AVERAGE TRAPPING TIME ON THE LEVEL-3 SIERPINSKI GASKET

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Abstract. In this paper, we consider the unbiased random walk on the level-3 Sierpinski gasket (SG_3). Due to the self-similar structure and iterative mechanism of the network, we obtain the exact analytic expression of the average trapping time (ATT) on SG_3. By comparing with the numerical result, we find that the analytical expression is very consistent with the corresponding numerical solution. Further, the obtained results indicate that ATT scales superlinearly with network size.

Key words: Average trapping time; unbiased random walk; level-3 Sierpinski gasket.

1. INTRODUCTION

In recent years, more and more scholars from different scientific fields pay attention to the research on complex networks. Random walk and diffusion on the network often take place in various physical environments, such as the propagation of porous media, heterogeneous catalysis, chemical reactions on related substrates, etc. As one of the research contents of random walk and diffusion, the average trapping time (ATT) has been widely studied by many scholars in the past few years [1–9]. ATT is the mean of the first-passage time from any node to a trap (a perfect absorber). It describes the diffusion efficiency of the whole network. When ATT is smaller, the diffusion efficiency of the whole network is higher [10].

This paper focuses on a particular fractal network, the level-3 Sierpinski gasket (SG_3). It is a extension of Sierpinski gasket and belongs to postcritically finite fractals (p.c.f.). This class of fractals have good self-similar structure, so they have gained much interest. Kigami [11, 12] defined the Dirichlet type operator and Laplacian on p.c.f. by using the functional analysis tools. Then, R. Strichartz [13, 14], H. Qiu [15], P. Alonso-Ruiz [16] and other scholars further study the Green function, Poisson equation, heat kernels and other problems on p.c.f., which promotes the study of fractal dynamics. In this paper, we consider the unbiased random walk on the fractal structure. We expect to get the exact analytic expression of ATT on such fractal networks. J. Kozak and V. Balakrishnan have studied the trapping problem on Sierpinski gasket (SG) [17]. Through the numerical calculation of FPTs on the previous several generations of SG, the recurrence relation of FPTs between two successive generations is observed, and finally the exact analytic expression of ATT on SG is obtained. As far as we know, the trapping problem on SG_3 has not been studied. Compared with the degree homogeneity of SG (except that the degree of the initial triangle nodes is 2, the degree of all other nodes is 4), SG_3 has no such structure (nodes with degree of 6 will appear in each iteration), so the research method of numerical calculation on SG cannot be extended to SG_3. In this paper, the exact analytical expression of ATT is obtained by using the relationship between the mean first-passage time of two successive generations and the method of probability generating function. Further, based on the numerical experiments, we find that the analytical solution and the numerical result are in good agreement.

2. THE CONSTRUCTION OF THE LEVEL-3 SIERPINSKI GASKET (SG_3)

In this section, we will introduce the construction and give some related topological properties of SG_3. We denote $SG_3(t)$ as the $t-$th generation of the network, where t represents the number of iterations. The network is constructed by the following iterative way:
(1) For $t = 0$, the initial network $SG_3(0)$ is an equilateral triangle consisting of three nodes and three edges. We label the three nodes as 1, 2, and 3.
(2) For $t \geq 1$, $SG_3(t)$ is generated from $SG_3(t-1)$ by performing the following operation. Each edge of each small equilateral triangle in the $SG_3(t-1)$ is trisected and the equilateral nodes in the triangle are connected in a manner parallel to the opposite edge.

Figure 1 illustrates the iteration of $SG_3(t)$ for $t = 0$, $t = 1$ and $t = 2$. Denote the small equilateral triangle on the $t-$th generation $SG_3(t)$ as $\triangle_t$. Then, the number of $\triangle_t$ on $SG_3(t)$ is 6^t. Therefore, by the number of small equilateral triangles $\triangle_t$, we can calculate the number of nodes and the number of edges for $SG_3(t)$, which can be denoted as N_t and E_t, respectively.

$$E_t = 3 \cdot \triangle_t = 3 \cdot 6^t \text{ and } N_t = \frac{7}{5}(6^t - 1) + 3. \tag{1}$$

Furthermore, every nodes of $SG_3(t)$ will be labeled sequentially from the top to the bottom by site index i. Taking the first-generation $SG_3(1)$ as an example, the network labeled with nodes is shown in Fig.1. Here, we set the trap at the apex of $SG_3(t)$, *i.e.*, at site 1. The left-hand corner node of the bottom row for $SG_3(t)$ is labeled as $i = N_t - 3^t$, while the right-hand corner node is labeled as $i = N_t$.

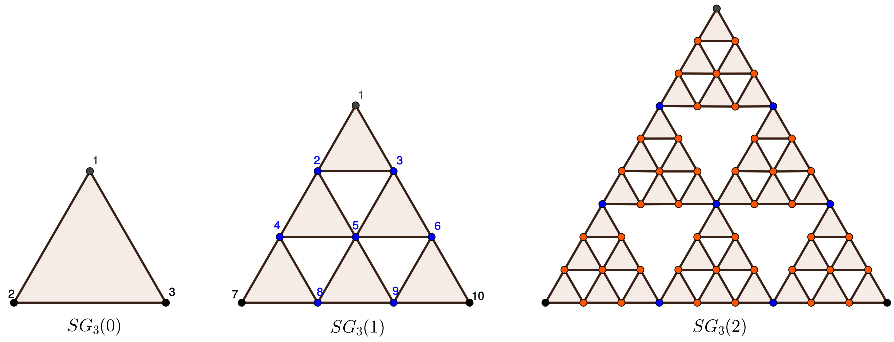

Fig. 1 – (Color online). Iterative method of $SG_3(t)$ for $t=0$, $t=1$ and $t=2$. The initial nodes are marked as black. The nodes generated after the first iteration and the second iteration are marked as blue and red, respectively.

From the above construction process, SG_3 has good self similarity. Therefore, the network can also be defined as follows. The network $SG_3(t)$ of generation t can be obtained by splicing the network $SG_3(t-1)$ of generation $t-1$. The stitching method is to replace each small equilateral triangle in the network $SG_3(1)$ with the gasket $SG_3(t-1)$ of $t-1$ generation. Here, we denote the 6 gaskets $SG_3(t-1)$ used for stitching as $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ respectively, which is illustrated in Fig. 2. Each one of these regions is a copy of $SG_3(t-1)$.

3. FORMULATION OF ATT ON SG_3

In this section, we consider an unbiased Markovian random walk on the $SG_3(t)$. Starting from any site other than the trap, the walker can jump to any of its nearest-neighbor nodes with equal probabilities at each time step. Therefore, the probability of transition at each step is the reciprocal of the degree of the node. Here, the degree of site i can be denoted as d_i, then, the probability can be expressed as follow:

$$P_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } i \sim j \\ 0, & \text{others} \end{cases}$$

where p_{ij} is the probability that the walker jump from the site i to the site j, and $i \sim j$ means that the site i is directly connected to the site j.

We use T_i^t to represent the mean first-passage time (MFPT), which is the expected time for a walker starting from a site i to the trap. Here we choose the apex of $SG_3(t)$ as the trap, which is node 1 in Figure 1 or node A in Figure 2. For any initial site i, the probability of the walker surviving after n jumps (that is, not being absorbed by the trap node) satisfies the discrete version of the backward Kolmogorov

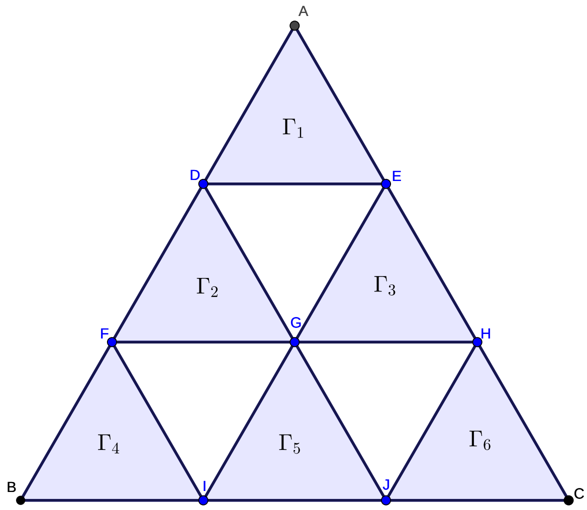

Fig. 2 – (Color online). Self-similar structure of $SG_3(t)$. Each region $\Gamma_i(i=1,2,\cdots,6)$ in the diagram is a copy of $SG_3(t-1)$.

equation. Naturally, T_i^t will satisfy the following relationship:

$$T_i^t = \sum_{j=2}^{N_t} p_{ij} T_j^t + 1. \tag{2}$$

Then, we can define the transition probability matrix with no trap as:

$$M = (p_{ij})_{N_t \times N_t}.$$

Since the gasket we consider here is fixed a trap at site 1, we can define the transition probability matrix of this gasket as M_T which is obtained by removing the row and column corresponding to site 1 in matrix M, namely the first row and the first column. Then, let $T = (T_2^t, T_3^t, \cdots, T_{N_t}^t)$. The relationship of Eq.(2) can be expressed as the vector relation:

$$T = M_T T + e, \tag{3}$$

where T and e is $N_t - 1-$dimensional column vector and all elements of e is 1.

Eq.(3) can be transformed as

$$T = (I - M_T)^{-1} e, \tag{4}$$

where I is the identity matrix of order $N_t - 1$.

Therefore, from the definition of the ATT, namely the average value of MFPTs from the every site in $SG_3(t)$ except site 1 to the trap, the following relationship can be deduced:

$$\langle T \rangle_t = \frac{1}{N_t - 1} \sum_{i=2}^{N_t} T_i^t = \frac{1}{N_t - 1} \sum_{i=1}^{N_t - 1} \sum_{j=1}^{N_t - 1} a_{ij}, \tag{5}$$

where we use the $\langle T \rangle_t$ to denote the ATT on $SG_3(t)$, and a_{ij} represents the element at i row and j column in matrix $(I - M_T)^{-1}$.

The above method can be used to calculate the ATT, but since the order of the matrix to be calculated is $N_t - 1$, it will be difficult to calculate when the number of iterations is large, and it is even more impossible to analyze the situation when the number of iterations approaches infinity. However, it can be used to check the exact analytical expression of ATT obtained in the next section.

4. ANALYTICAL SOLUTION OF ATT ON SG_3

In this section, we will obtain the exact analytical expression of ATT on the Level-3 Sierpinski Gasket. Firstly, we define that $T_{t,total}^t = \sum_{i=2}^{N_t} T_i^t$. Then, the ATT on t generation $SG_3(t)$ satisfies that

$$\langle T \rangle_t = \frac{1}{N_t - 1} T_{t,total}^t. \tag{6}$$

So, as long as we figure out $T_{t,total}^t$, we can obtain $\langle T \rangle_t$.

Next, we will calculate the value of $T_{t,total}^t$. Denote the set composed of all nodes in the network $SG_3(t)$ of generation t as Ω_t, and all newly generated nodes after the t iteration as $\bar{\Omega}_t$. Let

$$\bar{T}_{t,total}^t = \sum_{i \in \bar{\Omega}_t} T_i^t \quad \text{and} \quad T_{t-1,total}^t = \sum_{i \in \Omega_{t-1}} T_i^t.$$

Then, it can be easily obtained that:

$$T_{t,total}^t = \bar{T}_{t,total}^t + T_{t-1,total}^t. \tag{7}$$

So, we just need to calculate $\bar{T}_{t,total}^t$ and $T_{t-1,total}^t$ separately to get $T_{t,total}^t$.

For convenience, we use the node notation in Fig. 2. Due to the symmetry of $SG_3(t)$, we give the following definitions of MFPTs. The MFPT of a walker from node A to node B or C is denoted as F. The MFPT from node A to node D or E is denoted as F'. The MFPTs of node D, F, G and I to node B or C are denoted as F_D, F_F, F_G and F_I respectively. Based on the unbiased Markovian random walk definition and the symmetry on the gasket network, we can establish the following underlying equations:

$$\begin{cases} F = F' + F_D \\ F_D = \frac{1}{4}(F'+F) + \frac{1}{4}(F'+F_D) + \frac{1}{4}(F'+F_F) + \frac{1}{4}(F'+F_G) \\ F_F = \frac{1}{4}F' + \frac{1}{4}(F'+F_D) + \frac{1}{4}(F'+F_G)\frac{1}{4}(F'+F_I) \\ F_G = \frac{1}{3}(F'+F_D) + \frac{1}{3}(F'+F_F) + \frac{1}{3}(F'+F_I) \\ F_I = \frac{1}{4}F' + \frac{1}{4}(F'+F_F) + \frac{1}{4}(F'+F_G) + \frac{1}{4}(F'+F_I). \end{cases}$$

By solving the above equations, we can get $F = \frac{90}{7}F'$. From the self-similarity of the network $SG_3(t)$, it can be obtained that F' is also the MFPT of particles from node A to node B or C in the $t-1$ generation $SG_3(t-1)$. According to the properties of the gasket network structure, after the t-th iteration, the MFPT between all the node pairs in Ω_{t-1} is $\frac{90}{7}$ times that in $t-1$ generation. Therefore, the following relationship can be established:

$$T^{t}_{t-1,total} = \frac{90}{7} T^{t-1}_{t-1,total}. \tag{8}$$

Next, we will consider the newly generated points after iteration t, namely $\bar{\Omega}_t$. According to the description of the iterative process in the second section, all small equilateral triangles will be replaced by $SG_3(1)$ after one iteration. Therefore, each small triangle will produce seven corresponding nodes after one iteration. If the walker starts from these 7 nodes to reach the trap node, it must pass through the vertex of the triangle containing these nodes. Since the seven nodes are symmetrical in the triangle, the probability of passing through the three vertices of the triangle is equal if one is randomly selected as the initial node. In addition, since the degree of the vertex in $SG_3(t)$ determines the number of triangles that the node can connect at the same time, the higher the degree of the vertex, the more times the node appears in the trapping path of the newly generated node. Based on the above description, we can make the following derivation.

With reference to the $SG_3(1)$ mark in Fig.1, let the MFPT of node i except the vertexes to any vertex in the triangle be T'_i. And let $T' = \sum_{i\in\bar{\Omega}_1} T'_i$. The following conclusions can be obtained from the symmetry of the $SG_3(1)$.

$$T'_2 = T'_3 = T'_4 = T'_6 = T'_8 = T'_9.$$

Then, the following underlying equations can be established:

$$\begin{cases} T'_2 = \frac{1}{4} + \frac{1}{4}(T'_5+1) + \frac{1}{2}(T'_2+1) \\ T'_5 = T'_2 + 1. \end{cases}$$

By solving the above equations, we can get $T'_2 = 3$ and $T'_5 = 4$. So, it can be deduced that $T' = 22$. Then, the following relationship can be derived:

$$\bar{T}^{t}_{t,total} = T' \cdot \triangle_{t-1} + \frac{7}{3} \cdot \sum_{i\in\Omega_{t-1}} \frac{d_i}{2} T^{t}_i = T' \cdot \triangle_{t-1} + \frac{7}{3} \cdot \frac{90}{7} \cdot \sum_{i\in\Omega_{t-1}} \frac{d_i}{2} T^{t-1}_i,$$

where $\frac{d_i}{2}$ represents the number of equilateral triangles connected by node i. Since the second term of the above equation is weighted sum of the T_i^{t-1} of node i according to the degree of its nodes, it can be correlated with the mean global first-passage time, and it can be obtained:

$$\bar{T}_{t,total}^t = T' \cdot \triangle_{t-1} + \frac{7}{3} \cdot \frac{90}{7} \cdot E_{t-1}\big[\langle GFPT_{t-1}\rangle - \frac{1}{E_{t-1}}\langle FRT_{t-1}\rangle\big] \tag{9}$$

where $\langle FRT_t\rangle$ is the mean first return time with node A and $\langle GFPT_t\rangle$ represents the mean global first-passage time for node A, which are defined as:

$$\langle GFPT_t\rangle = \sum_{i=2}^{N_t} \frac{d_i}{2E_t} T_i^t + \frac{d_1}{2E_t}\langle FRT_t\rangle.$$

It can be proved that:

$$\langle GFPT_t\rangle = \frac{7}{3} + \frac{141}{83}\big[(\frac{90}{7})^t - 1\big], \tag{10}$$

$$\langle FRT_t\rangle = 3 \cdot 6^t. \tag{11}$$

Please, refer to Appendix for the specific proof process. By substituting Eq.(1), Eq.(11) and Eq.(10) into Eq.(9) the following relation can be obtained:

$$\bar{T}_{t,total}^t = \frac{141}{83} \cdot 90 \cdot 6^{t-1} \cdot (\frac{90}{7})^{t-1} - \frac{904}{83} \cdot 6^{t-1}. \tag{12}$$

So far, we have obtained the expressions of $\bar{T}_{t,total}^t$ and the iterative relation of $T_{t-1,total}^t$ respectively. In addition, in generation 0, namely $SG_3(0)$ is the primary equilateral triangle, it is easy to prove that $T_{0,total}^0 = 4$. Therefore, by substituting the Eq.(8) and Eq.(9) in the Eq.(7), it can be obtained that:

$$\begin{aligned}
T_{t,total}^t &= T_{t,total}^{t-1} + \bar{T}_{t,total}^t \\
&= \frac{90}{7} T_{t-1,total}^{t-1} + \frac{141}{83} \cdot 90 \cdot 6^{t-1} \cdot (\frac{90}{7})^{t-1} - \frac{904}{83} \cdot 6^{t-1} \\
&= (\frac{90}{7})^2 T_{t-2,total}^{t-2} + \frac{141}{83} \cdot 90 \cdot (\frac{90}{7})^{t-1} \cdot \big[6^{t-1} + 6^{t-2}\big] - \frac{904}{83} \cdot \big[6^{t-1} + 6^{t-2} \cdot \frac{90}{7}\big] \\
&= (\frac{90}{7})^t T_{0,total}^0 + \frac{141}{83} \cdot 90 \cdot (\frac{90}{7})^{t-1} \cdot \sum_{i=0}^{t-1} 6^i - \frac{904}{83} \cdot \sum_{i=0}^{t-1} 6^i \cdot (\frac{90}{7})^{t-1-i} \\
&= \frac{2538}{83} \cdot (\frac{90}{7})^{t-1} \cdot 6^t + \frac{791}{498} \cdot 6^t + \frac{3}{7} \cdot (\frac{90}{7})^{t-1}
\end{aligned}$$

In order to calculate $\langle T\rangle_t$, we just need to substitute Eq.(1) and Eq.(13) into Eq.(6) and get:

$$\langle T\rangle_t = \frac{1}{N_t - 1} T_{t,total}^t = \frac{2115 \cdot (\frac{90}{7})^{t-1} \cdot 6^{t+2} + 3955 \cdot 6^t + 83 \cdot (\frac{90}{7})^t}{83 \cdot (7 \cdot 6^{t+1} + 18)}.$$

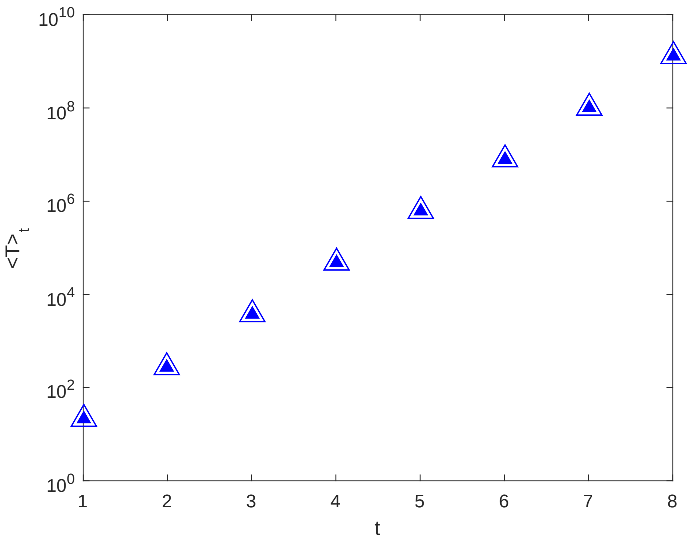

Fig. 3 – (Color online). $\langle T \rangle_t$ as a function of t. The hollow triangles are the data obtained by Eq. (5). The solid triangles are the data obtained by Eq. (13).

Then, if we use the number of nodes in the network to represent the size of the network, so the scaling expression of $\langle T \rangle_t$ with network size obeys:

$$\langle T \rangle_t \sim (\frac{90}{7})^t \sim N_t^{\frac{\ln 90 - \ln 7}{\ln 6}},$$

where $1 < \frac{\ln 90 - \ln 7}{\ln 6} \approx 1.43 < 2$, indicating that $\langle T \rangle_t$ scales superlinearly with network size N_t. We will check the above analytical result. It can be seen from Fig. 3 that the analytical solution obtained by us is very consistent with the numerical solution.

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A. $\langle GFPT_t \rangle$ AND $\langle FRT_t \rangle$ OF THE LEVEL-3 SIERPINSKI GASKET

In this appendix, we will briefly introduce the formulas for $\langle GFPT_t \rangle$ and $\langle FRT_t \rangle$.

Let $F_{ij}(t)$ be the first-passage time (FPT) for random walk from the starting node i to the ending node j on $t-$th generation $SG_3(t)$ for the first time. Consequently, $F_{ii}(t)$ can be defined as the first return time (FRT) on $SG_3(t)$ for random walk of initial node i. Then, let the probability of $F_{ij}(t) = k$ be $P\{F_{ij}(t) = k\}$. Thus, the probability generating function of FPT from node i to node j can be written as:

$$\Phi_{FPT}(t,z) = \sum_{k=0}^{+\infty} z^k P\{F_{ij}(t) = k\}.$$

Here, we consider the first-passage time between two special nodes, that is, the FPT from node A to either nodes B or C. We use $\Phi_{FPT'}(t,z)$ to denote its probability generating function.

Similarly, the probability generating function of FRT for given hub i can be given by

$$\Phi_{FRT}(t,z) = \sum_{k=0}^{+\infty} z^k P\{F_{ii}(t) = k\}.$$

Then, we can define $F_j(t)$ as the global first-passage time (GFPT) to the given site j in the gasket. In the steady state, since the probability of finding the initial random walker at node i can be represent as $\frac{d_i}{2E_t}$, the $F_j(t)$, averaging $F_{ij}(t)$ over all initial node i, can be expressed as

$$F_j(t) = \sum_{i}^{N_t} \frac{d_i}{2E_t} F_{ij}.$$

Therefore, on the base of the definition, the probability generating function of GFPT to site j can be given by

$$\Phi_{GFPT}(t,z) = \sum_{k=0}^{+\infty} z^k P\{F_j(t) = k\}.$$

Furthermore, for analysing the recurrence relation of $\Phi_{FRT}(t,z)$ and $\Phi_{GFPT}(t,z)$, we have to introduce return time (RT). Different from FRT, if RT with initial node i is denoted as $T_i(t)$ on $SG_3(t)$, then maybe it is not the first time that the random walker reach the starting node i. The probability generating function of RT for hub i can be denoted as $\Phi_{RT}(t,z)$, which is defined by

$$\Phi_{RT}(t,z) = \sum_{k=0}^{+\infty} z^k P(T_i = k),$$

where $P(T_i = k)$ represent the probability of finding the random walker starting from site i on the identical site i at time k.

For any $t > 0$, it is known that [18]

$$\Phi_{FRT}(t,z) = 1 - \frac{1}{\Phi_{RT}(t,z)}, \tag{13}$$

and

$$\Phi_{GFPT}(t,z) = \frac{z}{1-z} \cdot \frac{d_i}{2E_t} \cdot \frac{1}{\Phi_{RT}(t,z)}. \tag{14}$$

Therefore, $\Phi_{FRT}(t,z)$ and $\Phi_{GFPT}(t,z)$ can be obtained by calculating $\Phi_{RT}(t,z)$.

Due to the self-similarity of the level-3 Sierpinski gasket, according to Ref. [19], the following equation can be established.

$$\Phi_{RT}(t,z) = \frac{\Phi_{RT}(t-1,z)}{\varphi(\Phi_{FPT'}(t-1,z))}, \quad t > 0, \tag{15}$$

where $\varphi(z)$ satisfies that

$$\varphi(z) \equiv \frac{\Phi_{RT}(0,z)}{\Phi_{RT}(1,z)}.$$

By substituting Eq. (15) into Eq. (13) and Eq. (14), we can obtain the recursive relation of $\Phi_{FRT}(t,z)$ and $\Phi_{GFPT}(t,z)$.

$$\Phi_{FRT}(t,z) = 1 - \varphi(\Phi_{FPT'}(t-1,z))(1 - \Phi_{FRT}(t-1,z)), \tag{16}$$

$$\Phi_{GFPT}(t,z) = \frac{1}{6}\varphi(\Phi_{FPT'}(t-1,z))\Phi_{GFPT}(t-1,z). \tag{17}$$

According to the property of probability generating function, it can be obtained that:

$$\langle GFPT_t \rangle = \frac{\partial}{\partial z}\Phi_{GFPT}(t,z)\Big|_{z=1} \quad \text{and} \quad \langle FRT_t \rangle = \frac{\partial}{\partial z}\Phi_{FRT}(t,z)\Big|_{z=1}.$$

Similarly, as can be seen from Ref. [19], the FPT' satisfies the iterative relationship:

$$\Phi_{FPT'}(t,z) = \Phi_{FPT'}(1,x)|_{x=\Phi_{FPT}(t-1,z)}.$$

It is easy to verify that $\frac{\partial}{\partial z}\Phi_{FPT'}(1,z)\Big|_{z=1} = \frac{90}{7}$. By integrating the above conditions into Eq. (16) and Eq. (17), it can be obtained that

$$\langle FRT_t \rangle = 3 \cdot 6^t \quad \text{and} \quad \langle GFPT_t \rangle = \frac{7}{3} + \frac{141}{83}\left[(\frac{90}{7})^t - 1\right].$$

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