that on a time-scale of few cycles of rotation, each $\Omega_i \cong \Omega_i(x, y, z)$ (not depending on time t). Besides, here ρ is the fluid density, $p = p(\vec{r}, t)$ is the pressure, $\Phi(\vec{r})$ is the total gravitational potential, and $\chi(\vec{r})$ is the centrifugal potential (due to planetary rotation); $2\vec{\Omega} \times \vec{v}$ is the Coriolis acceleration (likewise, due to planetary rotation); $\alpha = \text{const}$ is the corresponding coefficient in Rayleigh dissipation term ($\alpha < 0$). As for the domain in which the flow occurs and the boundary conditions, let us consider only the Cauchy problem in the whole space.

Furthermore, according to [1], we have neglected the compressibility of the fluid in the Ocean (it means that the aforesaid fluid density $\rho = \text{const}$).

2. GENERAL PRESENTATION OF NON-HOMOGENEOUS SOLUTION FOR VELOCITY FIELD

The momentum equation (2) is known to be the system of three *linear* partial differential equations, PDEs (with regard to the time t) for three unknown functions [11–15]: v_1 , v_2 , and v_3 . Let us consider the variables $\{x, y, z\}$ as variable parameters below.

In this respect, (2) should be considered as the system of three linear differential equations with all the coefficients depending on time t . In accordance with [16] (p. 71), the general solution of such aforementioned system should be given as below:

$$\chi_s(t) = \sum_{v=1}^3 \zeta_{v,s} \cdot \left(\int \left(\frac{\Delta_v}{\Delta} \right) dt + C_v \right), \quad s = 1, 2, 3, \quad \Delta = \begin{vmatrix} \zeta_{1,1} & \zeta_{1,2} & \zeta_{1,3} \\ \zeta_{2,1} & \zeta_{2,2} & \zeta_{2,3} \\ \zeta_{3,1} & \zeta_{3,2} & \zeta_{3,3} \end{vmatrix} \quad (3)$$

where $\{\chi_s\}$ are the fundamental system of solutions of the three equations (2) for components v_1 , v_2 , and v_3 with regard to the time t ; $\{\zeta_{v,s}\}$ are the fundamental system of solutions $\vec{v}|_0 = \{U, V, W\}$ for the corresponding homogeneous variant of (2),

see Eqs. (4) below; $\{C_v\}$ are the set of functions, not depending on time t ; besides, here we denote as below:

$$\Delta_1 = \begin{vmatrix} f_x & f_y & f_z \\ \zeta_{2,1} & \zeta_{2,2} & \zeta_{2,3} \\ \zeta_{3,1} & \zeta_{3,2} & \zeta_{3,3} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \zeta_{1,1} & \zeta_{1,2} & \zeta_{1,3} \\ f_x & f_y & f_z \\ \zeta_{3,1} & \zeta_{3,2} & \zeta_{3,3} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \zeta_{1,1} & \zeta_{1,2} & \zeta_{1,3} \\ \zeta_{2,1} & \zeta_{2,2} & \zeta_{2,3} \\ f_x & f_y & f_z \end{vmatrix}.$$

It means that the system of Eqs. (2), being presented as a system of equations (4)

$$\begin{cases} \frac{\partial v_1}{\partial t} = (v_2 \cdot (2\Omega_3) - v_3 \cdot (2\Omega_2)) + \alpha \cdot v_1 + f_1, \\ \frac{\partial v_2}{\partial t} = (v_3 \cdot (2\Omega_1) - v_1 \cdot (2\Omega_3)) + \alpha \cdot v_2 + f_2, \\ \frac{\partial v_3}{\partial t} = v \cdot (v_1 \cdot (2\Omega_2) - v_2 \cdot (2\Omega_1)) + \alpha \cdot v_3 + f_3, \end{cases} \quad (4)$$

should be considered as having been solved if we obtain a general solution of *the corresponding homogeneous* system for (4), $\vec{v}|_0 = \{U, V, W\}$ with respect to time t :

$$\frac{d\vec{v}|_0}{dt} = \vec{v}|_0 \times (2\vec{\Omega}) + \alpha \vec{v}|_0, \quad \Rightarrow \quad \begin{cases} \frac{dU}{dt} = V \cdot (2\Omega_3) - W \cdot (2\Omega_2) + \alpha U, \\ \frac{dV}{dt} = W \cdot (2\Omega_1) - U \cdot (2\Omega_3) + \alpha V, \\ \frac{dW}{dt} = U \cdot (2\Omega_2) - V \cdot (2\Omega_1) + \alpha W. \end{cases} \quad (5)$$

Let us present in the next Section the aforementioned general solution of the corresponding *homogeneous* system (5) with respect to time t .

Also, according to the *continuity* equation (1), the appropriate restriction should be valid for identifying of the spatial components of velocity field \vec{v} (*non-homogeneous* solution) along with the set of functions $\{C_v(x, y, z)\}$ in (3):

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0 \quad (6)$$

which is a PDE of the first kind; $\{v_i\}$ ($i = 1, 2, 3$) depend on variables (x, y, z, t) .

3. PRESENTATION OF THE TIME-DEPENDENT PART OF SOLUTION FOR EQS. (5)

For the sake of simplicity of the presentation of the solution, let us denote in (5) the appropriate constants (with respect to the time t) as below:

$$a = \alpha, \quad \gamma = 2\Omega_3, \quad \beta = 2\Omega_2, \quad b = 2\Omega_1 \quad \Rightarrow \quad \begin{cases} \frac{dU}{dt} = a \cdot U + \gamma \cdot V - \beta \cdot W, \\ \frac{dV}{dt} = -\gamma \cdot U + a \cdot V + b \cdot W, \\ \frac{dW}{dt} = \beta \cdot U - b \cdot V + a \cdot W. \end{cases} \quad (7)$$

The characteristic equation for the system (7) should be presented accordingly (see [16] (paragraph 13.1 p. 86) as below:

$$\begin{vmatrix} (a-s) & \gamma & -\beta \\ -\gamma & (a-s) & b \\ \beta & -b & (a-s) \end{vmatrix} = 0 \quad (8)$$

where s is the proper root of algebraic equation (8).

So, we obtain from Eqn. (8):

$$\begin{aligned} & \begin{vmatrix} (a-s) & \gamma & -\beta \\ -\gamma & (a-s) & b \\ \beta & -b & (a-s) \end{vmatrix} = 0 \\ \Rightarrow & (a-s) \cdot ((a-s)^2 + b^2) - \gamma(-\gamma(a-s) - \beta \cdot b) - \beta \cdot (\gamma \cdot b - \beta \cdot (a-s)) = 0, \quad (9) \\ \Rightarrow & (a-s) \cdot ((a-s)^2 + b^2 + \gamma^2 + \beta^2) = 0 \end{aligned}$$

It means that one of the roots s is real ($s_1 = a$), but two others belong to imaginary numbers:

$$s_{2,3} = a \pm i \cdot \sqrt{b^2 + \gamma^2 + \beta^2} \quad (10)$$

So, according to the approach suggested in [16] (paragraph 13.1) we should substitute each of solutions components (11.1) – (11.3) the appropriate expressions in (7) (here below the upper symbol “ i ” denotes that this constant belongs to

imaginary numbers; solutions (11.1)–(11.3) are proved in [16] to be linearly independent):

$$\begin{cases} U = (A_{11} + B_{11} \cdot t + C_{11} \cdot t^2) \cdot \exp(a \cdot t), \\ V = (A_{12} + B_{12} \cdot t + C_{12} \cdot t^2) \cdot \exp(a \cdot t), \\ W = (A_{13} + B_{13} \cdot t + C_{13} \cdot t^2) \cdot \exp(a \cdot t), \end{cases} \quad (11.1)$$

$$\begin{cases} U = (A_{21}^i + B_{21}^i \cdot t + C_{21}^i \cdot t^2) \cdot \exp((a + i \cdot \sqrt{b^2 + \gamma^2 + \beta^2}) \cdot t), \\ V = (A_{22}^i + B_{22}^i \cdot t + C_{22}^i \cdot t^2) \cdot \exp((a + i \cdot \sqrt{b^2 + \gamma^2 + \beta^2}) \cdot t), \\ W = (A_{23}^i + B_{23}^i \cdot t + C_{23}^i \cdot t^2) \cdot \exp((a + i \cdot \sqrt{b^2 + \gamma^2 + \beta^2}) \cdot t), \end{cases} \quad (11.2)$$

$$\begin{cases} U = (A_{31}^i + B_{31}^i \cdot t + C_{31}^i \cdot t^2) \cdot \exp((a - i \cdot \sqrt{b^2 + \gamma^2 + \beta^2}) \cdot t), \\ V = (A_{32}^i + B_{32}^i \cdot t + C_{32}^i \cdot t^2) \cdot \exp((a - i \cdot \sqrt{b^2 + \gamma^2 + \beta^2}) \cdot t), \\ W = (A_{33}^i + B_{33}^i \cdot t + C_{33}^i \cdot t^2) \cdot \exp((a - i \cdot \sqrt{b^2 + \gamma^2 + \beta^2}) \cdot t). \end{cases} \quad (11.3)$$

Thus, we obtain from (7) for solution (11.1) as below (the mathematical procedure of obtaining expressions for the coefficients in (11.1) has been moved to an **Appendix A1**, with only the resulting formulae left in the main text):

$$\Rightarrow \begin{cases} \begin{cases} b \cdot A_{12} = \beta \cdot A_{11} \\ b \cdot A_{13} = \gamma \cdot A_{11} \end{cases} \\ \\ \begin{cases} B_{11} = \gamma \cdot A_{12} - \beta \cdot A_{13} = 0 \\ B_{12} = -\gamma \cdot A_{11} + b \cdot A_{13} = 0 \\ B_{13} = \beta \cdot A_{11} - b \cdot A_{12} = 0 \end{cases} \\ \\ C_{11} = C_{12} = C_{13} \equiv 0, \end{cases} \quad (12)$$

Let us denote, just for simplicity

$$\sqrt{b^2 + \gamma^2 + \beta^2} = \omega \quad (*)$$

Thus, we obtain for coefficients of solution (11.2) from equations of system (7) as below:

$$\begin{cases} U = A_{11} \cdot \exp(a \cdot t), \\ V = \frac{\beta}{b} A_{11} \cdot \exp(a \cdot t), \\ W = \frac{\gamma}{b} A_{11} \cdot \exp(a \cdot t), \end{cases} \quad (13)$$

$$\begin{cases} B_{21}^i + 2C_{21}^i \cdot t + i \cdot \omega \cdot A_{21}^i + i \cdot \omega \cdot B_{21}^i \cdot t + i \cdot \omega \cdot C_{21}^i \cdot t^2 = \\ = \gamma \cdot A_{22}^i + \gamma \cdot B_{22}^i \cdot t + \gamma \cdot C_{22}^i \cdot t^2 - \beta \cdot A_{23}^i - \beta \cdot B_{23}^i \cdot t - \beta \cdot C_{23}^i \cdot t^2, \\ B_{22}^i + 2C_{22}^i \cdot t + i \cdot \omega \cdot A_{22}^i + i \cdot \omega \cdot B_{22}^i \cdot t + i \cdot \omega \cdot C_{22}^i \cdot t^2 = \\ = -\gamma \cdot A_{21}^i - \gamma \cdot B_{21}^i \cdot t - \gamma \cdot C_{21}^i \cdot t^2 + b \cdot A_{23}^i + b \cdot B_{23}^i \cdot t + b \cdot C_{23}^i \cdot t^2, \\ B_{23}^i + 2C_{23}^i \cdot t + i \cdot \omega \cdot A_{23}^i + i \cdot \omega \cdot B_{23}^i \cdot t + i \cdot \omega \cdot C_{23}^i \cdot t^2 = \\ = \beta \cdot A_{21}^i + \beta \cdot B_{21}^i \cdot t + \beta \cdot C_{21}^i \cdot t^2 - b \cdot A_{22}^i - b \cdot B_{22}^i \cdot t - b \cdot C_{22}^i \cdot t^2. \end{cases} \quad (14)$$

The mathematical procedure of obtaining the proper expressions for coefficients in (11.2) has been moved to an **Appendix A2**, with only the resulting formulae (15)–(17) left in the main text:

$$\begin{cases} C_{22}^i = \frac{(\beta \cdot b - i \cdot \omega \cdot \gamma)}{(\gamma \cdot b + i \cdot \omega \cdot \beta)} \cdot C_{23}^i, \\ C_{21}^i = \frac{(\gamma^2 - \beta^2)}{(\gamma \cdot b + i \cdot \omega \cdot \beta)} \cdot C_{23}^i, \end{cases} \quad (15)$$

$$\begin{cases} B_{21}^i = \frac{b \cdot B_{23}^i - i \cdot \omega \cdot B_{22}^i - 2C_{22}^i}{\gamma}, \\ B_{22}^i = \frac{(-b \cdot \beta + i \cdot \omega \cdot \gamma) \cdot (2\gamma \cdot C_{21}^i - 2i \cdot \omega \cdot C_{22}^i) + (\beta \cdot \gamma + i \cdot \omega \cdot b) \cdot (-2\beta \cdot C_{22}^i - 2\gamma \cdot C_{23}^i)}{(\gamma^2 - \omega^2) \cdot (-b \cdot \beta + i \cdot \omega \cdot \gamma) + (b \cdot \gamma + i \cdot \omega \cdot \beta) \cdot (\beta \cdot \gamma + i \cdot \omega \cdot b)}, \\ B_{23}^i = \frac{(\gamma^2 - \omega^2) \cdot (2\beta \cdot C_{22}^i - 2\gamma \cdot C_{23}^i) - (b \cdot \gamma + i \cdot \omega \cdot \beta) \cdot (-2\gamma \cdot C_{21}^i - 2i \cdot \omega \cdot C_{22}^i)}{(\gamma^2 - \omega^2) \cdot (-b \cdot \beta + i \cdot \omega \cdot \gamma) + (b \cdot \gamma + i \cdot \omega \cdot \beta) \cdot (\beta \cdot \gamma + i \cdot \omega \cdot b)}, \end{cases} \quad (16)$$

$$\left\{ \begin{array}{l} A_{21}^i = \frac{b \cdot A_{23}^i - i \cdot \omega \cdot A_{22}^i - B_{22}^i}{\gamma}, \\ A_{22}^i = \frac{(i \cdot \omega \cdot \gamma - b \cdot \beta) \cdot (-\gamma \cdot B_{21}^i + i \cdot \omega \cdot B_{22}^i) + (\beta \cdot \gamma + b \cdot i \cdot \omega) \cdot (\beta \cdot B_{22}^i - \gamma \cdot B_{23}^i)}{(\omega^2 - \gamma^2) \cdot (i \cdot \omega \cdot \gamma - b \cdot \beta) - (i \cdot \omega \cdot \beta + b \cdot \gamma) \cdot (\beta \cdot \gamma + b \cdot i \cdot \omega)}, \\ A_{23}^i = \frac{(i \cdot \omega \cdot \beta + b \cdot \gamma) \cdot (-\gamma \cdot B_{21}^i + i \cdot \omega \cdot B_{22}^i) + (\omega^2 - \gamma^2) \cdot (\beta \cdot B_{22}^i + \gamma \cdot B_{23}^i)}{(\beta \cdot \gamma + b \cdot i \cdot \omega) \cdot (i \cdot \omega \cdot \beta + b \cdot \gamma) - (\omega^2 - \gamma^2) \cdot (i \cdot \omega \cdot \gamma - b \cdot \beta)}. \end{array} \right. \quad (17)$$

Taking into account denotation (*), the aforementioned formulae (15) – (17) mean that we have obtained a solution of a type (11.2), which should fully satisfy the equations of system (7). Let us note additionally that all the coefficients in (15)–(17) are to be of complex value.

Obviously, solution of a type (11.3) should be obtained in the same manner as the above algorithm (12) – (17) {we simply should change the sign in expression “+ $i \cdot \omega$ ” to opposite “- $i \cdot \omega$ ” in all the formulae (15) – (17)}.

4. DISCUSSION

As we can see from the derivation above, the system of Laplace tidal equations with the additional *linear dissipation* term (in momentum equation) is proved to be very unlikely to solve analytically.

Indeed, at the first step we should solve the *homogeneous* momentum equation (2) in a form (5) with regard to the time t (depending on three components of angular velocity $\vec{\Omega}$ of planetary rotation). In the general case, we have obtained here the fundamental system of solutions (11.1) – (11.3) according to the approach suggested in [16] (paragraph 13.1), which are linearly independent in accordance with the basic demand for such fundamental systems of solutions [16]; but meanwhile, solutions (11.2) – (11.3) are of complex value, nevertheless. So, we should find additional conditions, under which the aforesaid solutions should be converted to the solutions of real value. In this case, these solutions should have a clear physical sense.

Furthermore, at the second step, we should obtain the *non-homogeneous* part of the solution for velocity field \vec{v} in a form (4) (depending on pressure $p(\vec{r}, t)$, total gravitational potential $\Phi(\vec{r})$, and centrifugal potential $\chi(\vec{r})$ due to planetary rotation). We point out the appropriate algorithm in **Section 2** for obtaining the aforementioned *non-homogeneous* solution for velocity field \vec{v} .

Last but not least, at the third step we should use the continuity equation in a form (1) or (6) for identifying the spatial components of velocity field \vec{v} (*non-homogeneous* solution) along with the set of functions $\{C_\nu(x, y, z)\}$ in (3) and with the initial conditions for velocity field, but depending on the spatial parts of the components of angular velocity $\vec{\Omega}(x, y, z)$ of Earth’s rotation, spatial parts of pressure p , of total gravitational potential Φ , and centrifugal potential χ .

On the other hand, these fundamental equations (1)–(2) are usually examined with a proper initial and boundary conditions. In this research, we consider only the Cauchy problem in the whole space.

As for the physical relevance of this new type of solutions, let us discuss the aforementioned additional conditions, under which these solutions should be converted to the solutions of real value. It means the existence of the additional restrictions at choosing of the coefficients (15)–(17) (appropriate mathematical explanations have been moved to an **Appendix A3**).

Indeed, we could choose a proper *linear combination* of linearly independent solutions (11.2)–(11.3), which should transform appropriately *the partial* solution into the solution of real value (by transforming the combined left parts of appropriate conditions (18) in **Appendix A3** to the combined zero meanings).

So, such partial solution of real value should be presented as below ($\{D_i = \text{const}\}$, $i = 1, \dots, 6$):

$$\exp(a \cdot t) \cdot \left((D_1 + D_2 \cdot t + D_3 \cdot t^2) \cdot \cos(\omega \cdot t) - (D_4 + D_5 \cdot t + D_6 \cdot t^2) \cdot \sin(\omega \cdot t) \right) \quad (19)$$

Let us schematically show in Figs. 1–3, *e.g.* the appropriate component of the velocity field (19) (depending on time t), which corresponds to the aforementioned *partial* solution.

We should especially note that the variables $\{x, y, z\}$ should be considered as variable parameters in our derivation of solutions (in Fig. 1 we designate $x = t$ just for the aim of presenting the plot of solution).

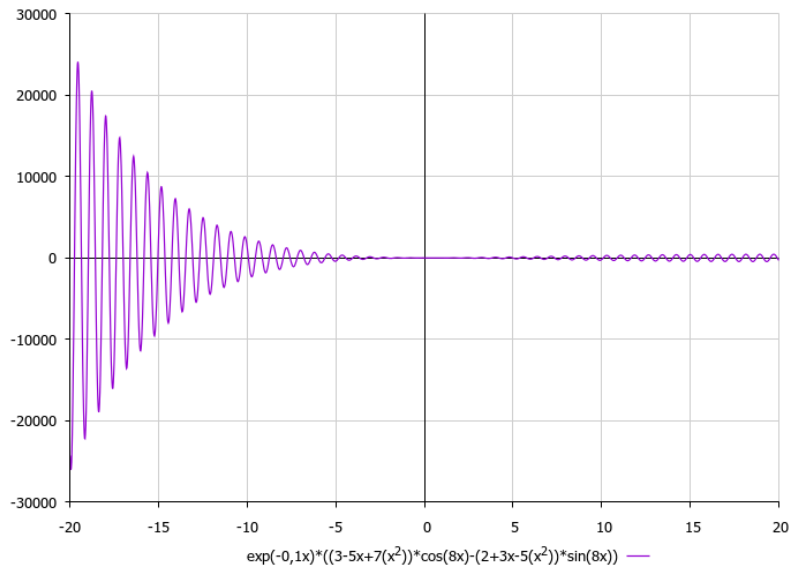


Fig. 1 – A *schematic* plot of the component (19) for partial solution (here we designate $x = t$ just for the aim of presenting the plot of solution).

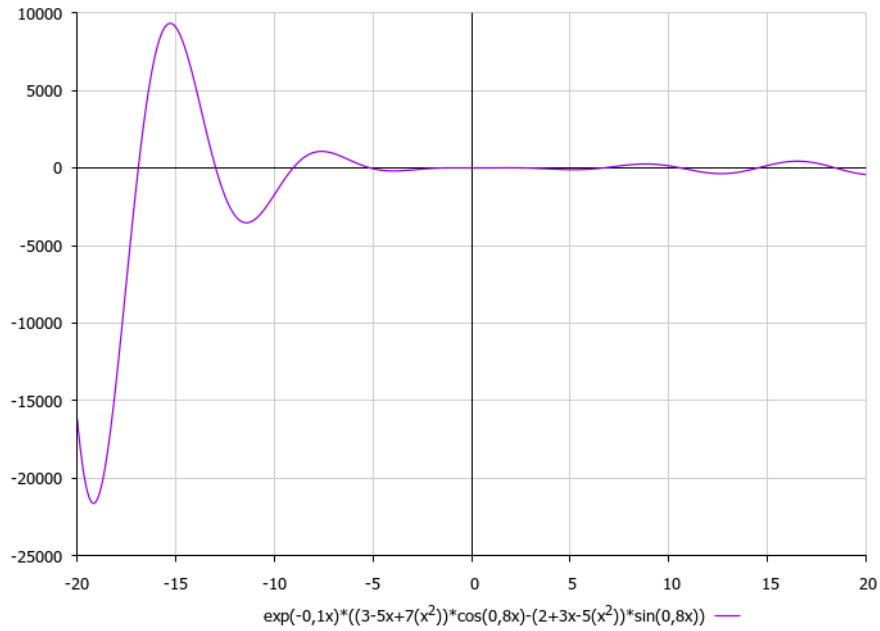


Fig. 2 – A schematic plot of the component (19) for partial solution (here we designate $x = t$ just for the aim of presenting the plot of solution).

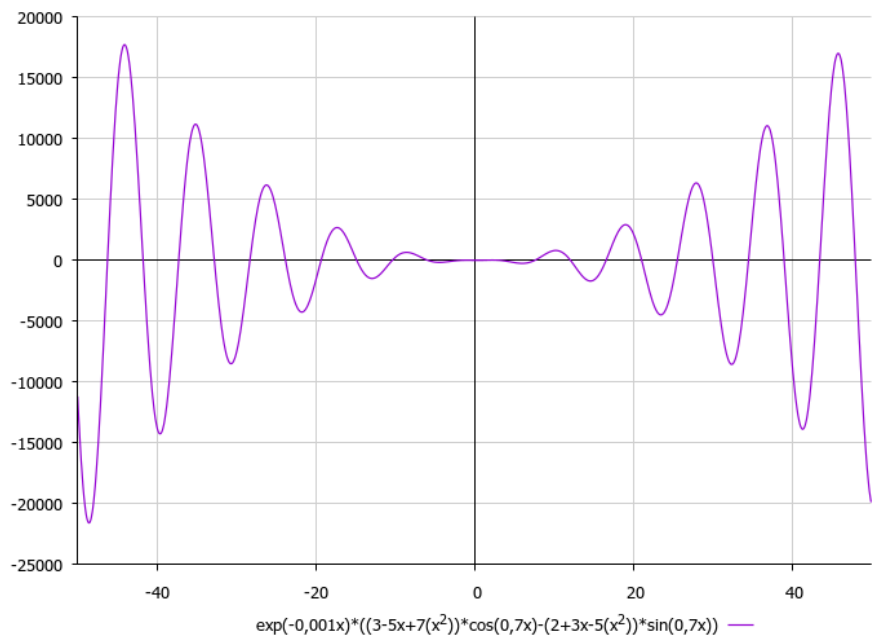


Fig. 3 – A schematic plot of the component (19) for partial solution (here we designate $x = t$ just for the aim of presenting the plot of solution).

5. CONCLUSION

As for the purpose of the current research, we can formulate it as follows: the main aim is to find a kind of exact solution (of real value) to the system of equations under consideration. Namely, each exact solution can clarify the structure, intrinsic code, and topology of the variety of possible solutions (from the mathematical point of view).

In this elegant analytic survey, we present a new approach for solving *Laplace tidal equations* (LTEs) with linear dissipation: the analytic algorithm is implemented here for solving the momentum equation of LTEs, where the dissipation term with a linear dependence on velocity field has been additionally taken into consideration. Basing on the results of the theory of ordinary differential equations, the aforementioned system of Laplace tidal equations (including the continuity equation) has been successfully explored with respect to the existence of an analytical way for presentation of the solution; moreover, the system of momentum equations has been successfully solved analytically: the analytic algorithm is pointed out for solving the momentum equation, which has been reduced to the analytical solution of three nonlinear ODEs with respect to the components of the flow velocity field, described by Laplace tidal equations. Moreover, absolutely new partial analytical solutions of real value have been obtained for the special case of nonzero fluid pressure in the Ocean, along with nonzero total gravitational potential and the centrifugal potential. Such new results, namely (19), can be used in calculations for modeling tidal waves on the surface of the real Ocean (with viscous dissipative effects at the bottom taken into account at spatial spreading of tidal waves deep inside the ocean medium), albeit they can be applied in ordinary way for testing the accuracy of numerical methods.

In the presented approach, the main motivation was the proper transformation of the previously presented system of equations to the convenient form, in which the minimum of numerical calculations are required to obtain the final solutions. We have presented the solution as a *linear combination* of linearly independent fundamental solutions (of real and complex value). Among others, we have pointed out also the elegant case of partial solution for velocity field of real value. Nevertheless, we should use the continuity equation for identifying the spatial components of velocity field in the case of nonzero fluid pressure in the Ocean, along with nonzero total gravitational potential and the centrifugal potential (due to planetary rotation). It means that the system of Laplace tidal equations with additional influence of the *linear dissipation* could not be solved analytically in the general case.

Also, some remarkable articles should be cited, which concern the problem under consideration, see Refs. [17–25].

Appendix A1

Obtaining of the coefficients for solution (11.1)

$$\left\{ \begin{array}{l} \frac{d((A_{11} + B_{11} \cdot t + C_{11} \cdot t^2) \cdot \exp(a \cdot t))}{dt} = a \cdot (A_{11} + B_{11} \cdot t + C_{11} \cdot t^2) \cdot \exp(a \cdot t) + \\ + \gamma \cdot (A_{12} + B_{12} \cdot t + C_{12} \cdot t^2) \cdot \exp(a \cdot t) - \beta \cdot (A_{13} + B_{13} \cdot t + C_{13} \cdot t^2) \cdot \exp(a \cdot t), \\ \frac{d((A_{12} + B_{12} \cdot t + C_{12} \cdot t^2) \cdot \exp(a \cdot t))}{dt} = -\gamma \cdot (A_{11} + B_{11} \cdot t + C_{11} \cdot t^2) \cdot \exp(a \cdot t) + \\ + a \cdot (A_{12} + B_{12} \cdot t + C_{12} \cdot t^2) \cdot \exp(a \cdot t) - b \cdot (A_{13} + B_{13} \cdot t + C_{13} \cdot t^2) \cdot \exp(a \cdot t), \\ \frac{d((A_{13} + B_{13} \cdot t + C_{13} \cdot t^2) \cdot \exp(a \cdot t))}{dt} = \beta \cdot (A_{11} + B_{11} \cdot t + C_{11} \cdot t^2) \cdot \exp(a \cdot t) - \\ - b \cdot (A_{12} + B_{12} \cdot t + C_{12} \cdot t^2) \cdot \exp(a \cdot t) + a \cdot (A_{13} + B_{13} \cdot t + C_{13} \cdot t^2) \cdot \exp(a \cdot t). \end{array} \right. \Rightarrow$$

$$\left\{ \begin{array}{l} B_{11} + 2C_{11} \cdot t + A_{11} \cdot a + B_{11} \cdot a \cdot t + C_{11} \cdot t^2 \cdot a = a \cdot A_{11} + a \cdot B_{11} \cdot t + a \cdot C_{11} \cdot t^2 + \\ + \gamma \cdot A_{12} + \gamma \cdot B_{12} \cdot t + \gamma \cdot C_{12} \cdot t^2 - \beta \cdot A_{13} - \beta \cdot B_{13} \cdot t - \beta \cdot C_{13} \cdot t^2, \\ B_{12} + 2C_{12} \cdot t + A_{12} \cdot a + B_{12} \cdot a \cdot t + C_{12} \cdot t^2 \cdot a = -\gamma \cdot A_{11} - \gamma \cdot B_{11} \cdot t - \gamma \cdot C_{11} \cdot t^2 + \\ + a \cdot A_{12} + a \cdot B_{12} \cdot t + a \cdot C_{12} \cdot t^2 + b \cdot A_{13} + b \cdot B_{13} \cdot t + b \cdot C_{13} \cdot t^2, \\ B_{13} + 2C_{13} \cdot t + A_{13} \cdot a + B_{13} \cdot a \cdot t + C_{13} \cdot t^2 \cdot a = \beta \cdot A_{11} + \beta \cdot B_{11} \cdot t + \beta \cdot C_{11} \cdot t^2 - \\ - b \cdot A_{12} - b \cdot B_{12} \cdot t - b \cdot C_{12} \cdot t^2 + a \cdot A_{13} + a \cdot B_{13} \cdot t + a \cdot C_{13} \cdot t^2. \end{array} \right. \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{l} B_{11} = \gamma \cdot A_{12} - \beta \cdot A_{13}, \\ 2C_{11} = \gamma \cdot B_{12} - \beta \cdot B_{13}, \\ \beta \cdot C_{13} = \gamma \cdot C_{12}, \\ \\ B_{12} = -\gamma \cdot A_{11} + b \cdot A_{13}, \\ 2C_{12} = -\gamma \cdot B_{11} + b \cdot B_{13}, \\ b \cdot C_{13} = \gamma \cdot C_{11}, \\ \\ b \cdot C_{12} = \beta \cdot C_{11}, \\ B_{13} = \beta \cdot A_{11} - b \cdot A_{12}, \\ 2C_{13} = \beta \cdot B_{11} - b \cdot B_{12}, \end{array} \right.$$

$$\begin{aligned}
B_{11} &= \gamma \cdot A_{12} - \beta \cdot A_{13}, \quad B_{12} = -\gamma \cdot A_{11} + b \cdot A_{13}, \quad B_{13} = \beta \cdot A_{11} - b \cdot A_{12}, \\
C_{11} &= \frac{1}{2}(\gamma \cdot (-\gamma \cdot A_{11} + b \cdot A_{13}) - \beta \cdot (\beta \cdot A_{11} - b \cdot A_{12})) = \frac{1}{2}(-(\gamma^2 + \beta^2) \cdot A_{11} + \beta \cdot b \cdot A_{12} + \gamma \cdot b \cdot A_{13}), \\
\Rightarrow C_{12} &= \frac{1}{2}(-\gamma \cdot B_{11} + b \cdot B_{13}) = \frac{1}{2}(-(\gamma^2 + b^2) \cdot A_{12} + \beta \cdot b \cdot A_{11} + \gamma \cdot \beta \cdot A_{13}), \quad \Rightarrow \\
C_{13} &= \frac{1}{2}(\beta \cdot B_{11} - b \cdot B_{12}) = \frac{1}{2}(-(\beta^2 + b^2) \cdot A_{13} + \gamma \cdot b \cdot A_{11} + \gamma \cdot \beta \cdot A_{12}), \\
C_{11} &= C_{12} = C_{13} \equiv 0, \\
\begin{cases} -(\gamma^2 + \beta^2) \cdot A_{11} + \beta \cdot b \cdot A_{12} + \gamma \cdot b \cdot A_{13} = 0 \\ \beta \cdot b \cdot A_{11} - (\gamma^2 + b^2) \cdot A_{12} + \gamma \cdot \beta \cdot A_{13} = 0 \\ \gamma \cdot b \cdot A_{11} + \gamma \cdot \beta \cdot A_{12} - (\beta^2 + b^2) \cdot A_{13} = 0 \end{cases} &\Rightarrow \begin{cases} -\beta(\gamma^2 + \beta^2) \cdot A_{11} + \beta^2 \cdot b \cdot A_{12} + \beta \cdot \gamma \cdot b \cdot A_{13} = 0 \\ \beta \cdot b^2 \cdot A_{11} - b \cdot (\gamma^2 + b^2) \cdot A_{12} + \gamma \cdot \beta \cdot b \cdot A_{13} = 0 \\ \gamma \cdot b \cdot A_{11} + \gamma \cdot \beta \cdot A_{12} - (\beta^2 + b^2) \cdot A_{13} = 0 \end{cases} \\
\Rightarrow &\begin{cases} b \cdot A_{12} = \beta \cdot A_{11} \\ b \cdot A_{13} = \gamma \cdot A_{11} \\ B_{11} = \gamma \cdot A_{12} - \beta \cdot A_{13} = 0 \\ B_{12} = -\gamma \cdot A_{11} + b \cdot A_{13} = 0 \\ B_{13} = \beta \cdot A_{11} - b \cdot A_{12} = 0 \\ C_{11} = C_{12} = C_{13} \equiv 0, \end{cases} \quad (12)
\end{aligned}$$

Appendix A2

Obtaining of the coefficients for solution (11.2)

Let us denote, just for simplicity

$$\sqrt{b^2 + \gamma^2 + \beta^2} = \omega$$

Thus, we obtain for coefficients of solution (11.2) from equations of system (7) as below

$$\begin{cases} B_{21}^i + 2C_{21}^i \cdot t + i \cdot \omega \cdot A_{21}^i + i \cdot \omega \cdot B_{21}^i \cdot t + i \cdot \omega \cdot C_{21}^i \cdot t^2 = \\ = \gamma \cdot A_{22}^i + \gamma \cdot B_{22}^i \cdot t + \gamma \cdot C_{22}^i \cdot t^2 - \beta \cdot A_{23}^i - \beta \cdot B_{23}^i \cdot t - \beta \cdot C_{23}^i \cdot t^2, \\ B_{22}^i + 2C_{22}^i \cdot t + i \cdot \omega \cdot A_{22}^i + i \cdot \omega \cdot B_{22}^i \cdot t + i \cdot \omega \cdot C_{22}^i \cdot t^2 = \\ = -\gamma \cdot A_{21}^i - \gamma \cdot B_{21}^i \cdot t - \gamma \cdot C_{21}^i \cdot t^2 + b \cdot A_{23}^i + b \cdot B_{23}^i \cdot t + b \cdot C_{23}^i \cdot t^2, \\ B_{23}^i + 2C_{23}^i \cdot t + i \cdot \omega \cdot A_{23}^i + i \cdot \omega \cdot B_{23}^i \cdot t + i \cdot \omega \cdot C_{23}^i \cdot t^2 = \\ = \beta \cdot A_{21}^i + \beta \cdot B_{21}^i \cdot t + \beta \cdot C_{21}^i \cdot t^2 - b \cdot A_{22}^i - b \cdot B_{22}^i \cdot t - b \cdot C_{22}^i \cdot t^2. \end{cases} \quad (14)$$

⇒

$$\begin{cases} B_{21}^i + i \cdot \omega \cdot A_{21}^i = \gamma \cdot A_{22}^i - \beta \cdot A_{23}^i, \\ 2C_{21}^i \cdot t + i \cdot \omega \cdot B_{21}^i \cdot t = \gamma \cdot B_{22}^i \cdot t - \beta \cdot B_{23}^i \cdot t, \\ i \cdot \omega \cdot C_{21}^i \cdot t^2 = \gamma \cdot C_{22}^i \cdot t^2 - \beta \cdot C_{23}^i \cdot t^2, \\ \\ B_{22}^i + i \cdot \omega \cdot A_{22}^i = -\gamma \cdot A_{21}^i + b \cdot A_{23}^i, \\ 2C_{22}^i \cdot t + i \cdot \omega \cdot B_{22}^i \cdot t = -\gamma \cdot B_{21}^i \cdot t + b \cdot B_{23}^i \cdot t, \\ i \cdot \omega \cdot C_{22}^i \cdot t^2 = -\gamma \cdot C_{21}^i \cdot t^2 + b \cdot C_{23}^i \cdot t^2, \\ \\ B_{23}^i + i \cdot \omega \cdot A_{23}^i = \beta \cdot A_{21}^i - b \cdot A_{22}^i, \\ 2C_{23}^i \cdot t + i \cdot \omega \cdot B_{23}^i \cdot t = \beta \cdot B_{21}^i \cdot t - b \cdot B_{22}^i \cdot t, \\ i \cdot \omega \cdot C_{23}^i \cdot t^2 = \beta \cdot C_{21}^i \cdot t^2 - b \cdot C_{22}^i \cdot t^2, \end{cases} \Rightarrow \begin{cases} B_{21}^i + i \cdot \omega \cdot A_{21}^i = \gamma \cdot A_{22}^i - \beta \cdot A_{23}^i, \\ B_{22}^i + i \cdot \omega \cdot A_{22}^i = -\gamma \cdot A_{21}^i + b \cdot A_{23}^i, \\ B_{23}^i + i \cdot \omega \cdot A_{23}^i = \beta \cdot A_{21}^i - b \cdot A_{22}^i, \\ \\ 2C_{21}^i + i \cdot \omega \cdot B_{21}^i = \gamma \cdot B_{22}^i - \beta \cdot B_{23}^i, \\ 2C_{22}^i + i \cdot \omega \cdot B_{22}^i = -\gamma \cdot B_{21}^i + b \cdot B_{23}^i, \\ 2C_{23}^i + i \cdot \omega \cdot B_{23}^i = \beta \cdot B_{21}^i - b \cdot B_{22}^i, \\ \\ i \cdot \omega \cdot C_{21}^i = \gamma \cdot C_{22}^i - \beta \cdot C_{23}^i, \\ i \cdot \omega \cdot C_{22}^i = -\gamma \cdot C_{21}^i + b \cdot C_{23}^i, \\ i \cdot \omega \cdot C_{23}^i = \beta \cdot C_{21}^i - b \cdot C_{22}^i, \end{cases}$$

where the last *linear* algebraic sub-system yields as below:

$$\begin{cases} i \cdot \omega \cdot C_{21}^i = \gamma \cdot C_{22}^i - \beta \cdot C_{23}^i, \\ i \cdot \omega \cdot C_{22}^i = -\gamma \cdot C_{21}^i + b \cdot C_{23}^i, \\ i \cdot \omega \cdot C_{23}^i = \beta \cdot C_{21}^i - b \cdot C_{22}^i, \end{cases} \Rightarrow \begin{cases} i \cdot \omega \cdot C_{21}^i - \gamma \cdot C_{22}^i + \beta \cdot C_{23}^i = 0, \\ \gamma \cdot C_{21}^i + i \cdot \omega \cdot C_{22}^i - b \cdot C_{23}^i = 0, \\ -\beta \cdot C_{21}^i + b \cdot C_{22}^i + i \cdot \omega \cdot C_{23}^i = 0, \end{cases}$$

$$\Rightarrow \begin{cases} i \cdot \omega \cdot C_{21}^i - \gamma \cdot C_{22}^i + \beta \cdot C_{23}^i = 0, \\ \beta \cdot \gamma \cdot C_{21}^i + i \cdot \omega \cdot C_{22}^i - \beta \cdot b \cdot C_{23}^i = 0, \\ -\beta \cdot \gamma \cdot C_{21}^i + \gamma \cdot b \cdot C_{22}^i + i \cdot \omega \cdot \gamma \cdot C_{23}^i = 0, \end{cases} \Rightarrow \begin{cases} C_{22}^i = \frac{(\beta \cdot b - i \cdot \omega \cdot \gamma)}{(\gamma \cdot b + i \cdot \omega \cdot \beta)} \cdot C_{23}^i, \\ C_{21}^i = \frac{(\gamma^2 - \beta^2)}{(\gamma \cdot b + i \cdot \omega \cdot \beta)} \cdot C_{23}^i, \end{cases} \quad (15)$$

So, we obtain

$$\begin{cases} 2C_{21}^i + i \cdot \omega \cdot B_{21}^i = \gamma \cdot B_{22}^i - \beta \cdot B_{23}^i, \\ 2C_{22}^i + i \cdot \omega \cdot B_{22}^i = -\gamma \cdot B_{21}^i + b \cdot B_{23}^i, \\ 2C_{23}^i + i \cdot \omega \cdot B_{23}^i = \beta \cdot B_{21}^i - b \cdot B_{22}^i, \end{cases} \Rightarrow \begin{cases} i \cdot \omega \cdot B_{21}^i - \gamma \cdot B_{22}^i + \beta \cdot B_{23}^i = -2C_{21}^i, \\ \gamma \cdot B_{21}^i + i \cdot \omega \cdot B_{22}^i - b \cdot B_{23}^i = -2C_{22}^i, \\ -\beta \cdot B_{21}^i + b \cdot B_{22}^i + i \cdot \omega \cdot B_{23}^i = -2C_{23}^i, \end{cases}$$

$$\Rightarrow \begin{cases} (\gamma^2 - \omega^2) \cdot B_{22}^i - (\beta \cdot \gamma + i \cdot \omega \cdot b) \cdot B_{23}^i = 2\gamma \cdot C_{21}^i - 2i \cdot \omega \cdot C_{22}^i, \\ (b \cdot \gamma + i \cdot \omega \cdot \beta) \cdot B_{22}^i + (-b \cdot \beta + i \cdot \omega \cdot \gamma) \cdot B_{23}^i = -2\beta \cdot C_{22}^i - 2\gamma \cdot C_{23}^i, \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} B_{21}^i = \frac{b \cdot B_{23}^i - i \cdot \omega \cdot B_{22}^i - 2C_{22}^i}{\gamma}, \\ B_{22}^i = \frac{(-b \cdot \beta + i \cdot \omega \cdot \gamma) \cdot (2\gamma \cdot C_{21}^i - 2i \cdot \omega \cdot C_{22}^i) + (\beta \cdot \gamma + i \cdot \omega \cdot b) \cdot (-2\beta \cdot C_{22}^i - 2\gamma \cdot C_{23}^i)}{(\gamma^2 - \omega^2) \cdot (-b \cdot \beta + i \cdot \omega \cdot \gamma) + (b \cdot \gamma + i \cdot \omega \cdot \beta) \cdot (\beta \cdot \gamma + i \cdot \omega \cdot b)}, \\ B_{23}^i = \frac{(\gamma^2 - \omega^2) \cdot (2\beta \cdot C_{22}^i - 2\gamma \cdot C_{23}^i) - (b \cdot \gamma + i \cdot \omega \cdot \beta) \cdot (-2\gamma \cdot C_{21}^i - 2i \cdot \omega \cdot C_{22}^i)}{(\gamma^2 - \omega^2) \cdot (-b \cdot \beta + i \cdot \omega \cdot \gamma) + (b \cdot \gamma + i \cdot \omega \cdot \beta) \cdot (\beta \cdot \gamma + i \cdot \omega \cdot b)}, \end{cases} \quad (16)$$

and, besides, we also obtain

$$\begin{cases} B_{21}^i + i \cdot \omega \cdot A_{21}^i = \gamma \cdot A_{22}^i - \beta \cdot A_{23}^i, \\ B_{22}^i + i \cdot \omega \cdot A_{22}^i = -\gamma \cdot A_{21}^i + b \cdot A_{23}^i, \\ B_{23}^i + i \cdot \omega \cdot A_{23}^i = \beta \cdot A_{21}^i - b \cdot A_{22}^i, \end{cases} \Rightarrow \begin{cases} i \cdot \omega \cdot A_{21}^i - \gamma \cdot A_{22}^i + \beta \cdot A_{23}^i = -B_{21}^i, \\ \gamma \cdot A_{21}^i + i \cdot \omega \cdot A_{22}^i - b \cdot A_{23}^i = -B_{22}^i, \\ -\beta \cdot A_{21}^i + b \cdot A_{22}^i + i \cdot \omega \cdot A_{23}^i = -B_{23}^i, \end{cases} \Rightarrow$$

$$\begin{cases} (\omega^2 - \gamma^2) \cdot A_{22}^i + (\beta \cdot \gamma + b \cdot i \cdot \omega) \cdot A_{23}^i = -\gamma \cdot B_{21}^i + i \cdot \omega \cdot B_{22}^i, \\ (i \cdot \omega \cdot \beta + b \cdot \gamma) \cdot A_{22}^i + (i \cdot \omega \cdot \gamma - b \cdot \beta) \cdot A_{23}^i = -\beta \cdot B_{22}^i - \gamma \cdot B_{23}^i, \end{cases} \Rightarrow$$

$$\begin{cases} A_{21}^i = \frac{b \cdot A_{23}^i - i \cdot \omega \cdot A_{22}^i - B_{22}^i}{\gamma}, \\ A_{22}^i = \frac{(i \cdot \omega \cdot \gamma - b \cdot \beta) \cdot (-\gamma \cdot B_{21}^i + i \cdot \omega \cdot B_{22}^i) + (\beta \cdot \gamma + b \cdot i \cdot \omega) \cdot (\beta \cdot B_{22}^i - \gamma \cdot B_{23}^i)}{(\omega^2 - \gamma^2) \cdot (i \cdot \omega \cdot \gamma - b \cdot \beta) - (i \cdot \omega \cdot \beta + b \cdot \gamma) \cdot (\beta \cdot \gamma + b \cdot i \cdot \omega)}, \\ A_{23}^i = \frac{(i \cdot \omega \cdot \beta + b \cdot \gamma) \cdot (-\gamma \cdot B_{21}^i + i \cdot \omega \cdot B_{22}^i) + (\omega^2 - \gamma^2) \cdot (\beta \cdot B_{22}^i + \gamma \cdot B_{23}^i)}{(\beta \cdot \gamma + b \cdot i \cdot \omega) \cdot (i \cdot \omega \cdot \beta + b \cdot \gamma) - (\omega^2 - \gamma^2) \cdot (i \cdot \omega \cdot \gamma - b \cdot \beta)}. \end{cases} \quad (17)$$

Appendix A3

Additional conditions for solutions (11.2) of real value

Let us note that the result of multiplying of two arbitrary complex numbers $(k + l \cdot i)$ and $(m + n \cdot i)$ should be a real number if the additional condition is valid as below:

$$(k + l \cdot i) \cdot (m + n \cdot i) = (k \cdot m - l \cdot n) + (l \cdot m + k \cdot n) \cdot i$$

$$\Leftrightarrow (l \cdot m + k \cdot n) = 0$$

So, we obtain the additional restrictions for coefficients (15)–(17) under which the solution (11.2) below:

$$\begin{cases} U = (A_{21}^i + B_{21}^i \cdot t + C_{21}^i \cdot t^2) \cdot \exp(a \cdot t) \cdot (\cos(\omega \cdot t) + \sin(\omega \cdot t) \cdot i), \\ V = (A_{22}^i + B_{22}^i \cdot t + C_{22}^i \cdot t^2) \cdot \exp(a \cdot t) \cdot (\cos(\omega \cdot t) + \sin(\omega \cdot t) \cdot i), \\ W = (A_{23}^i + B_{23}^i \cdot t + C_{23}^i \cdot t^2) \cdot \exp(a \cdot t) \cdot (\cos(\omega \cdot t) + \sin(\omega \cdot t) \cdot i), \end{cases}$$

should be converted to the solution of real value (taking into account (*)):

$$\begin{cases} \text{Im}\{(A_{21}^i + B_{21}^i \cdot t + C_{21}^i \cdot t^2)\} \cdot \cos(\omega \cdot t) + \text{Re}\{(A_{21}^i + B_{21}^i \cdot t + C_{21}^i \cdot t^2)\} \cdot \sin(\omega \cdot t) = 0, \\ \text{Im}\{(A_{22}^i + B_{22}^i \cdot t + C_{22}^i \cdot t^2)\} \cdot \cos(\omega \cdot t) + \text{Re}\{(A_{22}^i + B_{22}^i \cdot t + C_{22}^i \cdot t^2)\} \cdot \sin(\omega \cdot t) = 0, \\ \text{Im}\{(A_{23}^i + B_{23}^i \cdot t + C_{23}^i \cdot t^2)\} \cdot \cos(\omega \cdot t) + \text{Re}\{(A_{23}^i + B_{23}^i \cdot t + C_{23}^i \cdot t^2)\} \cdot \sin(\omega \cdot t) = 0. \end{cases} \quad (18)$$

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In this research, Sergey Ershkov is responsible for the general ansatz and the solving procedure, simple algebra manipulations, calculations, results of the article in Sections 1–4 and Appendixes A1–A3, and is also responsible for the search of exact solutions.

Dmytro Leshchenko is responsible for theoretical investigations as well as for the deep survey in literature on the problem under consideration and Ayrat Giniyatullin is responsible for the plots and graphical solutions.

All authors agreed with the results and conclusions of each other in Sections 1–5.

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