

THIRD ORDER ANOMALIES IN THE CAUSAL APPROACH

D. R. GRIGORE

Department of Theoretical Physics,
Horia Hulubei National Institute for Physics and Nuclear Engineering,
Reactorului 30, RO-077125, P.O.B. MG-6, Bucharest-Magurele, Romania
E-mail: grigore@theory.nipne.ro

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Abstract. We consider gauge models in the causal approach and study the third order of the perturbation theory. We are interested in the computation of the anomalies in this order of the perturbation theory and for this purpose we analyse in detail the causal splitting of the distributions with causal support relevant to tree and loop anomalies.

Key words: Perturbative quantum field theory, gauge models in the causal approach, loop anomalies.

1. INTRODUCTION

The most natural way to arrive at the Bogoliubov axioms of perturbative quantum field theory (pQFT) is by analogy with non-relativistic quantum mechanics [8, 15]. Suppose that we have a time-dependent interaction potential V . Then one goes to the interaction picture and the time evolution is governed by the evolution equation:

$$\frac{d}{dt}U(t, s) = -iV_{\text{int}}(t)U(t, s); \quad U(s, s) = I. \quad (1)$$

This equation can be solved in some cases by a perturbative method, namely the series

$$U(t, s) \equiv \sum \frac{(-i)^n}{n!} \int_{\mathbb{R}^n} dt_1 \cdots dt_n T(t_1, \dots, t_n) \quad (2)$$

makes sense. The operators $T_n(t_1, \dots, t_n)$ are called *chronological products*; n is called the *order* of the perturbation theory. They verify the following properties:

- Initial condition

$$T_1(t) = V_{\text{int}}(t) \quad (3)$$

- Symmetry

$$T_n(t_{\pi(1)}, \dots, t_{\pi(n)}) = T_n(t_1, \dots, t_n) \quad (4)$$

for all permutations π of $\{1, \dots, n\}$.

- Causality

$$T_n(t_1, \dots, t_n) = T_m(t_1, \dots, t_m) T_{n-m}(t_{m+1}, \dots, t_n),$$

$$\text{for } t_j > t_k, \quad j = 1, \dots, m; k = m+1, \dots, n. \quad (5)$$

- Unitary

$$U(t, s)^\dagger U(t, s) = I \quad (6)$$

In terms of the chronological products, define the *anti-chronological products* as follows: because of the symmetry property we can write $T_I = T(i_1, \dots, i_k)$ for $I = \{i_1, \dots, i_k\}$. Then the anti-chronological products are

$$(-1)^n \bar{T}_n(t_1, \dots, t_n) = \sum_{m=1}^n (-1)^m \sum_{I_1, \dots, I_m \in \text{part}(I)} T_{I_1} \cdots T_{I_m} \quad (7)$$

where I_1, \dots, I_m is a partition of I . The the unitarity axiom is equivalent to

$$\bar{T}_n(t_1, \dots, t_n) = T_n(t_1, \dots, t_n)^\dagger \quad (8)$$

- Invariance properties

If the interaction potential is translation invariant then we have

$$T_n(t_1 + \tau, \dots, t_n + \tau) = T_n(t_1, \dots, t_n), \quad \forall \tau \in \mathbb{R} \quad (9)$$

We can write an explicit formula

$$T_n(t_1, \dots, t_n) = \sum \theta(t_{\pi(1)} - t_{\pi(2)}) \cdots \theta(t_{\pi(n-1)} - t_{\pi(n)}) V_{int}(t_1) \cdots V_{int}(t_n). \quad (10)$$

The purpose is to generalize this idea in the relativistic context especially the causality property. Essentially we try to substitute $t \in \mathbb{R}$ by a Minkowski variable $x \in \mathbb{R}^4$. The chronological operators will be some operators $T(x_1, \dots, x_n)$ and all the preceding axioms can be easily generalized: the symmetry and the unitarity axioms remain unchanged and for the invariance axiom we have to substitute the translation group with Poincaré group. The causally axiom is more subtle. We have to replace temporal succession $t_1 > t_2$ by causal succession $x_1 \succ x_2$ which means that x_1 should not be in the past causal shadow of x_2 *i.e.* $x_2 \cap (x_1 + \bar{V}^+) = \emptyset$. In formulas: if $x_i \succ x_j, \quad \forall i \leq k, \quad j \geq k+1$ then we have:

$$T(x_1, \dots, x_n) = T(x_1, \dots, x_k) T(x_{k+1}, \dots, x_n). \quad (11)$$

From here it follows that the “initial condition” $T(x)$ should satisfy

$$[T(x), T(y)] = 0, \quad (x - y)^2 < 0 \quad (12)$$

where for the Minkowski product we use the convention $1, -1, -1, -1$. It is surprisingly difficult to obtain solutions of the preceding equation. The relevant solution for

pQFT are in fact some distribution-valued operators, called Wick monomials. They can be associated to free fields (or generalized free fields) and act in some Hilbert space of the Fock type. This is in accord with our intuition that in pQFT we should be able to describe scattering processes with creation and annihilation of particles. However, in this case the formula (10) makes no sense. It involves an illegal operation: the multiplication of distributions. It is better to try to solve directly the axioms of pQFT in an recursive way.

So we start from Bogoliubov axioms [1, 7] as presented in [5]; for every set of Wick polynomials $A_1(x_1), \dots, A_n(x_n)$ acting in some Fock space \mathcal{H} one associates the operator-valued distributions $T^{A_1, \dots, A_n}(x_1, \dots, x_n)$ called chronological products; it will be convenient to use another notation: $T(A_1(x_1), \dots, A_n(x_n))$. The expression $T(x_1, \dots, x_n)$ corresponds to the choice $A_1 = \dots = A_n = T$ and the generalization to the case of distinct A_1, \dots, A_n is possible because the symmetry axioms suggests that a sort of polarization formula is possible.

The axioms for the chronological products remain unchanged, only the symmetry axiom should be replaced by skew-symmetry in all arguments: for arbitrary $A_1(x_1), \dots, A_n(x_n)$ we should have

$$T(\dots, A_i(x_i), A_{i+1}(x_{i+1}), \dots) = (-1)^{f_i f_{i+1}} T(\dots, A_{i+1}(x_{i+1}), A_i(x_i), \dots) \quad (13)$$

where f_i is the number of Fermi fields appearing in the Wick monomial A_i .

Even in the simplest case when the Fock space is generated by a real scalar field $\phi(x)$ and the interaction Lagrangian is a Wick monomial $T(x) =: \Phi^4(x)$: the construction of the chronological products is a surprisingly difficult problem.

There are, at least to our knowledge, three rigorous ways to do that; for completeness we remind them following [14]:

(a) *Hepp axioms* [15]: one rewrites Bogoliubov axioms in terms of vacuum averages of chronological products $\langle \Omega, T^{A_1, \dots, A_n}(x_1, \dots, x_n) \Omega \rangle$.

(b) *Polchinski flow equations* [18, 20]: one considers an ultra-violet cut-off Λ for the Feynman amplitudes and establishes some differential equations (in this parameter) for these amplitudes.

(c) *The causal approach* due to Epstein and Glaser [7, 8]: is a recursive procedure for the basic objects $T(A_1(x_1), \dots, A_n(x_n))$ and reduces the induction procedure to a distribution splitting of some distributions with causal support, or to the process of extension of distributions [19]. An equivalent point of view uses retarded products [24] instead of chronological products. The causal method is by far the most elementary, so we expect that this will stay true for more complicated models like gauge models.

In fact, a basic problem is the choice of the Fock space. Generally, we should consider some elementary particle described by some projective unitarity, irreducible representation of the Poincaré group, construct the associated Fock space (taking

into account the spin-statistics theorem) and build free fields as combinations of the creation and annihilation operators. Because the irreducible representation of the Poincaré group are unique, up to a unitary transformation, one would expect that the construction of the associated pQFT is also essentially unique. However, this is not so obvious. For instance, the scalar particles are usually described by a scalar function $\Phi : \mathbb{R}^4 \rightarrow \mathbb{C}$ verifying Klein-Gordon equation. But they can also be described by a skew-symmetric tensor $t : \mathbb{R}^4 \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^4$ verifying Dirac equation in both entries [25], pg. 360. It is not obvious that if we work in this representation for the scalar field we will obtain the same results as above. So it is a bit of art to choose a “nice” concrete representation of the Hilbert space of an elementary particle. This task is more difficult for gauge theories which describe particles of higher spin. If we describe a particle of spin 1 by a vector field and try to consider only the physical degrees of freedom (three for the massive case and only two for the massless case) we end up with non-renormalizable theories.

However, one can save renormalizability using ghost fields. There are two ways to do that:

(A) In BRST approach one introduces even and odd Grassmann classical fields; the odd fields are the so-called *ghost* (or Faddeev-Popov) fields. Then one can try to make sense of the formal path integral and ends up with some consistency relation - the master equation [17]. Presumably, if a solution of this equation can be found, one would be able to construct the chronological products with the desired properties, although a rigorous proof of this fact seems to be missing, at least to our knowledge. A supplementary problem in the functional formalism is that the Green functions are affected by infra-red divergences; an adiabatic limit must be performed and as it can be seen from the paper of Epstein and Glaser, this limit is not easy to perform.

(A') A variant of the preceding idea is the use of the Zinn-Justin relation [26].

(B) The *causal approach* of Scharf and collaborators [21, 22]. In this approach one makes sense of the ghost fields as well defined fields in some mathematical Fock space with physical and non-physical states. One has to select the physical states by a certain gauge condition and the chronological products should leave invariant the set of physical states.

We remind the details: the theories are defined in a Fock space \mathcal{H} with indefinite metric, and one selects the physical states assuming the existence of an operator Q called *gauge charge* which verifies $Q^2 = 0$ and such that the *physical Hilbert space* is by definition $\mathcal{H}_{\text{phys}} \equiv \text{Ker}(Q)/\text{Im}(Q)$. One assigns a *ghost number* $gh(f)$ to every field f and this gives a grading in the Hilbert space \mathcal{H} and in the space of Wick monomials in \mathcal{H} . If we consider that the gauge charge has ghost number 1 then the graded commutator d_Q of the gauge charge with any operator A of fixed grading

number

$$d_Q A = [Q, A] \quad (14)$$

makes sense and is raising the ghost number by a unit. It means that d_Q is a co-chain operator in the space of Wick polynomials. From now on $[\cdot, \cdot]$ denotes the graded commutator.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number $T(x)$ called *the interaction Lagrangian* such that

$$[Q, T] = i \partial_\mu T^\mu \quad (15)$$

for some other Wick polynomials T^μ . This relation means that the expression T leaves invariant the physical states, at least in the adiabatic limit. Indeed, if this is true we have:

$$T(f) \mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{phys}} \quad (16)$$

up to terms which can be made as small as desired (making the test function f flatter and flatter). In all known models one finds out that there exists a chain of Wick polynomials T^I (where I is a collection of indexes $I = [\nu_1, \dots, \nu_p]$, $p = 0, 1, \dots$ and the brackets emphasize the complete antisymmetry in these indexes) such that

$$\begin{aligned} T &\equiv T^\emptyset \\ \omega(T^I) &= \omega_0, \forall I \\ gh(T^I) &= |I| \end{aligned} \quad (17)$$

where $\omega(T^I)$ is the canonical dimension of T^I ; we have

$$d_Q T^I = i \partial_\mu T^{I\mu}. \quad (18)$$

It is clear that we should have $T^I = 0$, $|I| > 4$ but in the Yang-Mills case we have $T^I = 0$, $|I| > 2$.

Now we can construct the chronological products

$$T^{I_1, \dots, I_n}(x_1, \dots, x_n) \equiv T(T^{I_1}(x_1), \dots, T^{I_n}(x_n)) \quad (19)$$

according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (18):

$$d_Q T^{I_1, \dots, I_n} = i \sum_{l=1}^n (-1)^{s_l} \frac{\partial}{\partial x_l^\mu} T^{I_1, \dots, I_l \mu, \dots, I_n} \quad (20)$$

are true for all $n \in \mathbb{N}$ and all I_1, \dots, I_n . Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1} |I_j|. \quad (21)$$

In particular, the case $I_1 = \dots = I_n = \emptyset$ it is sufficient for the gauge invariance of the scattering matrix, at least in the adiabatic limit: we have the same argument as for relation (16).

Such identities can be usually broken by *anomalies i.e.* expressions of the type A^{I_1, \dots, I_n} which are quasi-local and might appear in the right-hand side of the relation (20). In a previous paper we have emphasized the cohomological structure of this problem [11]. We consider a *cochain* to be an ensemble of distribution-valued operators of the form $C^{I_1, \dots, I_n}(x_1, \dots, x_n)$, $n = 1, 2, \dots$ (usually we impose some supplementary symmetry properties) and define the derivative operator δ according to

$$(\delta C)^{I_1, \dots, I_n} = \sum_{l=1}^n (-1)^{s_l} \frac{\partial}{\partial x_l^\mu} C^{I_1, \dots, I_l \mu, \dots, I_n}. \quad (22)$$

We can prove that

$$\delta^2 = 0. \quad (23)$$

Next we define

$$s = d_Q - i\delta, \quad \bar{s} = d_Q + i\delta \quad (24)$$

and note that

$$s\bar{s} = \bar{s}s = 0. \quad (25)$$

We call *relative cocycles* the expressions C verifying

$$sC = 0 \quad (26)$$

and a *relative coboundary* an expression C of the form

$$C = \bar{s}B. \quad (27)$$

The relation (20) is simply the cocycle condition

$$sT = 0. \quad (28)$$

This cohomological structure is similar but different from the well-known cohomology of the BRS(T) operator [3]. Our BRST operator s is a linear operator so it make sense in a Hilbert space; the BRS(T) operator from [3] is a non-linear operator acting on polynomials in the classical fields and their derivatives. In fact, formally, our BRST operator is the linear part of the usual BRS expression.

If we can prove that this relation is valid up to the order $n - 1$ then in order n this relation is valid up to *anomalies*:

$$sT = \mathcal{A} \quad (29)$$

where the anomalies in the right hand side have the generic form

$$\mathcal{A}(x_1, \dots, x_n) = \sum p_i(\partial) \delta(x_1, \dots, x_n) W_i(x_1, \dots, x_n). \quad (30)$$

Here

$$\delta(x_1, \dots, x_n) = \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n), \quad (31)$$

p_i are polynomials in the partial derivatives and W_i are Wick polynomials. There is a bound on the number

$$\text{deg}(\mathcal{A}) \equiv \text{supp}_i \{ \text{deg}(p_i) + \omega(W_i) \} \quad (32)$$

coming from the power counting theorem; here $\text{deg}(p)$ is the degree of the polynomial p and $\omega(W)$ is the canonical dimension of the Wick polynomial W . We call this number the *canonical dimension of the anomaly*. For instance, if the interaction Lagrangian and the associated expressions T^I verify $\omega(T^I) = 4$ (as is the case of Yang-Mills models) then the canonical dimension of the anomaly is ≤ 5 . The contributions corresponding to maximal degree will be called *dominant*.

We note that from (28) it follows that the anomaly must verify a consistency relation of the Wess-Zumino type

$$\bar{s}\mathcal{A} = 0. \quad (33)$$

Such type of relations have intensively used to obtain the generic form of the anomalies in the causal approach in [9].

According to our knowledge, there is no rigorous proof of the equivalence between the functional formalism and the causal formalism which we use here.

A systematic study for the loop contributions in the third order of the perturbation theory in the causal approach appears in [12]. In this paper we consider the Yang-Mills models up to the third order studying all contributions: tree and loop; for the loop anomalies we present a simplified version. The basic idea is to isolate some typical numerical distributions with causal support appearing in the loop contributions in the second and the third order of the perturbation theory; these distributions are verifying some identities which must causally split. Some anomalies are produced in this way. We must collect all such anomalous terms and impose the condition that the total anomaly \mathcal{A} is a coboundary. We treat more carefully the contributions of the one-particle reducible graphs and the non-unicity of the anomalies due to the presence of the δ factors.

In the next Section we will give a minimal account of the gauge theories in the causal approach. Next we turn to the one-loop anomalies in the second and third order of perturbation theory in Sections 3 and 4.

2. GENERAL GAUGE THEORIES

2.1. PERTURBATION THEORY

The axioms of perturbation theory of pQFT in the Bogoliubov framework have been described in the introduction; for more details see [13]. We only remind two

supplementary axioms.

(a) *Wick expansion property*

It can be proved [7] that this system of axioms can be supplemented with

$$T(A_1(x_1), \dots, A_n(x_n)) = \sum \langle \Omega, T(A'_1(x_1), \dots, A'_n(x_n)) \Omega \rangle : A''_1(x_1), \dots, A''_n(x_n) : \quad (34)$$

where A'_i and A''_i are Wick submonomials of A_i such that $A_i =: A'_i A''_i$: and appropriate signs should be included if Fermi fields are present; here Ω is the vacuum state.

(b) *Power counting bound*

The order of singularity $\omega(d)$ of a distribution $d(x) \in \mathcal{S}'(\mathbb{R}^n)$ is defined in [7] (and slightly differently in [24]); essentially the Fourier transform $\tilde{d}(p)$ behaves for large momenta as $|p|^{\omega(d)}$.

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials A_1, \dots, A_n ; explicitly:

$$\omega(\langle \Omega, T^{A_1, \dots, A_n}(x_1, \dots, x_n) \Omega \rangle) \leq \sum_{l=1}^n \omega(A_l) - 4(n-1) \quad (35)$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(A)$ we mean the canonical dimension of the Wick monomial A . The contributions saturating the inequality (*i.e.* corresponding to the equal sign) will be called *dominant*; they will produce dominant anomalies.

Up to now, we have defined the chronological products only for self-adjoint Wick monomials A_1, \dots, A_n but we can extend the definition for arbitrary Wick polynomials by linearity.

One can modify the chronological products without destroying the basic property of causality using transformations of the type:

$$T(A_1(x_1), \dots, A_n(x_n)) \rightarrow T(A_1(x_1), \dots, A_n(x_n)) + \sum P_j(\partial) \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n) W_j(x_1, \dots, x_n) \quad (36)$$

with P_j monomials in the partial derivatives and W_j are Wick monomials. Some restrictions are following from power counting, Lorentz covariance and unitarity.

From now on we consider that we work in the four-dimensional Minkowski space and we have the Wick polynomials A, B , etc. such that we have

$$A(x) B(y) = (-1)^{|A||B|} B(y) A(x), \quad \forall x \sim y \quad (37)$$

i.e. for $x - y$ space-like these expressions causally commute in the graded sense. The chronological products $T(A_1(x_1), \dots, A_n(x_n))$ are constructed according re-

cursively using the causal commutators.

The basic recursive idea of Epstein and Glaser starts from the chronological products

$$T(A_1(x_1), \dots, A_m(x_m)) \quad m = 1, 2, \dots$$

up to order $n - 1$ and constructs a causal commutator in order n . For instance, for $n = 2$ the *causal commutator* is defined according to:

$$D(A(x), B(y)) = A(x) B(y) - (-1)^{|A||B|} B(y) A(x) \quad (38)$$

and after the operation of causal splitting one can obtain the second order chronological products. Generalizations of this formula are available for higher orders of the perturbation theory. In particular we have in the third order

$$D(A(x), B(y); C(z)) \equiv -[\bar{T}(A(x), B(y)), C(z)] \\ + (-1)^{|B||C|} [T(A(x), C(z)), B(y)] + (-1)^{|A|(|B|+|C|)} [T(B(y), C(z)), A(x)] \quad (39)$$

where all commutators are understood to be graded.

2.2. GAUGE THEORIES

We will be interested in the following by Yang-Mills models; by this we mean the most general interaction between particles of spin 0, 1/2 and 1. The fields of spin 1 are described using ghost fields and a suitable gauge operator. The Hilbert space of the model is generated by quantum free fields associated to the following types of particles:

1. Particles of null mass and helicity 1 (photons and gluons). They are described by the vector fields v_a^μ (with Bose statistics) and the scalar fields u_a, \tilde{u}_a (with Fermi statistics): $a \in I_1$.

2. Particles of positive mass and spin 1 (heavy Bosons). They are described by the vector fields v_a^μ (with Bose statistics) and the scalar fields u_a, \tilde{u}_a (with Fermi statistics) and scalar fields Φ_a with Bose statistics: $a \in I_2$.

3. Scalar particles (the Higgs particle) Φ_a with Bose statistics: $a \in I_3$.

4. Dirac fields ψ_A with Fermi statistics: $A \in I_4$.

To describe completely the model we need to give the following elements:

- The 2-point functions; then we can generate the n -point functions using as a guide Wick theorem.

- A Hermiticity structure.

All these elements can be found in preceding publications for instance [9]. One can use the formalism described there to obtain in an unique way the expression of the interaction Lagrangian T : it is (relatively) cohomologous to a non-trivial co-cycle

of the form:

$$\begin{aligned}
T &= f_{abc} \left(\frac{1}{2} v_{a\mu} v_{b\nu} F_c^{\nu\mu} + u_a v_b^\mu \partial_\mu \tilde{u}_c \right) \\
&+ f'_{abc} [\Phi_a (\partial^\mu \Phi_b - m_b v_b^\mu) v_{c\mu} + m_b \Phi_a \tilde{u}_b u_c] \\
&+ \frac{1}{3!} f''_{abc} \Phi_a \Phi_b \Phi_c + j_a^\mu v_{a\mu} + j_a \Phi_a.
\end{aligned} \tag{40}$$

The first line give the pure Yang-Mills interaction, the second line is the vector-scalar interaction, then comes the pure scalar interaction and the last two terms give the interaction of the Dirac fields with the vector and resp. scalar particles mediated by the vector and scalar currents

$$j_a^\mu = \sum_{\epsilon=\pm} \bar{\psi} t_a^\epsilon \otimes \gamma^\mu \gamma_\epsilon \psi \quad j_a = \sum_{\epsilon=\pm} \bar{\psi} s_a^\epsilon \otimes \gamma_\epsilon \psi \quad \gamma_\epsilon = \frac{1}{2}(I + \epsilon \gamma_5). \tag{41}$$

Here $t_a = (t_a)_{AB}$, $s_a = (s_a)_{AB}$ are matrices of dimension I_4 , ($A, B \in I_4$) and we group the Dirac fields in a vector column $\psi = (\psi)_A$, ($A \in I_4$). The expression T above is constrained by Lorentz invariance and the bound ≤ 4 on the canonical dimension. One can also give explicit formulas for the associated expressions $T^\mu, T^{\mu\nu}$ (see [9]).

There are some linear relations fulfilled by the coefficients from (40). We mention only the fact that f_{abc} is completely antisymmetric and that f'_{abc} is antisymmetric in a, b .

2.3. DISTRIBUTIONS WITH CAUSAL SUPPORT AND CAUSAL SPLITTING

We will use many times the so-called *central splitting of causal distributions* [22]. We remind the reader the basic formula. Let $d \in (\mathcal{S}^{4n})'$ be a distribution in the variables x_1, \dots, x_n from the Minkowski space. Suppose that d has causal support *i.e.*

$$\text{supp}(d) \in \{(x_1, \dots, x_n) | x_j - x_n \in V^+ \cup V^-, j = 1, \dots, n-1\} \tag{42}$$

and has the order of causality $\omega = \omega(d) \in \mathbb{N}$; essentially this means that the Fourier transform \tilde{d} of d behaves for large momenta as p^ω . It is a standard theorem in distribution theory that we can split

$$d = a - r \tag{43}$$

where

$$\begin{aligned}
\text{supp}(a) &\in \{(x_1, \dots, x_n) | x_j - x_n \in V^+, j = 1, \dots, n-1\} \\
\text{supp}(r) &\in \{(x_1, \dots, x_n) | x_j - x_n \in V^-, j = 1, \dots, n-1\}
\end{aligned} \tag{44}$$

are called the *advanced* and resp. *retarded* components of d ; moreover, the splitting does not increases the order of singularity. If $\omega(d) < 0$ then a and r are uniquely

determined; formally we have

$$\begin{aligned} a(x) &= \theta^+(x) d(x) \\ r(x) &= \theta^-(x) d(x) \end{aligned} \quad (45)$$

where θ^\pm are some Heaviside functions separating the two pieces of the light cone. Let us suppose that $0 \notin \text{supp}(\tilde{d})$; then taking the Fourier transform we get for:

$$\tilde{a}(p) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\tilde{d}(tp)}{1-t+i0}, \quad p \in V^+ \cup V^- \quad (46)$$

and the integral is convergent. If $\omega(d) \geq 0$ then the integral is not convergent any more and (as for the subtracted Cauchy formula) we have:

$$\tilde{a}(p) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\tilde{d}(tp)}{(t-i0)^\omega (1-t+i0)} \quad (47)$$

and the integral is again convergent. This is the so-called *central* solution of the splitting problem. The general solution is given by adding a polynomial in p of maximal degree $\omega(d)$.

3. SECOND ORDER CAUSAL DISTRIBUTIONS

In second order we have some typical distributions. We remind the fact that the Pauli-Villars distribution is defined by

$$D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x) \quad (48)$$

where

$$D_m^{(\pm)}(x) = \pm \frac{i}{(2\pi)^3} \int dp e^{ip \cdot x} \theta(\pm p_0) \delta(p^2 - m^2) \quad (49)$$

such that

$$D^{(-)}(x) = -D^{(+)}(-x). \quad (50)$$

This distribution has causal support. In fact, it can be causally split (uniquely) into an advanced and a retarded part:

$$D = D^{\text{adv}} - D^{\text{ret}} \quad (51)$$

and then we can define the Feynman propagator and anti-propagator

$$D^F = D^{\text{ret}} + D^{(+)}, \quad \bar{D}^F = D^{(+)} - D^{\text{adv}}. \quad (52)$$

All these distributions have singularity order $\omega(D) = -2$.

These distributions do appear in the tree contributions to the chronological products. One can have anomalies due to the following fact. From the gauge in-

variance (18) we can prove that

$$\begin{aligned} & sD(T^I(x), T^J(y)) \equiv \\ & d_Q D(T^I(x), T^J(y)) - i \partial_\mu^1 D(T^{I\mu}(x), T^J(y)) - (-1)^{|I|} i \partial_\mu^2 D(T^I(x), T^{J\mu}(y)) = 0. \end{aligned} \quad (53)$$

Use must be made of the Klein-Gordon equation

$$(\square + m^2) D_m(x) = 0. \quad (54)$$

Indeed, we have to find the terms from $D(T^{I\mu}(x), T^J(y))$ having a factor $\partial_1^\mu D(x - y)$ and the terms from $D(T^I(x), T^{J\mu}(y))$ having a factor $\partial_2^\mu D(x - y)$ and we must use the Klein-Gordon equation from above to eliminate some terms. However, if we use the causal splitting and replace $D_m(x) \rightarrow D_m^{adv, ret, F}(x)$ in the causal commutator, we are faced with the fact that the Klein-Gordon equation cannot be causally split: we have

$$(\square + m^2) D_m^F(x) = \delta(x). \quad (55)$$

These anomalies have been investigated in detail: see [10] and references quoted there; in this reference we have used an alternative method, namely the *off-shell* analysis. The main result is that the gauge invariance at the second-order tree level can be restored if one redefines the chronological products $T(A(x), B(y)) \rightarrow T(A(x), B(y)) + \delta(x - y) N^{A,B}(x)$ where the Wick polynomials $N^{A,B}(x)$ can be obtained from the quadri-linear terms of the classical Yang-Mills Lagrangian with the classical fields replaced by quantum fields and afterwards Wick ordering is applied. To be able to perform such a redefinition of the chronological products some bilinear relations must be obeyed by the coefficients from (40). We mention only the fact that: (a) f_{abc} should fulfill the Jacobi identity, so they are the structure constants of some Lie algebra; (b) $(f'_a)_{bc} \equiv -f'_{bca}$ and t_a^ϵ should be representations of the Lie algebra emerging above; (c) s_a^ϵ are tensor operators. We give below the expressions for the finite renormalizations of the chronological products:

$$\begin{aligned} N^{\emptyset\emptyset} &= \frac{i}{2} f_{abe} f_{cde} v_a^\mu v_b^\nu v_{c\mu} v_{d\nu} - i f'_{eab} f'_{ecd} \Phi_a v_{b\mu} \Phi_c v_d^\mu + \frac{i}{24} \sum_{a,b,c,d} g_{abcd} \Phi_a \Phi_b \Phi_c \Phi_d \\ N^{[\mu]\emptyset} &= -i f_{abe} f_{cde} u_a v_b^\nu v_{c\nu} v_d^\mu - i f'_{eab} f'_{ecd} \Phi_a u_b \Phi_c v_d^\mu \\ N^{[\mu][\nu]} &= -i f_{abe} f_{cde} u_a v_b^\nu u_c v_d^\mu \\ N^{[\mu\nu]\emptyset} &= -\frac{i}{2} f_{abe} f_{cde} u_a u_b v_c^\mu v_d^\nu \\ N^{[\mu\nu][\rho]} &= 0, \end{aligned} \quad (56)$$

where

$$F_{abcd} \equiv \begin{cases} \frac{1}{m_a} \mathcal{S}_{bcd} \left(f'_{eba} f''_{ecd} \right) & \text{if } a \in I_2 \\ 0 & \text{if } a \in I_1 \cup I_3 \end{cases} . \quad (57)$$

For one-loop contributions in the second order we need the basic distributions

$$d_{D_1, D_2}(x) \equiv \frac{1}{2} [D_1^{(+)}(x) D_2^{(+)}(x) - D_1^{(-)}(x) D_2^{(-)}(x)] \quad (58)$$

where $D_j = D_{m_j}$ which also with causal support. This expression is linear in D_1 and D_2 . We will also use the notation

$$d_{12} \equiv d(D_1, D_2) \equiv d_{D_1, D_2} \quad (59)$$

and when no confusion about the distributions $D_j = D_{m_j}$ can appear, we skip all indexes altogether. The causal split

$$d_{12} = d_{12}^{adv} - d_{12}^{ret} \quad (60)$$

is not unique because $\omega(d_{12}) = 0$ so we make the redefinitions

$$d_{12}^{adv(ret)}(x) \rightarrow d_{12}^{adv(ret)}(x) + c \delta(x) \quad (61)$$

without affecting the support properties and the order of singularity. The corresponding Feynman propagators can be defined as above and will be denoted as d_{12}^F . Another way to construct them is to define for $x \neq 0$ the distribution

$$d_{12}^{(0)}(x) \equiv \frac{1}{2} D_1^F(x) D_2^F(x) \quad (62)$$

and to extend it to the whole domain using a standard result in distribution theory (see the preceding Section).

In [12] one can find the expressions of the dominant one-loop contributions from the chronological products. It is necessary to consider the case $D_1 = D_2 = D_m$ and determine its Fourier transform. By direct computations it can be obtained that

$$\tilde{d}_{m,m}(k) \equiv \frac{1}{(2\pi)^2} \int dx e^{ik \cdot x} d_{m,m}(x) = -\frac{1}{8(2\pi)^3} \varepsilon(k_0) \theta(k^2 - m^2) \sqrt{1 - \frac{4m^2}{k^2}}. \quad (63)$$

To obtain the dominant contribution in the causal commutator $D_{\text{dominant}}^{JJ}(x, y)$ we write for every mass m in the game

$$D_m = D_M + (D_M - D_m) \quad (64)$$

and consider only the first term.

We also have two-loops contributions in $D^{\emptyset\emptyset}$; the relevant distributions are

$$d_{m_1, m_2, m_3}(x) \equiv \frac{1}{3!} [D_{m_1}^{(+)}(x) D_{m_2}^{(+)}(x) D_{m_3}^{(+)}(x) - D_{m_1}^{(-)}(x) D_{m_2}^{(-)}(x) D_{m_3}^{(-)}(x)]. \quad (65)$$

4. THIRD ORDER CAUSAL DISTRIBUTIONS

We must start from (39) and use the complete formula for the second order causal products. Generically we have

$$T(A(x), B(y)) =: A(x)B(y) : + T_{(0)}(A(x), B(y)) + T_{(1)}(A(x), B(y)) \\ + T_{(2)}(A(x), B(y)) + \delta(x - y)N^{A, B}(x) \quad (66)$$

where the contributions $T_{(l)}$, $l = 0, 1, 2$ are the tree, one-loop and two-loops contributions and the last term is the finite renormalization which must be done to eliminate the anomalies coming from the tree contributions - see (56). The two-loop term $T_{(2)}$ from (66) does not contribute to the causal commutator (39) because it is a c -number.

We remain with two distinct types of terms in (39): tree and loop graphs.

4.1. TREE GRAPHS

The first possibility is to consider the first term from the preceding formula of the generic form

$$: A(x)B(y) := \sum : a_1(x)a_2(x)a_3(x)b_1(y)b_2(y)b_3(y) : \quad (67)$$

When we commute this operator with $C(z) =: c_1(z)c_2(z)c_3(z) :$ we can take a contraction of a factor a with a factor c and a contraction of a factor b with another factor c .

Another possibility comes from the second term of (66) with the generic form

$$T_{(0)}(A(x), B(y)) = \sum p_j(\partial) D_m^F(x - y) : a_1(x)a_2(x)b_1(y)b_2(y) : \quad (68)$$

When we commute this operator with $C(z) =: c_1(z)c_2(z)c_3(z) :$ one possibility is to contract one of the factors a (or one of the factors b) with a factor c .

These relevant causal distributions are:

$$\begin{aligned}
d_{D_1, D_2}^{(1)}(x, y, z) &\equiv \bar{D}_1^F(x-y)D_2(z-x) - D_1(x-y)D_2^F(z-x) \\
&\quad + D_1^{(-)}(x-y)D_2^{(+)}(z-x) - D_1^{(+)}(x-y)D_2^{(-)}(z-x) \\
d_{D_1, D_2}^{(2)}(x, y, z) &\equiv -\bar{D}_1^F(x-y)D_2(y-z) + D_1(x-y)D_2^F(y-z) \\
&\quad + D_1^{(+)}(x-y)D_2^{(-)}(y-z) - D_1^{(-)}(x-y)D_2^{(+)}(y-z) \\
d_{D_1, D_2}^{(3)}(x, y, z) &\equiv D_1^F(z-x)D_2(y-z) - D_1(z-x)D_2^F(y-z) \\
&\quad + D_1^{(-)}(z-x)D_2^{(+)}(y-z) - D_1^{(+)}(z-x)D_2^{(-)}(y-z)
\end{aligned} \tag{69}$$

where the dominant contribution corresponds to the choice $D_1 = D_2 = D_m$. As in the previous section we will use the alternative notation:

$$d^{(j)}(D_1, D_2) = d_{D_1, D_2}^{(j)}. \tag{70}$$

The causal support properties follow from the alternative formulas

$$\begin{aligned}
d_{D_1, D_2}^{(1)}(x, y, z) &= D_1^{\text{ret}}(x-y)D_2^{\text{ret}}(z-x) - D_1^{\text{adv}}(x-y)D_2^{\text{adv}}(z-x) \\
d_{D_1, D_2}^{(2)}(x, y, z) &= D_1^{\text{ret}}(y-x)D_2^{\text{ret}}(z-y) - D_1^{\text{adv}}(y-x)D_2^{\text{adv}}(z-y) \\
d_{D_1, D_2}^{(3)}(x, y, z) &= D_1^{\text{ret}}(z-x)D_2^{\text{ret}}(y-z) - D_1^{\text{adv}}(z-x)D_2^{\text{adv}}(y-z).
\end{aligned} \tag{71}$$

The order of singularity of these distributions is $\omega = -2$. We can define associated distributions if we replace $D_1 \mapsto \partial_\alpha D_1$, etc.

$$\begin{aligned}
\mathcal{D}_\alpha^2 d_{D_1, D_2}^{(1)} &= d_{D_1, \partial_\alpha D_2}^{(1)}, & \mathcal{D}_\alpha^3 d_{D_1, D_2}^{(1)} &= d_{\partial_\alpha D_1, D_2}^{(1)}, \\
\mathcal{D}_\alpha^1 d_{D_1, D_2}^{(2)} &= d_{D_1, \partial_\alpha D_2}^{(2)}, & \mathcal{D}_\alpha^3 d_{D_1, D_2}^{(2)} &= d_{\partial_\alpha D_1, D_2}^{(2)}, \\
\mathcal{D}_\alpha^1 d_{D_1, D_2}^{(3)} &= d_{D_1, \partial_\alpha D_2}^{(3)}, & \mathcal{D}_\alpha^2 d_{D_1, D_2}^{(3)} &= d_{\partial_\alpha D_1, D_2}^{(3)}.
\end{aligned} \tag{72}$$

The causal splitting of the distributions $d^{(j)}$ is elementary:

$$\begin{aligned}
d_{D_1, D_2}^{(1)adv}(x, y, z) &= D_1^{\text{ret}}(x-y)D_2^{\text{ret}}(z-x), \\
d_{D_1, D_2}^{(1)ret}(x, y, z) &= D_1^{\text{adv}}(x-y)D_2^{\text{adv}}(z-x), \\
d_{D_1, D_2}^{(2)adv}(x, y, z) &= D_1^{\text{ret}}(y-x)D_2^{\text{ret}}(z-y), \\
d_{D_1, D_2}^{(2)ret}(x, y, z) &= D_1^{\text{adv}}(y-x)D_2^{\text{adv}}(z-y), \\
d_{D_1, D_2}^{(3)adv}(x, y, z) &= D_1^{\text{ret}}(z-x)D_2^{\text{ret}}(y-z), \\
d_{D_1, D_2}^{(3)ret}(x, y, z) &= D_1^{\text{adv}}(z-x)D_2^{\text{adv}}(y-z)
\end{aligned} \tag{73}$$

and similar relations for the associated distributions $\mathcal{D}_\alpha^2 d_{D_1, D_2}^{(1)}$, etc. For the the Feyn-

man propagators we have

$$\begin{aligned} d_{D_1, D_2}^{(1)F}(x, y, z) &= D_1^F(x-y)D_2^F(z-x) \\ d_{D_1, D_2}^{(2)F}(x, y, z) &= D_1^F(y-x)D_2^F(y-z) \\ d_{D_1, D_2}^{(3)F}(x, y, z) &= D_1^F(z-x)D_2^F(y-z) \end{aligned} \quad (74)$$

and it follows that these contributions do not produce anomalies.

Another type of tree contributions comes from the last term of (66) *i.e.* the finite renormalizations. We have the generic form

$$N^{A,B}(x) = \sum : a_1(x)a_2(x)a_3(x)a_4(x) : \quad (75)$$

and when commuting with $C(z) =: c_1(z)c_2(z)c_3(z) :$ we can have one, two or three contractions corresponding to tree, one-loop and two-loops contributions

$$\begin{aligned} T^N(A(x), B(y), C(z)) &= T_{(0)}^N(A(x), B(y), C(z)) + T_{(1)}^N(A(x), B(y), C(z)) \\ &\quad + T_{(2)}^N(A(x), B(y), C(z)) + \dots \end{aligned} \quad (76)$$

where \dots are the un-contracted terms. The relevant distributions for the tree contributions are

$$\begin{aligned} d_1(x, y, z) &= \delta(y-z) D_m(x-y) \\ d_2(x, y, z) &= \delta(z-x) D_m(y-z) \\ d_3(x, y, z) &= \delta(x-y) D_m(y-z) \end{aligned} \quad (77)$$

where the dominant contributions correspond to the same positive mass. These contributions can produce anomalies by the same mechanism as for the tree contribution from the second order of perturbation theory.

4.2. ONE-LOOP GRAPHS: TRIANGLE TYPE

We consider again the tree contribution given by (68). When we commute this operator with $C(z) =: c_1(z)c_2(z)c_3(z) :$ we can contract a factor a with one of the factors c and one of the factors b with another c ; in terms of Feynman graphs it corresponds to triangle graphs. We describe the relevant distributions with causal support.

First, we take $D_j = D_{m_j}, j = 1, 2, 3$ and define

$$\begin{aligned} d_{D_1, D_2, D_3}(x, y, z) &\equiv \bar{D}_3^F(x-y)[D_2^{(-)}(z-x)D_1^{(+)}(y-z) - D_2^{(+)}(z-x)D_1^{(-)}(y-z)] \\ &\quad + D_1^F(y-z)[D_3^{(-)}(x-y)D_2^{(+)}(z-x) - D_3^{(+)}(x-y)D_2^{(-)}(z-x)] \\ &\quad + D_2^F(z-x)[D_1^{(-)}(y-z)D_3^{(+)}(x-y) - D_1^{(+)}(y-z)D_3^{(-)}(x-y)] \end{aligned} \quad (78)$$

which are with causal support; indeed we have the alternative forms

$$\begin{aligned}
d_{D_1, D_2, D_3}(x, y, z) = & \\
& - D_3^{\text{ret}}(x-y)[D_2^{(-)}(z-x)D_1^{(+)}(y-z) - D_2^{(+)}(z-x)D_1^{(-)}(y-z)] \\
& + D_1^{\text{adv}}(y-z)[D_3^{(-)}(x-y)D_2^{(+)}(z-x) - D_3^{(+)}(x-y)D_2^{(-)}(z-x)] \\
& + D_2^{\text{adv}}(z-x)[D_1^{(-)}(y-z)D_3^{(+)}(x-y) - D_1^{(+)}(y-z)D_3^{(-)}(x-y)] \quad (79)
\end{aligned}$$

and

$$\begin{aligned}
d_{D_1, D_2, D_3}(x, y, z) = & \\
& - D_3^{\text{adv}}(x-y)[D_2^{(-)}(z-x)D_1^{(+)}(y-z) - D_2^{(+)}(z-x)D_1^{(-)}(y-z)] \\
& + D_1^{\text{ret}}(y-z)[D_3^{(-)}(x-y)D_2^{(+)}(z-x) - D_3^{(+)}(x-y)D_2^{(-)}(z-x)] \\
& + D_2^{\text{ret}}(z-x)[D_1^{(-)}(y-z)D_3^{(+)}(x-y) - D_1^{(+)}(y-z)D_3^{(-)}(x-y)] \quad (80)
\end{aligned}$$

from which it follows that the distribution $d_{D_1, D_2, D_3}(x, y, z)$ is null outside the causal cone $\{(x, y, z) | x-z \in V^+, y-z \in V^+\} \cup \{(x, y, z) | x-z \in V^-, y-z \in V^-\}$. These distributions have the singularity order $\omega(d_{D_1, D_2, D_3}) = -2$.

As in the previous Section we use the alternative notation

$$d_{123} \equiv d(D_1, D_2, D_3) \equiv d_{D_1, D_2, D_3} \quad (81)$$

and when there is no ambiguity about the distributions D_j we simply denote $d = d_{123}$. There are some associated distributions obtained from $d_{D_1, D_2, D_3}(x, y, z)$ applying derivatives on the factors $D_j = D_{m_j}, j = 1, 2, 3$. We also denote

$$\begin{aligned}
\mathcal{D}_1^\mu d_{D_1, D_2, D_3} &\equiv d_{\partial^\mu D_1, D_2, D_3}, \\
\mathcal{D}_2^\mu d_{D_1, D_2, D_3} &\equiv d_{D_1, \partial^\mu D_2, D_3}, \\
\mathcal{D}_3^\mu d_{D_1, D_2, D_3} &\equiv d_{D_1, D_2, \partial^\mu D_3},
\end{aligned} \quad (82)$$

and so on for more derivatives ∂_α distributed in an arbitrary way on the factors $D_j = D_{m_j}, j = 1, 2, 3$.

It is known that these distributions can be causally split in such a way that the order of singularity, translation invariance and Lorentz covariance are preserved. The same will be true for the corresponding Feynman distributions. Because $\omega(d_{123}) = -2$ and $\omega(\mathcal{D}_i^\mu d_{123}) = -1$ the corresponding advanced, retarded and Feynman distributions are unique. For more derivatives we have some freedom of redefinition.

As in the previous Section, let us consider the case $D_1 = D_2 = D_3 = D_m, m > 0$ and study the corresponding distribution $d_{m,m,m}$. We consider it as distribution in two variables $X \equiv x-z, Y \equiv y-z$ and we will need its Fourier transform. The

computation is essentially done in [21] and gives the following formula:

$$\tilde{d}_{m,m,m}(p,q) = \frac{1}{8(2\pi)^5} \frac{1}{\sqrt{N}} [\epsilon(p_0)\theta(p^2 - 4m^2) \ln_1 + \epsilon(q_0)\theta(q^2 - 4m^2) \ln_2 + \epsilon(P_0)\theta(P^2 - 4m^2) \ln_3], \quad (83)$$

where

$$\begin{aligned} \ln_1 &\equiv \ln \left(\frac{P \cdot q + \sqrt{N(1 - 4m^2/p^2)}}{P \cdot q - \sqrt{N(1 - 4m^2/p^2)}} \right) \\ \ln_2 &\equiv \ln \left(\frac{P \cdot p + \sqrt{N(1 - 4m^2/q^2)}}{P \cdot p - \sqrt{N(1 - 4m^2/q^2)}} \right) \\ \ln_3 &\equiv \ln \left(\frac{-p \cdot q + \sqrt{N(1 - 4m^2/P^2)}}{-p \cdot q - \sqrt{N(1 - 4m^2/P^2)}} \right) \end{aligned} \quad (84)$$

with the notations $P = p + q$ and $N \equiv (p \cdot q)^2 - p^2 q^2$.

We give here an example of the use of such a causal distribution. By direct computation we can prove

Theorem 4.1 *In the Yang-Mills sector the dominant contribution (i.e. of maximal order of singularity) for one-loop triangle graphs is:*

$$\begin{aligned} D_{YM}^{[\mu],[\nu],[\rho]}(x,y,z)_{(1)} &= i[\mathcal{D}_1^\mu \mathcal{D}_1^\nu \mathcal{D}_2^\rho + \mathcal{D}_1^\rho \mathcal{D}_2^\mu \mathcal{D}_2^\nu + \mathcal{D}_1^\mu \mathcal{D}_1^\rho \mathcal{D}_3^\nu \\ &\quad + \mathcal{D}_1^\nu \mathcal{D}_3^\mu \mathcal{D}_3^\rho + \mathcal{D}_2^\nu \mathcal{D}_2^\rho \mathcal{D}_3^\mu + \mathcal{D}_2^\mu \mathcal{D}_3^\nu \mathcal{D}_3^\rho \\ &\quad + \mathcal{D}_1^\nu \mathcal{D}_1^\rho \mathcal{D}_2^\mu + \mathcal{D}_1^\nu \mathcal{D}_2^\mu \mathcal{D}_2^\rho + \mathcal{D}_1^\nu \mathcal{D}_1^\rho \mathcal{D}_3^\mu \\ &\quad + \mathcal{D}_1^\rho \mathcal{D}_3^\mu \mathcal{D}_3^\nu + \mathcal{D}_2^\mu \mathcal{D}_2^\rho \mathcal{D}_3^\nu + \mathcal{D}_2^\rho \mathcal{D}_3^\mu \mathcal{D}_3^\nu \\ &\quad - 2\mathcal{D}_1^\mu \mathcal{D}_2^\nu \mathcal{D}_3^\rho \\ &\quad + 2(\mathcal{D}_1^\mu \mathcal{D}_2^\rho \mathcal{D}_3^\nu + \mathcal{D}_1^\rho \mathcal{D}_2^\nu \mathcal{D}_3^\mu + \mathcal{D}_1^\nu \mathcal{D}_2^\mu \mathcal{D}_3^\rho) \\ &\quad + \eta^{\mu\nu} \mathcal{D}_1^\rho \mathcal{D}_1 \cdot \mathcal{D}_2 + \eta^{\mu\nu} \mathcal{D}_2^\rho \mathcal{D}_1 \cdot \mathcal{D}_2 + \eta^{\mu\rho} \mathcal{D}_1^\nu \mathcal{D}_1 \cdot \mathcal{D}_3 \\ &\quad + \eta^{\mu\rho} \mathcal{D}_3^\nu \mathcal{D}_1 \cdot \mathcal{D}_3 + \eta^{\nu\rho} \mathcal{D}_2^\mu \mathcal{D}_2 \cdot \mathcal{D}_3 + \eta^{\nu\rho} \mathcal{D}_3^\mu \mathcal{D}_2 \cdot \mathcal{D}_3 \\ &\quad - (\eta^{\mu\nu} \mathcal{D}_3^\rho \mathcal{D}_1 \cdot \mathcal{D}_3 + \eta^{\mu\nu} \mathcal{D}_3^\rho \mathcal{D}_2 \cdot \mathcal{D}_3 + \eta^{\mu\rho} \mathcal{D}_2^\nu \mathcal{D}_1 \cdot \mathcal{D}_2 \\ &\quad + \eta^{\mu\rho} \mathcal{D}_2^\nu \mathcal{D}_2 \cdot \mathcal{D}_3 + \eta^{\nu\rho} \mathcal{D}_1^\mu \mathcal{D}_1 \cdot \mathcal{D}_2 + \eta^{\nu\rho} \mathcal{D}_1^\mu \mathcal{D}_1 \cdot \mathcal{D}_3)] d(x,y,z) \\ &\quad f_{abc}^{(0)} u_a(x) u_b(y) u_c(z) \end{aligned} \quad (85)$$

where

$$f_{[abc]}^{(0)} = f_{eap} f_{ebq} f_{cpq}. \quad (86)$$

4.3. ONE-LOOP GRAPHS: ONE-PARTICLE REDUCIBLE TYPE

Such contributions have two sources: (a) from the one-loop contribution of (66) of the generic form

$$T(A(x), B(y))_{(1)} = \sum p_j(\partial) d_2^F(x-y) : a(x)b(y) : \quad (87)$$

Commuting with $C(z) =: c_1(z)c_2(z)c_3(z) :$ we contract the factor a (or the factor b) with one of the factors c ; (b) from (68) commuting with $C(z) =: c_1(z)c_2(z)c_3(z) :$ we contract two factors a (or two factors b) with the factors c . These relevant causal distributions are of the type (69) namely

$$\begin{aligned} d_{m,m,m}^{(j)} &= d_{D_m, d_{m,m}}^{(j)} = d^{(j)}(D_m, d_{m,m}), \\ f_{m,m,m}^{(j)} &= d_{d_{m,m}, D_m}^{(j)} = d^{(j)}(d_{m,m}, D_m), \quad j = 1, 2, 3 \end{aligned} \quad (88)$$

where $d_{m,m}$ is defined by (58) for equal masses $m_1 = m_2 = m$.

We illustrate the use of these distributions by the following

Theorem 4.2 *In the Yang-Mills sector the dominant contribution (i.e. of maximal order of singularity) for one-loop, one-particle reducible graphs is*

$$\begin{aligned} D(T^\mu(x), T^\nu(y); T^\rho(z))_{1PR} &= -\frac{i}{3} (f_{abc}^{(0)} + f_{abc}^{(3)} + f_{abc}^{(4)}) \\ &\left[\mathcal{D}_2^\mu (\mathcal{D}_1^\nu \mathcal{D}_1^\rho - \eta^{\nu\rho} \mathcal{D}_1^2) d^{(3)}(x, y, z) u_a(x) u_b(x) u_c(y) \right. \\ &+ \mathcal{D}_1^\nu (\mathcal{D}_2^\mu \mathcal{D}_2^\rho - \eta^{\mu\rho} \mathcal{D}_2^2) f^{(3)}(x, y, z) u_a(x) u_b(y) u_c(y) \\ &+ \mathcal{D}_3^\mu (\mathcal{D}_1^\nu \mathcal{D}_1^\rho - \eta^{\nu\rho} \mathcal{D}_1^2) d^{(2)}(x, y, z) u_a(x) u_b(x) u_c(z) \\ &+ \mathcal{D}_1^\rho (\mathcal{D}_3^\mu \mathcal{D}_3^\nu - \eta^{\mu\nu} \mathcal{D}_3^2) f^{(2)}(x, y, z) u_a(x) u_b(z) u_c(z) \\ &+ \mathcal{D}_3^\nu (\mathcal{D}_2^\mu \mathcal{D}_2^\rho - \eta^{\mu\rho} \mathcal{D}_2^2) d^{(1)}(x, y, z) u_a(x) u_b(z) u_c(z) \\ &\left. + \mathcal{D}_2^\rho (\mathcal{D}_3^\mu \mathcal{D}_3^\nu - \eta^{\mu\nu} \mathcal{D}_3^2) f^{(1)}(x, y, z) u_a(y) u_b(z) u_c(z) \right] \end{aligned} \quad (89)$$

where $f_{abc}^{(0)}$ has been defined in the previous theorem and

$$f_{[abc]}^{(3)} = f'_{epa} f'_{eqb} f'_{pqc}, \quad f_{[abc]}^{(4)} = i \text{Tr}([t_a^\epsilon, t_b^\epsilon] t_c^\epsilon) = f_{abd} g_{cd}^{(2)}. \quad (90)$$

We also have loop contributions of one-particle reducible type associated to the finite renormalizations (the last term) from (66). We commute an expression of the type (75) with $C(z) =: c_1(z)c_2(z)c_3(z) :$ and take two contractions and obtain $T_{(1)}^N(A(x), B(y), C(z))$.

The relevant causal distributions are

$$\begin{aligned} f_1(x, y, z) &= \delta(y-z) d_{m,m}(x-y) \\ f_2(x, y, z) &= \delta(z-x) d_{m,m}(y-z) \\ f_3(x, y, z) &= \delta(x-y) d_{m,m}(y-z) \end{aligned} \quad (91)$$

with

$$\omega(f_j) = 0. \quad (92)$$

We consider them (as before) as distributions in two variables $X \equiv x - z$, $Y \equiv y - z$ and the Fourier transforms are:

$$\tilde{f}_1(p, q) = \frac{1}{(2\pi)^2} \tilde{d}_{m,m}(p), \quad \tilde{f}_2(p, q) = \frac{1}{(2\pi)^2} \tilde{d}_{m,m}(q), \quad \tilde{f}_3(p, q) = \frac{1}{(2\pi)^2} \tilde{d}_{m,m}(P) \quad (93)$$

but these contributions do not produce anomalies. The same is true for the last contribution in (76). We will need in the next Section the distributions:

$$\begin{aligned} f'_1(x, y, z) &= \delta(y - z) d'_{m,m}(x - y) \\ f'_2(x, y, z) &= \delta(z - x) d'_{m,m}(y - z) \\ f'_3(x, y, z) &= \delta(x - y) d'_{m,m}(y - z). \end{aligned} \quad (94)$$

4.4. TWO-LOOP GRAPHS

The associated causal distributions are $d(d_{m,m}, D_m, D_m)$, $d(D_m, d_{m,m}, D_m)$, $d(D_m, D_m, d_{m,m})$ in the notation (81). There are also some disconnected contributions involving the distribution (65) which do not give anomalies.

5. CAUSAL SPLITTING IN THE THIRD ORDER FOR TRIANGLE CONTRIBUTIONS

We denote for simplicity

$$\begin{aligned} d_i^\mu &\equiv \mathcal{D}_i^\mu d, \\ d_{ij}^{\mu\nu} &\equiv \mathcal{D}_i^\mu \mathcal{D}_j^\nu d, & d_{ij} &\equiv \eta_{\mu\nu} d_{ij}^{\mu\nu} \\ d_{ijk}^{\mu\nu\rho} &\equiv \mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d, & d_{ijk}^\mu &\equiv \eta_{\nu\rho} d_{ijk}^{\mu\nu\rho} \end{aligned} \quad (95)$$

and we have the following orders of singularity:

$$\omega(d_j^\mu) = -1, \quad \omega(d_{jk}^{\mu\nu}) = 0, \quad \omega(d_{jkl}^{\mu\nu\rho}) = 1. \quad (96)$$

To perform the computation we need an explicit formula for the Fourier transform of these distributions. In [12] one can find the explicit expressions of these Fourier transforms. From Lorentz covariance considerations the Fourier transform should be of the form:

$$\tilde{d}_j^\mu(p, q) = -i [p^\mu \tilde{A}_j(p, q) + q^\mu \tilde{B}_j(p, q)] \quad (97)$$

where the scalar functions \tilde{A}_j and \tilde{B}_j depend in fact only on the Lorentz invariants: $p^2, q^2, p \cdot q$. Similarly

$$\begin{aligned} \tilde{d}_{jk}^{\mu\nu}(p, q) &= -[p^\mu p^\nu \tilde{A}_{jk}(p, q) + q^\mu q^\nu \tilde{B}_{jk}(p, q) + p^\mu q^\nu \tilde{C}_{jk}^{(1)}(p, q) + q^\mu p^\nu \tilde{C}_{jk}^{(2)}(p, q)] \\ &\quad + \eta^{\mu\nu} \tilde{D}_{jk}(p, q), \end{aligned} \quad (98)$$

where, as before, the scalar functions A, B, C, D depend only on the Lorentz invariants. The Fourier transform is

$$\begin{aligned}\tilde{A}_3(p, q) &= -\frac{q^2 p \cdot P}{2N} \tilde{d}_{m,m,m}(p, q) + \frac{q^2}{N} [\tilde{f}_3(p, q) - \tilde{f}_2(p, q)] + \frac{p \cdot q}{N} [\tilde{f}_3(p, q) - \tilde{f}_1(p, q)] \\ \tilde{B}_3(p, q) &= -\tilde{A}_3(q, p).\end{aligned}\tag{99}$$

Using these formulas we will be able to perform the central causal splitting.

We start with the simplest case.

Theorem 5.1 *The following relations are true*

$$\frac{\partial}{\partial x^\rho} \mathcal{D}_1^\rho d = \mathcal{D}_1 \cdot \mathcal{D}_3 d - \mathcal{D}_1 \cdot \mathcal{D}_2 d \tag{100}$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_1^\rho d = -\mathcal{D}_1 \cdot \mathcal{D}_3 d - m^2 d + 2 f_1, \tag{101}$$

$$\frac{\partial}{\partial z^\rho} \mathcal{D}_1^\rho d = \mathcal{D}_1 \cdot \mathcal{D}_2 d + m^2 d - 2 f_1 \tag{102}$$

and another two sets of relations which can be obtained by circular permutations.

After the central causal splitting we obtain:

$$\frac{\partial}{\partial x^\rho} (\mathcal{D}_1^\rho d)^F = (\mathcal{D}_1 \cdot \mathcal{D}_3 d)^F - (\mathcal{D}_1 \cdot \mathcal{D}_2 d)^F \tag{103}$$

$$\frac{\partial}{\partial y^\rho} (\mathcal{D}_1^\rho d)^F = -(\mathcal{D}_1 \cdot \mathcal{D}_3 d)^F - m^2 d^F + 2 f_1^F + A \delta, \tag{104}$$

$$\frac{\partial}{\partial z^\rho} (\mathcal{D}_1^\rho d)^F = (\mathcal{D}_1 \cdot \mathcal{D}_2 d)^F + m^2 d^F - 2 f_1^F - A \delta \tag{105}$$

and another two sets of relations which can be obtained by circular permutations; here $\delta = \delta(x, y, z) = \delta(x - z)\delta(y - z)$ and $A = \frac{i}{8(2\pi)^2}$.

Proof: We illustrate the idea using the relation (101); after we perform a Fourier transform:

$$-i q_\mu \tilde{d}_1^\mu = -\tilde{d}_{13} - m^2 \tilde{d} + 2 \tilde{f}_1. \tag{106}$$

Because of (96) and (92) we must causally split d and d_1^μ with formula (46) and d_{13} and f_1 with formula (47). It follows that the anomaly

$$\tilde{A}_1 \equiv -i q_\rho \tilde{a}_1^\rho + \tilde{a}_{13} + m^2 \tilde{a} - 2 \tilde{f}_1^{adv} \tag{107}$$

is given by

$$\tilde{A}_1 = -\frac{im^2}{2\pi} \int \frac{dt}{t} \tilde{d}(tp, tq) = \frac{i}{8(2\pi)^6} \tag{108}$$

and this gives (104). All other relations are causally split in the same way. ■

Next, we have a more complicated case.

Theorem 5.2 *The following relations are true*

$$\frac{\partial}{\partial x^\rho} \mathcal{D}_i^\mu \mathcal{D}_1^\rho d = \mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_3 d - \mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_2 d \quad (109)$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_i^\mu \mathcal{D}_1^\rho d = -\mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_3 d - m^2 \mathcal{D}_i^\mu d + f_{i1}^\mu, \quad (110)$$

$$\frac{\partial}{\partial z^\rho} \mathcal{D}_i^\mu \mathcal{D}_1^\rho d = \mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_2 d + m^2 \mathcal{D}_i^\mu d - f_{i1}^\mu \quad (111)$$

and another two sets of relations which can be obtained by circular permutations. Here

$$f_{11}^\mu = (\partial_1^\mu + 2\partial_2^\mu) f_1, \quad f_{21}^\mu = -\partial_1^\mu f_1, \quad f_{31}^\mu = \partial_1^\mu f_1 \quad (112)$$

and the rest by circular permutations.

After central causal splitting we obtain:

$$\frac{\partial}{\partial x^\rho} (\mathcal{D}_i^\mu \mathcal{D}_1^\rho d)^F = (\mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_3 d)^F - (\mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_2 d)^F \quad (113)$$

$$\frac{\partial}{\partial y^\rho} (\mathcal{D}_i^\mu \mathcal{D}_1^\rho d)^F = -(\mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_3 d)^F - m^2 (\mathcal{D}_i^\mu d)^F + f_{i1}^{\mu,F} + A_1^\mu, \quad (114)$$

$$\frac{\partial}{\partial z^\rho} (\mathcal{D}_i^\mu \mathcal{D}_1^\rho d)^F = (\mathcal{D}_i^\mu \mathcal{D}_1 \cdot \mathcal{D}_2 d)^F + m^2 (\mathcal{D}_i^\mu d)^F - f_{i1}^{\mu,F} - A_1^\mu \quad (115)$$

and another two sets of relations which can be obtained by circular permutations. Here

$$A_1^\mu = B(\partial_2^\mu - \partial_3^\mu) \delta = B(\partial_1^\mu + 2\partial_2^\mu) \delta \quad (116)$$

and the rest by circular permutations. We have defined $B \equiv \frac{1}{3} A$.

Proof: We consider (110): after the Fourier transform, we end up, as before, with the anomaly

$$\tilde{\mathcal{A}}_{j1} = -i q_\rho \tilde{a}_{j1}^{\mu\rho} + \tilde{a}_{j13}^\mu + m^2 \tilde{a}_j^\mu - \tilde{f}_{j1}^{\mu,\text{adv}} \quad (117)$$

After the causal splitting we find out that

$$\tilde{\mathcal{A}}_{j1} = -\frac{im^2}{2\pi} \int \frac{dt}{t^2} (1+t) \tilde{a}_j^\mu(tp, tq) \quad (118)$$

depends only on j . We must use the formula (97) and we obtain:

$$\tilde{\mathcal{A}}_j^\mu(p, q) = -\frac{m^2}{2\pi} \left[p^\mu \int \frac{dt}{t} \tilde{A}_j(tp, tq) + q^\mu \int \frac{dt}{t} \tilde{B}_j(tp, tq) \right]. \quad (119)$$

To compute the two integrals above we must use the formulas (99). For instance we have:

$$\int \frac{dt}{t} \tilde{A}_3(tp, tq) = -\frac{q^2 p \cdot P}{2N} \int \frac{dt}{t} \tilde{d}_{m,m,m}(tp, tq) + \frac{q^2}{N} \int \frac{dt}{t^3} [\tilde{f}_3(tp, tq) - \tilde{f}_2(tp, tq)] + \frac{p \cdot q}{N} \int \frac{dt}{t^3} [\tilde{f}_3(tp, tq) - \tilde{f}_1(tp, tq)]. \quad (120)$$

The first integral has been already computed at the preceding theorem. If we use the expressions (93) then we get

$$\int \frac{dt}{t^3} \tilde{f}_1(tp, tq) = b(p^2), \quad \int \frac{dt}{t^3} \tilde{f}_2(tp, tq) = b(q^2), \quad \int \frac{dt}{t^3} \tilde{f}_3(tp, tq) = b(P^2) \quad (121)$$

where

$$b(k) \equiv \frac{1}{(2\pi)^2} \int \frac{dt}{t^3} \tilde{d}_{m,m}(tk). \quad (122)$$

The preceding integral can be computed using the explicit expression and the result is

$$b(k) = b k^2, \quad b \equiv -\frac{1}{48(2\pi)^5 m^2}. \quad (123)$$

so after some simple substitutions we obtain the formulas from the statement. ■

Finally we have:

Theorem 5.3 *The following relations are true*

$$\frac{\partial}{\partial x^\rho} \mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1^\rho d = \mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_3 d - \mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_2 d \quad (124)$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1^\rho d = -\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_3 d - m^2 \mathcal{D}_i^\mu \mathcal{D}_j^\nu d + f_{ij1}^{\mu\nu} - \frac{2m^2}{3} C_1^{\mu\nu} f_1', \quad (125)$$

$$\frac{\partial}{\partial z^\rho} \mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1^\rho d = \mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_2 d + m^2 \mathcal{D}_i^\mu \mathcal{D}_j^\nu d - f_{ij1}^{\mu\nu} + \frac{2m^2}{3} C_1^{\mu\nu} f_1' \quad (126)$$

and another two sets of relations which can be obtained by circular permutations.

Here

$$\begin{aligned} f_{221}^{\mu\nu} &= f_{331}^{\mu\nu} = A_1^{\mu\nu} f_1, & f_{231}^{\mu\nu} &= -B_1^{\mu\nu} f_1, \\ f_{131}^{\mu\nu} &= (\partial_1^\nu \partial_2^\mu + A_1^{\mu\nu}) f_1, & f_{121}^{\mu\nu} &= -(\partial_1^\nu \partial_2^\mu + B_1^{\mu\nu}) f_1, \\ f_{111}^{\mu\nu} &= (\partial_1^\nu \partial_2^\mu + \partial_1^\mu \partial_2^\nu + 2\partial_2^\mu \partial_2^\nu + A_1^{\mu\nu}) f_1 \end{aligned} \quad (127)$$

and the rest by circular permutations. Here we have defined

$$\begin{aligned} A_j^{\mu\nu} &\equiv \frac{2}{3} \left(\partial_j^\mu \partial_j^\mu - \frac{1}{4} \eta^{\mu\nu} \square_j \right) \\ B_j^{\mu\nu} &\equiv \frac{1}{3} \left(\partial_j^\mu \partial_j^\mu + \frac{1}{2} \eta^{\mu\nu} \square_j \right) \\ C_j^{\mu\nu} &\equiv (\partial_j^\mu \partial_j^\mu - \eta^{\mu\nu} \square_j). \end{aligned} \quad (128)$$

After the central causal splitting we obtain

$$\frac{\partial}{\partial x^\rho} (\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1^\rho d)^F = (\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_3 d)^F - (\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_2 d)^F \quad (129)$$

$$\begin{aligned} \frac{\partial}{\partial y^\rho} (\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1^\rho d)^F &= -(\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_3 d)^F - m^2 (\mathcal{D}_i^\mu \mathcal{D}_j^\nu d)^F + f_{ij1}^{\mu\nu,F} \\ &\quad - \frac{2m^2}{3} C_1^{\mu\nu} f_1^{\prime,F} + A_{ij1}^{\mu\nu}, \end{aligned} \quad (130)$$

$$\begin{aligned} \frac{\partial}{\partial z^\rho} (\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1^\rho d)^F &= (\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_1 \cdot \mathcal{D}_2 d)^F + m^2 (\mathcal{D}_i^\mu \mathcal{D}_j^\nu d)^F - f_{ij1}^{\mu\nu,F} \\ &\quad + \frac{2m^2}{3} C_1^{\mu\nu} f_1^{\prime,F} - A_{ij1}^{\mu\nu} \end{aligned} \quad (131)$$

and another two sets of relations which can be obtained by circular permutations. Here

$$A_{ijk}^{\mu\nu} \equiv C \left(a_{jk}^{\mu\nu} + \frac{2}{3} C_i^{\mu\nu} \right) \delta \quad (132)$$

Here we have defined the differential operators

$$\begin{aligned} a_{11}^{\mu\nu} &\equiv \partial_2^\mu \partial_2^\nu + \partial_3^\mu \partial_3^\nu - \frac{1}{2} (\partial_2^\mu \partial_3^\nu + \partial_3^\mu \partial_2^\nu) - \frac{1}{2} \eta^{\mu\nu} (\square_2 + \square_3 + \partial_2 \cdot \partial_3) \\ a_{12}^{\mu\nu} &\equiv -\partial_1^\mu \partial_1^\nu - \partial_2^\mu \partial_2^\nu - \frac{1}{2} (\partial_1^\mu \partial_2^\nu + \partial_2^\mu \partial_1^\nu) - \frac{1}{2} \eta^{\mu\nu} (\square_1 + \square_2 + \partial_1 \cdot \partial_2) \end{aligned} \quad (133)$$

and $a_{22}^{\mu\nu}, a_{33}^{\mu\nu}, a_{23}^{\mu\nu}, a_{31}^{\mu\nu}$ by circular permutations and $C = \frac{1}{6} A$.

Proof: We consider the relation (125). The anomaly is, in momentum space:

$$\tilde{\mathcal{A}}_{ij1}^{\mu\nu} \equiv -i q_\rho \tilde{a}_{ij1}^{\mu\nu\rho} + \tilde{a}_{ij13}^{\mu\nu} + m^2 \tilde{a}_{ij}^{\mu\nu} - (\tilde{f}_{ij1}^{\mu\nu})^{\text{adv}} - \frac{2m^2}{3} \tilde{C}_1^{\mu\nu} (\tilde{f}'_1)^{\text{adv}} \quad (134)$$

where $\tilde{C}_1^{\mu\nu}$ is obtained from $C_1^{\mu\nu}$ making $\partial_1 \rightarrow p, \partial_2 \rightarrow q$. By the same mechanism as before we have:

$$\tilde{\mathcal{A}}_{ij1}^{\mu\nu}(p, q) = -\frac{im^2}{2\pi} \int \frac{dt}{t^3} (1+t) \tilde{a}_{ij}^{\mu\nu}(tp, tq) + \frac{2im^2}{6\pi} \tilde{C}_1^{\mu\nu} \int \frac{dt}{t} \tilde{f}'_1(tp, tq). \quad (135)$$

If we use (98) we obtain:

$$\begin{aligned} \tilde{\mathcal{A}}_{ij1}^{\mu\nu}(p, q) &= \frac{im^2}{2\pi} p^\mu p^\nu \int \frac{dt}{t} \tilde{A}_{ij}(tp, tq) + \frac{im^2}{2\pi} q^\mu q^\nu \int \frac{dt}{t} \tilde{B}_{ij}(tp, tq) \\ &+ \frac{im^2}{2\pi} p^\mu q^\nu \int \frac{dt}{t} \tilde{C}_{ij}^{(1)}(tp, tq) + \frac{im^2}{2\pi} q^\mu p^\nu \int \frac{dt}{t} \tilde{C}_{ij}^{(2)}(tp, tq) \\ &- \eta^{\mu\nu} \frac{im^2}{2\pi} \int \frac{dt}{t^3} \tilde{D}_{ij}(tp, tq) + \frac{im^2}{3\pi} (p^\mu p^\nu - \eta^{\mu\nu} p^2) \int \frac{dt}{t} \tilde{f}'_1(tp, tq). \end{aligned} \quad (136)$$

If we substitute the formulas for the functions $\tilde{A}_{jk}(p, q)$, etc. obtained previously then we need a few more integrals; the first is:

$$a' \equiv \int \frac{dt}{t^3} \tilde{d}_{m,m,m}(tp, tq). \quad (137)$$

Proceeding as in [21] we obtain

$$a' = \frac{b}{m^2} (p^2 + q^2 + p \cdot q). \quad (138)$$

Finally we need

$$\int \frac{dt}{t} \tilde{f}'_j(tp, tq) = b. \quad (139)$$

Using all these formulas we obtain the result from the statement. ■

Now we have relations similar to those from the previous theorems for the one-particle reducible distributions of the type (88).

Theorem 5.4 *The following relations are true*

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_2^\rho d^{(3)} = -m^2 d^{(3)} - f_2, \quad (140)$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_1^\mu \mathcal{D}_2^\rho d^{(3)} = -m^2 \mathcal{D}_1^\mu d^{(3)} + \partial_2^\mu f_2, \quad (141)$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_2^\mu \mathcal{D}_2^\rho d^{(3)} = -m^2 \mathcal{D}_2^\mu d^{(3)} - \partial_1^\mu f_2, \quad (142)$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_1^\mu \mathcal{D}_1^\nu \mathcal{D}_2^\rho d^{(3)} = -m^2 \mathcal{D}_1^\mu \mathcal{D}_1^\nu d^{(3)} + \partial_2^\mu \partial_2^\nu f_2, \quad (143)$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_2^\mu \mathcal{D}_2^\nu \mathcal{D}_2^\rho d^{(3)} = -m^2 \mathcal{D}_2^\mu \mathcal{D}_2^\nu d^{(3)} + \partial_1^\mu \partial_1^\nu f_3, \quad (144)$$

$$\frac{\partial}{\partial y^\rho} \mathcal{D}_1^\mu \mathcal{D}_2^\nu \mathcal{D}_2^\rho d^{(3)} = -m^2 \mathcal{D}_1^\mu \mathcal{D}_2^\nu d^{(3)} - \partial_2^\mu \partial_1^\nu f_3 \quad (145)$$

and similar relations for the other five distributions of this type. These relations can be causality split without anomalies.

Proof: We can proceed as in the proceeding theorems but there is a simple way, already noticed before: see (73) and (74). ■

Remark 5.5 *Based on previous experience, for instance (54) versus (55) or theorem 5.1, etc. we might be inclined to think that the origin of the anomalies is the presence of mass factors multiplying distributions of lower order of singularity as the rest of the equations. However, the preceding theorem is a counter-example to this idea. This point shows how difficult is to decide a priori which differential equations involving causal distributions will produce anomalies.*

6. ANOMALIES IN THE THIRD ORDER OF THE PERTURBATION THEORY

6.1. TREE ANOMALIES

We have mentioned in the first subsection of the previous section that we have third order anomalies of tree type. These anomalies can be obtained as the tree anomalies from the second order of the perturbation theory.

Theorem 6.1 *Let us consider the causal commutators $D_{(0)}^N(T^I(x), T^J(y), T^K(z))$ and perform the causal splitting, i.e. we obtain the chronological products $T_{(0)}^N(T^I(x), T^J(y), T^K(z))$ by making $D(x) \rightarrow D^F(x)$ as in the second order of the perturbation theory. Then we have the anomalies*

$$\mathcal{A}^{IJK}(x, y, z) \equiv sT_{(0)}^N(T^I(x), T^J(y), T^K(z)). \quad (146)$$

Only in the case $I = J = K = \emptyset$ the anomaly is non-trivial, namely

$$\mathcal{A}^{\emptyset\emptyset\emptyset}(x, y, z) = \delta(x - z) \delta(y - z) W(z) + \dots, \quad (147)$$

where

$$W = -\frac{1}{2} g_{abcp} f'_{dpe} \Phi_a \Phi_b \Phi_c \Phi_d u_e \quad (148)$$

and \dots are anomalies of lower canonical dimension. So, we do not have anomalies of canonical dimension 5 iff

$$\mathcal{S}_{abcd} (g_{abcp} f'_{dpe}) = 0. \quad (149)$$

Proof: Let us consider the case $I = [\mu], J = [\nu], K = \emptyset$ when we have

$$\begin{aligned} sD_{(0)}^N(T^\mu(x), T^\nu(y), T(z)) &= d_Q D_{(0)}^N(T^\mu(x), T^\nu(y), T(z)) \\ &\quad - i [\partial_\rho^1 D_{(0)}^N(T^{\mu\rho}(x), T^\nu(y), T(z)) - (x \leftrightarrow y, \mu \leftrightarrow \nu)] \\ &\quad - i \partial_\mu^3 D_{(0)}^N(T^\mu(x), T^\nu(y), T^\rho(z)) \end{aligned} \quad (150)$$

and the anomalies are produced by the terms with derivatives from the right hand side. From the general expression (39) we have

$$D_{(0)}^N(T^{\mu\rho}(x), T^\nu(y), T(z)) = \delta(x-y)[N^{[\mu\rho][\nu]}(y), T(z)] \\ - \delta(x-z)[N^{[\mu\rho]\emptyset}(z), T^\nu(y)] - \delta(y-z)[N^{[\nu]\emptyset}(z), T^{\mu\rho}(x)] \quad (151)$$

and we need the contributions with the factor $\partial_1^\rho D$ from this expression. Only the last term gives such a contribution and in the end we find out:

$$D_{(0)}^N(T^{\mu\rho}(x), T^\nu(y), T(z)) = \delta(y-z) \partial^\rho D(x-z) W_1^{\mu\nu}(x, z) + \dots \quad (152)$$

where the Wick polynomial $W_1^{\mu\nu}(x, z)$ can be written explicitly and \dots are the terms without the derivative ∂_1^ρ . Similarly

$$D_{(0)}^N(T^\mu(x), T^\nu(y), T^\rho(z)) = \delta(x-y)[N^{[\mu][\nu]}(y), T^\rho(z)] \\ + \delta(x-z)[N^{[\mu][\rho]}(z), T^\nu(y)] - \delta(y-z)[N^{[\nu][\rho]}(z), T^\mu(x)] \quad (153)$$

and we need the terms with the factor $\partial_3^\rho D$. Only the first term can produce such a combination. In the end we get

$$D_{(0)}^N(T^\mu(x), T^\nu(y), T^\rho(z)) = \delta(x-y) \partial^\rho D(x-z) W_2^{\mu\nu}(y, z) + \dots \quad (154)$$

where the Wick polynomial $W_2^{\mu\nu}(x, z)$ can be written explicitly. The relation

$$sD_{(0)}^N(T^\mu(x), T^\nu(y), T(z)) = 0 \quad (155)$$

is true because we use the Klein-Gordon equation, like in the second order of perturbation theory. If we make the causal decomposition, then we get the anomaly

$$\mathcal{A}^{[\mu][\nu]\emptyset}(x, y, z) \equiv sT_{(0)}^N(T^\mu(x), T^\nu(y), T(z)) = -i \delta(x-z) \delta(y-z) W^{\mu\nu}(z) \quad (156)$$

where

$$W^{\mu\nu}(z) \equiv W_1^{\mu\nu}(z, z) - W_1^{\nu\mu}(z, z) - W_2^{\mu\nu}(z, z) \quad (157)$$

corresponding to the three terms from the right hand side of (150). An explicit computation gives $W^{\mu\nu} = 0$ so the anomaly is null. The other cases are considered similarly. ■

As in the second order of the perturbation theory, we can derive the preceding result using the off-shell method [10]. In the particular case of the standard model, from the relation (149) one can obtain the usual form of the Higgs coupling [22].

6.2. LOOP ANOMALIES

We need some definitions. In the Yang-Mills sector we need

$$f_{[abc]}^{(0)} = f_{eap} f_{ebq} f_{cpq} \quad (158)$$

and

$$A_{abc} \equiv \sum_{\epsilon} \epsilon \operatorname{Tr}(\{t_a^{\epsilon}, t_b^{\epsilon}\} t_c^{\epsilon}). \quad (159)$$

In the scalar sector we will need:

$$\begin{aligned} f_{abc}^{(1)} &= f'_{pae} f'_{qbe} f'_{pqc}, \\ f_{abc}^{(2)} &= f'_{epa} f'_{ebq} f'_{cpq} = \frac{1}{2} f'_{abd} g_{cd} \quad f_{[abc]}^{(3)} = f'_{epa} f'_{eqb} f'_{pqc} \end{aligned} \quad (160)$$

and in the Dirac sector

$$f_{[abc]}^{(4)} = i \operatorname{Tr}([t_a^{\epsilon}, t_b^{\epsilon}] t_c^{\epsilon}) = f_{abd} g_{cd}^{(2)} \quad (161)$$

and

$$t_{a\epsilon}^{(2)} = \sum_b t_b^{\epsilon} t_a^{\epsilon} t_b^{\epsilon}. \quad (162)$$

It is also useful to denote

$$F_{abc} \equiv -\frac{4C}{3} (7f_{abc}^{(0)} + 2f_{abc}^{(3)} + 4f_{abc}^{(4)}). \quad (163)$$

We have the following result.

Theorem 6.2 *Let us perform the central causal splitting for all distributions appearing in the third order causal products. Then we obtain the following anomalies:*

$$sT(T^I(x), T^J(y), T^K(z)) = \mathcal{A}^{IJK}(x, y, z) \quad (164)$$

where:

(a) *In the Yang-Mills sector we have*

- *the even part:*

$$\begin{aligned} \mathcal{A}_{\text{even}}^{[\mu][\nu]\emptyset}(x, y, z) &= [\partial_1^{\mu} \partial_1^{\nu} - \partial_2^{\mu} \partial_2^{\nu} - \eta^{\mu\nu} (\square_1 - \square_2)] \\ &\quad \delta(x, y, z) F_{abc} u_a(x) u_b(y) u_c(z) \end{aligned} \quad (165)$$

$$\mathcal{A}_{\text{even}}^{\emptyset\emptyset[\mu\nu]} = 0 \quad (166)$$

$$\begin{aligned} \mathcal{A}_{\text{even}}^{\emptyset\emptyset[\mu]}(x, y, z) &= \left\{ [\partial_2^{\mu} \partial_2^{\nu} + \partial_1^{\mu} \partial_2^{\nu} + \partial_1^{\nu} \partial_2^{\mu} - \eta^{\mu\nu} (\square_2 + 2\partial_1 \cdot \partial_2)] \delta(x, y, z) \right. \\ &\quad \left. F_{abc} v_{a\nu}(x) u_b(y) u_c(z) \right. \\ &\quad \left. + B(\partial_1 + 2\partial_2)_{\nu} \delta(x, y, z) f_{abc}^{(0)} F_a^{\mu\nu}(x) u_b(y) u_c(z) \right\} + (x \leftrightarrow y) \end{aligned} \quad (167)$$

and

$$\mathcal{A}_{\text{even}}^{\emptyset\emptyset\emptyset}(x, y, z) = \mathcal{A}_{(3, YM)}^{\emptyset\emptyset\emptyset}(x, y, z) + (x \leftrightarrow z) + (y \leftrightarrow z), \quad (168)$$

where

$$\begin{aligned} \mathcal{A}_{(3,YM)}^{\theta\theta\theta}(x,y,z) &\equiv [\partial_1^\mu \partial_1^\nu - \partial_2^\mu \partial_2^\nu - \eta^{\mu\nu}(\square_1 - \square_2)] \\ &\quad \delta(x,y,z) F_{abc} v_{a\nu}(x) v_{b\nu}(y) u_c(z) \\ &\quad + B[(\partial_1^\rho + 2\partial_2^\rho)\delta(x,y,z) f_{abc}^{(0)} F_{a\rho\sigma}(x) v_b^\sigma(y) u_c(z) + (x \leftrightarrow y)] \end{aligned} \quad (169)$$

- in the odd part:

$$\begin{aligned} \mathcal{A}_{\text{odd}}^{[\mu][\nu]\theta}(x,y,z) &= -8iC \varepsilon^{\mu\nu\rho\sigma} \partial_{1\rho} \partial_{2\sigma} \delta(x,y,z) A_{abc} u_a(x) u_b(y) u_c(z) \\ \mathcal{A}_{\text{odd}}^{\theta\theta[\mu]}(x,y,z) &= -8iC \varepsilon^{\mu\nu\rho\sigma} \partial_{1\rho} \partial_{2\sigma} \delta(x,y,z) A_{abc} u_a(x) v_{b\nu}(y) u_c(z) + (x \leftrightarrow y) \\ \mathcal{A}_{\text{odd}}^{\theta\theta\theta}(x,y,z) &= -8iC \varepsilon^{\mu\nu\rho\sigma} \partial_{1\rho} \partial_{2\sigma} \delta(x,y,z) A_{abc} v_{a\mu}(x) v_{b\nu}(y) u_c(z) \\ &\quad + (x \leftrightarrow z) + (y \leftrightarrow z) \end{aligned} \quad (170)$$

(b) In the scalar sector we have only an even part. The non-zero contributions appears only in

$$\begin{aligned} \mathcal{A}_{(3,\text{scalar})}^{\theta\theta\theta}(x,y,z) &= -f_{abc}^{(1)} \{ [2B(\partial_1^\mu + 2\partial_2^\mu)\delta(x,y,z) [\partial_\mu \Phi_a(x) \Phi_b(y) u_c(z) + (x \leftrightarrow y)] \\ &\quad - 2C(\square_1 - \square_2)\delta(x,y,z) \Phi_a(x) \Phi_b(y) u_c(z) \} \\ &\quad - f_{abc}^{(2)} [-B(2\partial_1^\mu + \partial_2^\mu)\delta(x,y,z) u_c(x) \partial_\mu \Phi_a(y) \Phi_b(z) \\ &\quad + 2(\square_1 - \square_2)\delta(x,y,z) u_c(x) \Phi_a(y) \Phi_b(z) + (x \leftrightarrow y)] \end{aligned} \quad (171)$$

(c) The Dirac sector has even and odd sectors grouped as follows:

$$\mathcal{A}_{(3,\text{Dirac})}^{\theta\theta\theta}(x,y,z) = 4B[(2\partial_1^\mu + \partial_2^\mu)\delta(x,y,z) u_a(z) \bar{\psi}(x) t_{a\epsilon}^{(2)} \otimes \gamma_\mu \gamma_\epsilon \psi(y) + (x \leftrightarrow y)] \quad (172)$$

Proof: By definition

$$\begin{aligned} sT(T^\mu(x), T^\nu(y), T(z)) &= d_Q T(T^\mu(x), T^\nu(y), T(z)) \\ &\quad - i [\partial_\rho^1 T(T^{\mu\rho}(x), T^\nu(y), T(z)) + (x \leftrightarrow y)] \\ &\quad - i \partial_\rho^3 T(T^\mu(x), T^\nu(y), T^\rho(z)) \end{aligned} \quad (173)$$

Let us investigate the anomalies produced by the derivative terms in the right hand side. For simplicity we consider only the Yang-Mills sector. We must use formula (85). We remind again the origin of the anomalies. To prove $sD(T^\mu(x), T^\nu(y), T(z)) = 0$ we must use the first three relations of theorem 5.3. However, after we perform the central causal splitting $D(A(x), B(y), C(z)) \rightarrow T(A(x), B(y), C(z))$ we obtain anomalies according to (129) - (131). The anomaly produced by the last

term of the relation (173) is

$$\begin{aligned} \mathcal{A}_{1,YM}^{[\mu][\nu]\emptyset}(x,y,z) = & [A_{112}^{\mu\nu} - A_{221}^{\mu\nu} - A_{311}^{\nu\mu} + A_{232}^{\nu\mu} - A_{112}^{\nu\mu} + A_{122}^{\nu\mu} \\ & - A_{311}^{\mu\nu} - A_{331}^{\mu\nu} + A_{232}^{\mu\nu} + A_{332}^{\mu\nu} + 2(A_{312}^{\nu\mu} - A_{231}^{\nu\mu}) \\ & + \eta^{\mu\nu} \eta_{\rho\sigma} (-A_{121}^{\rho\sigma} + A_{122}^{\rho\sigma})] f_{abc}^{(0)} u_a(x) u_b(y) u_c(z). \end{aligned} \quad (174)$$

In the same way we obtain anomalies from the terms $\partial_\rho^1 T(T^{\mu\rho}(x), T^\nu(y), T(z)) + (x \leftrightarrow y)$ so in the end, we obtain in the Yang-Mills sector the anomaly:

$$\begin{aligned} \mathcal{A}_{1,YM}^{[\mu][\nu]\emptyset}(x,y,z) = & C \left\{ \left[a_{11}^{\mu\nu} - a_{12}^{\mu\nu} + a_{12}^{\nu\mu} + a_{23}^{\nu\mu} - \frac{16}{3} C_1^{\mu\nu} - \eta^{\mu\nu} \eta_{\rho\sigma} \left(a_{23}^{\rho\sigma} + \frac{2}{3} C_1^{\rho\sigma} \right) \right] \right. \\ & \left. \delta(X) \delta(Y) - (x \leftrightarrow y, \mu \leftrightarrow \nu) \right\} f_{abc}^{(0)} u_a(x) u_b(y) u_c(z) \end{aligned} \quad (175)$$

The end result is

$$\begin{aligned} \mathcal{A}_{YM}^{[\mu][\nu]\emptyset}(x,y,z) = & -\frac{28C}{3} [\partial_1^\mu \partial_1^\nu - \partial_2^\mu \partial_2^\nu - \eta^{\mu\nu} (\square_1 - \square_2)] \delta(X) \delta(Y) \\ & f_{abc}^{(0)} u_a(x) u_b(y) u_c(z) \end{aligned} \quad (176)$$

The scalar and Dirac contributions can be computed in the same way and we get the first formula from the statement. The other two formulas are obtained similarly. ■

The preceding expressions are not unique because of the presence of the delta distribution $\delta(x,y,z) = \delta(x-z) \delta(y-z)$. We can re-express the anomalies in an unique form of the type $p(\partial_1, \partial_2) \delta W(z)$.

Theorem 6.3 *The anomalies $\mathcal{A}^{IJK}(x,y,z)$ can be uniquely written as follows:*

(a) *In the Yang-Mills sector we have*

- *the even part:*

$$\begin{aligned} \mathcal{A}_{\text{even}}^{[\mu][\nu]\emptyset}(x,y,z) = & F_{abc} [(\partial_1^\mu \partial_1^\nu - \eta^{\mu\nu} \square_1) \delta(x,y,z) (u_a u_b u_c)(z) \\ & - \partial_1^\mu \delta(x,y,z) (\partial^\nu u_a u_b u_c)(z) - \partial_1^\nu \delta(x,y,z) (\partial^\mu u_a u_b u_c)(z) \\ & + 2 \eta^{\mu\nu} \partial_\rho^1 \delta(x,y,z) (\partial^\rho u_a u_b u_c)(z)] \\ & - (x \leftrightarrow y, \mu \leftrightarrow \nu) \end{aligned} \quad (177)$$

$$\mathcal{A}_{\text{even}}^{\emptyset\emptyset[\mu\nu]} = 0 \quad (178)$$

$$\begin{aligned}
\mathcal{A}_{\text{even}}^{\emptyset\emptyset[\mu]}(x, y, z) = & F_{abc} \{ [\partial_1^\mu \partial_1^\nu + \partial_2^\mu \partial_2^\nu + 2 (\partial_1^\mu \partial_2^\nu + \partial_1^\nu \partial_2^\mu) \\
& - \eta^{\mu\nu} (\square_1 + \square_2 + 4\partial_1 \cdot \partial_2)] \delta(x, y, z) (v_{a\nu} u_b u_c)(z) \\
& - (\partial_1^\mu + \partial_2^\mu) \delta(x, y, z) (2v_a^\nu \partial_\nu u_b u_c + \partial_\nu v_a^\nu u_b u_c)(z) \\
& + (\partial_1^\nu + \partial_2^\nu) \delta(x, y, z) (2v_a^\mu \partial^\nu u_b u_c - \partial^\mu v_a^\nu u_b u_c + 2\partial^\nu v_a^\mu u_b u_c)(z) \\
& + 2 \delta(x, y, z) (\partial^\mu v_a^\nu \partial_\nu u_b u_c - 2\partial^\nu v_a^\mu \partial_\nu u_b u_c + \partial_\nu v_a^\nu \partial^\mu u_b u_c + v_a^\nu \partial^\mu \partial_\nu u_b u_c)(z) \} \\
& + B f_{abc}^{(0)} [3(\partial_\nu^1 + \partial_\nu^2) \delta(x, y, z) (F_a^{\mu\nu} u_b u_c)(z) \\
& - 2\delta(x, y, z) (\partial^\mu \partial_\nu v_a^\nu u_b u_c + 2F_a^{\mu\nu} \partial_\nu u_b u_c)(z)] \quad (179)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}^{\emptyset\emptyset\emptyset}(x, y, z) = & 2\delta(x, y, z) [F_{abc} (\partial^\mu \partial_\nu v_a^\nu v_b^\mu u_c - \partial_\nu v_a^\mu v_b^\nu \partial_\mu u_c + 2\partial^\mu v_a^\nu v_{b\nu} \partial_\mu u_c \\
& - \partial_\nu v_a^\nu v_b^\mu \partial_\mu u_c) + 3B f_{abc}^{(0)} (F_a^{\mu\nu} v_{b\mu} \partial_\nu u_c)](z). \quad (180)
\end{aligned}$$

- in the odd part:

$$\begin{aligned}
\mathcal{A}_{\text{odd}}^{[\mu][\nu]\emptyset}(x, y, z) = & -8iC A_{abc} \delta(x, y, z) \varepsilon^{\mu\nu\rho\sigma} (\partial_\rho u_a \partial_\sigma u_b u_c)(z) \\
\mathcal{A}_{\text{odd}}^{\emptyset\emptyset[\mu]}(x, y, z) = & 8iC A_{abc} \varepsilon^{\mu\nu\rho\sigma} [(\partial_\nu^1 + \partial_\nu^2) \delta(x, y, z) (\partial_\rho u_a v_{b\sigma} u_c)(z) \\
& + \delta(x, y, z) (\partial_\nu u_a F_{b\rho\sigma} u_c)(z)] \\
\mathcal{A}_{\text{odd}}^{\emptyset\emptyset\emptyset}(x, y, z) = & -8iC A_{abc} \varepsilon_{\mu\nu\rho\sigma} \delta(x, y, z) \quad (181) \\
& \left(-\frac{1}{4} F_a^{\mu\nu} F_b^{\rho\sigma} u_c + \partial^\mu u_a v_b^\nu F_c^{\rho\sigma} \right) (z)
\end{aligned}$$

(b) In the scalar sector

$$\mathcal{A}_{\text{scalar}}^{\emptyset\emptyset\emptyset}(x, y, z) = (-2C f_{abc}^{(1)} + 3B f_{abc}^{(2)}) \delta(x, y, z) (\partial^\mu \Phi_a \Phi_b \partial_\mu u_c)(z) \quad (182)$$

(c) In the Dirac sector:

$$\mathcal{A}_{\text{Dirac}}^{\emptyset\emptyset\emptyset}(x, y, z) = -24B \delta(x, y, z) (\partial^\mu u_a \bar{\psi}_{a\epsilon}^{(2)} \otimes \gamma_\mu \gamma_\epsilon \psi)(z). \quad (183)$$

The next task is to investigate if the preceding anomalies can be eliminated by a redefinition of the chronological products. This can be done *iff* the anomalies can be written as a coboundary *i.e.*

$$\mathcal{A}^{IJK} = (sB)^{IJK} = d_Q B^{IJK} - i(\delta B)^{IJK} \quad (184)$$

where the expressions B^{IJK} are quasi-local, Lorentz covariant, of canonical dimension 4 and with the same (graded) symmetry in $(x, I), (y, J), (z, K)$ as the chronological products - see (13). We will write them in the unique form $p(\partial_1, \partial_2) \delta(x, y, z) W(z)$ used in the previous theorem.

Theorem 6.4 *The generic form of the coboundaries:*

(a) In the Yang-Mills sector

- the even part with respect to parity:

$$B^{\emptyset\emptyset[\mu\nu\rho]} = 0 \quad (185)$$

$$\begin{aligned} B^{[\mu][\nu][\rho]}(x, y, z) = & \\ k_{abc} \{ & [\eta^{\mu\nu} (\partial_1^\rho - \partial_2^\rho) - \eta^{\mu\rho} (2\partial_1^\nu + \partial_2^\nu) + \eta^{\nu\rho} (\partial_1^\mu + 2\partial_2^\mu)] \delta(x, y, z) (u_a u_b u_c)(z) \\ & + 3 \delta(x, y, z) (\eta^{\mu\rho} \partial^\nu u_a u_b u_c - \eta^{\nu\rho} \partial^\mu u_a u_b u_c)(z) \} \quad (186) \end{aligned}$$

$$\begin{aligned} B^{[\mu\nu][\rho]\emptyset}(x, y, z) = & \\ \eta^{\mu\rho} [& p_{abc}^1 \partial_1^\nu \delta(x, y, z) (u_a u_b u_c)(z) + p_{abc}^2 \partial_2^\nu \delta(x, y, z) (u_a u_b u_c)(z) \\ & + p_{abc}^3 \delta(x, y, z) (\partial^\nu u_a u_b u_c)(z)] - (\mu \leftrightarrow \nu) \quad (187) \end{aligned}$$

$$\begin{aligned} B^{[\mu][\nu]\emptyset}(x, y, z) = & q_{abc}^1 [\partial_1^\mu \delta(x, y, z) (v_a^\nu u_b u_c)(z) - \partial_2^\nu \delta(x, y, z) (v_a^\mu u_b u_c)(z)] \\ & + q_{abc}^2 [\partial_1^\nu \delta(x, y, z) (v_a^\mu u_b u_c)(z) - \partial_2^\mu \delta(x, y, z) (v_a^\nu u_b u_c)(z)] \\ & + q_{abc}^3 \eta^{\mu\nu} (\partial_1^\rho - \partial_2^\rho) \delta(x, y, z) (v_a^\rho u_b u_c)(z) \\ & + q_{abc}^4 \delta(x, y, z) (v_a^\mu \partial^\nu u_b u_c - v_a^\nu \partial^\mu u_b u_c)(z) \\ & + q_{abc}^5 \delta(x, y, z) (F_a^{\mu\nu} \partial^\nu u_b u_c)(z) \quad (188) \end{aligned}$$

$$\begin{aligned} B^{\emptyset\emptyset[\mu\nu]}(x, y, z) = & r_{abc}^1 [(\partial_1^\mu + \partial_2^\mu) \delta(x, y, z) (v_a^\nu u_b u_c)(z) \\ & - (\partial_1^\nu + \partial_2^\nu) \delta(x, y, z) (v_a^\mu u_b u_c)(z)] \\ & + r_{abc}^2 \delta(x, y, z) (v_a^\mu \partial^\nu u_b u_c - v_a^\nu \partial^\mu u_b u_c)(z) \\ & + r_{abc}^3 \delta(x, y, z) (F_a^{\mu\nu} \partial^\nu u_b u_c)(z) \quad (189) \end{aligned}$$

$$\begin{aligned} B^{\emptyset\emptyset[\mu]}(x, y, z) = & s_{abc}^1 (\partial_\nu^1 + \partial_\nu^2) \delta(x, y, z) (v_a^\mu v_b^\nu u_c)(z) \\ & + s_{abc}^2 (\partial_1^\mu + \partial_2^\mu) \delta(x, y, z) (u_a u_b \tilde{u}_c)(z) \\ & + s_{abc}^3 \delta(x, y, z) (\partial^\mu v_a^\nu v_{b\nu} u_c)(z) + s_{abc}^4 \delta(x, y, z) (\partial_\nu v_a^\mu v_b^\nu u_c)(z) \\ & + s_{abc}^5 \delta(x, y, z) (\partial_\nu v_a^\nu v_b^\mu u_c)(z) + s_{abc}^6 \delta(x, y, z) (v_a^\mu v_b^\nu \partial_\nu u_c)(z) \\ & + s_{abc}^7 \delta(x, y, z) (\partial^\mu u_a u_b \tilde{u}_c)(z) + s_{abc}^8 \delta(x, y, z) (u_a u_b \partial^\mu \tilde{u}_c)(z) \\ & + s_{abc}^9 (\partial_1^\mu + \partial_2^\mu) \delta(x, y, z) (v_a^\nu v_{b\nu} u_c)(z) + s_{abc}^{10} \delta(x, y, z) (v_a^\nu v_{b\nu} \partial^\mu u_c)(z) \quad (190) \end{aligned}$$

$$\begin{aligned} B^{\emptyset\emptyset\emptyset}(x, y, z) = & \delta(x, y, z) (t_{abc}^1 \partial^\mu v_a^\nu v_{b\mu} v_{c\nu} + t_{abc}^2 \partial_\nu v_a^\nu u_b \tilde{u}_c \\ & + t_{abc}^3 v_a^\mu \partial_\mu u_b \tilde{u}_c + t_{abc}^4 v_a^\mu u_b \partial_\mu \tilde{u}_c + t_{abc}^5 \partial_\mu v_a^\mu v_b^\nu v_{c\nu})(z) \quad (191) \end{aligned}$$

- the odd part with respect to parity

$$B^{\emptyset\emptyset[\mu\nu\rho]} = \varepsilon^{\mu\nu\rho\sigma} [d_{abc}^1 (\partial_\sigma^1 + \partial_\sigma^2) \delta(x, y, z) (u_a u_b u_c)(z) + d_{abc}^2 \delta(x, y, z) (\partial_\sigma u_a u_b u_c)(z)] \quad (192)$$

$$B^{[\mu][\nu][\rho]}(x, y, z) = e_{abc} \varepsilon^{\mu\nu\rho\sigma} \delta(x, y, z) (\partial_\sigma u_a u_b u_c)(z) \quad (193)$$

$$B^{[\mu\nu][\rho]\emptyset}(x, y, z) = \varepsilon^{\mu\nu\rho\sigma} [(f_{abc}^1 \partial_\sigma^1 + f_{abc}^2 \partial_\sigma^2) \delta(x, y, z) (u_a u_b u_c)(z) + f_{abc}^3 \delta(x, y, z) (\partial_\sigma u_a u_b u_c)(z)] \quad (194)$$

$$B^{[\mu][\nu]\emptyset}(x, y, z) = \varepsilon^{\mu\nu\rho\sigma} [g_{abc}^1 (\partial_\rho^1 + \partial_\rho^2) \delta(x, y, z) (v_{a\sigma} u_b u_c)(z) + \delta(x, y, z) (g_{abc}^2 F_{a\rho\sigma} u_b u_c + g_{abc}^3 v_{a\rho} \partial_\sigma u_b u_c)(z)] \quad (195)$$

$$B^{\emptyset\emptyset[\mu\nu]}(x, y, z) = \varepsilon^{\mu\nu\rho\sigma} [h_{abc}^1 (\partial_\rho^1 + \partial_\rho^2) \delta(x, y, z) (v_{a\sigma} u_b u_c)(z) + \delta(x, y, z) (h_{abc}^2 F_{a\rho\sigma} u_b u_c + h_{abc}^3 v_{a\rho} \partial_\sigma u_b u_c)(z)] \quad (196)$$

$$B^{\emptyset\emptyset[\mu]}(x, y, z) = \varepsilon^{\mu\nu\rho\sigma} [j_{abc}^1 (\partial_\nu^1 + \partial_\nu^2) \delta(x, y, z) (v_{a\rho} v_{b\sigma} u_c)(z) + \delta(x, y, z) (j_{abc}^2 v_{a\nu} F_{b\rho\sigma} u_c + j_{abc}^3 v_{a\rho} v_{b\sigma} \partial_\nu u_c)(z)] \quad (197)$$

$$B^{\emptyset\emptyset\emptyset}(x, y, z) = \varepsilon^{\mu\nu\rho\sigma} \delta(x, y, z) l_{abc} F_{a\mu\nu} v_{b\rho} v_{c\sigma} \quad (198)$$

(b) In the scalar sector we have only an even part:

$$B^{IJK} = 0, \quad |I| + |J| + |K| = 0, 2, 3 \quad (199)$$

$$B^{\emptyset\emptyset[\mu]}(x, y, z) = w_{abc}^1 (\partial_1^\mu + \partial_2^\mu) \delta(x, y, z) (\Phi_a \Phi_b u_c)(z) + w_{abc}^2 \delta(x, y, z) (\partial^\mu \Phi_a \Phi_b u_c)(z) + w_{abc}^3 \delta(x, y, z) (\Phi_a \Phi_b \partial^\mu u_c)(z) \quad (200)$$

(c) In the Dirac sector:

$$B^{IJK} = 0, \quad |I| + |J| + |K| = 2, 3 \quad (201)$$

$$B^{\emptyset\emptyset[\mu]}(x, y, z) = \delta(x, y, z) V^\mu(z) + B^{\emptyset\emptyset\emptyset}(x, y, z) \delta(x, y, z) V(z) \quad (202)$$

where

$$V^\mu = u_a \bar{\Psi} V_{a\epsilon} \otimes \gamma^\mu \gamma_\epsilon \Psi \\ V = u_a \bar{\Psi} V'_{a\epsilon} \otimes \gamma_\epsilon \Psi \quad (203)$$

In the preceding expressions we can suppose convenient (anti)symmetry properties of the coefficients.

Now we impose (184). In the even sector the anomaly has only the coefficients $F_{abc}, f_{abc}^{(0)}, f_{abc}^{(1)}$ and $f_{abc}^{(2)}$, which are completely antisymmetric in a, b, c so if we want to prove that we have a solution of the equation (184) in this sector, it is sufficient to suppose that all coefficients $k_{abc}, p_{abc}^j, q_{abc}^j, r_{abc}^j, s_{abc}^j, t_{abc}^j$ and w_{abc}^j are completely antisymmetric in a, b, c . In particular, it means that we can take $s_{abc}^j = 0, j = 9, 10, t_{abc}^5 = 0, w_{abc}^j = 0, j = 1, 3$.

One can prove easily that cohomological relation (184) has a solution. A interesting problem is if we really need to renormalize the expression $T(T(x), T(y), T(z))$ *i.e.* if we can take

$$B^{\emptyset\emptyset\emptyset} = 0 \quad (204)$$

or not. It can be proved that the preceding equality is equivalent to

$$F_{abc} = B f_{abc}^{(0)}. \quad (205)$$

In the odd sector, because the anomaly involves only the coefficient A_{abc} which is completely symmetric in a, b, c we can consider that all coefficients $d_{abc}^\alpha, e_{abc}, f_{abc}^\alpha, g_{abc}^\alpha, h_{abc}^\alpha, j_{abc}^\alpha, l_{abc}$ are completely symmetric in a, b, c .

In particular, it means that we can take $d_{abc}^\alpha, e_{abc}, f_{abc}^\alpha, g_{abc}^\alpha, \alpha = 1, 2, h_{abc}^\alpha, \alpha = 1, 2, j_{abc}^\alpha, \alpha = 1, 3$ and l_{abc} to be zero, *i.e.* only the coefficients g_{abc}^3, h_{abc}^3 and j_{abc}^2 survive.

It is sufficient to consider only the case $I = J = K = \emptyset$ of (184). We obtain:

$$\begin{aligned} j_2 &= 2CA \\ -2j_2 &= -8CA \end{aligned} \quad (206)$$

which are leading to the equality

$$A_{abc} = 0. \quad (207)$$

This is exactly the standard form (see for instance [16] formula (11.58)) for the cancellation of the axial anomaly.

Finally, in the Dirac sector we have the solution of (184)

$$V_{a\epsilon} = -24Bt_{a\epsilon}^{(2)}. \quad (208)$$

7. CONCLUSIONS

In the functional formalism one considers anomalies of the current conservation

$$\partial_\mu j_{\text{Axial}}^\mu - j_{\text{Axial}}^5 \quad (209)$$

or of the BRST invariance of the generating functional of the Green distributions

$$s_{BRST}\Gamma \quad (210)$$

(where s_{BRST} is the non-linear BRST operator from the functional formalism). In this formalism the anomaly has terms cubic and quartic in the fields - see for instance [2] formula (13). The cubic term $\varepsilon^{\mu\nu\rho\sigma} \partial_{1\rho} \partial_{2\sigma} \delta(X) \delta(Y) A_{abc} v_{a\mu}(x) v_{b\nu}(y) u_c(z)$ obtained above coincides with the first contribution of this formula, up to partial integration. To obtain the quartic terms one would have to go to the fourth order of the perturbation theory.

We have investigated the anomalies of the standard model of maximal canonical dimension $\omega = 5$ in the third order of the perturbation theory for tree and one-loop contributions. Anomalies of lower canonical dimension must be investigated separately using Wess-Zumino consistency relations (33). The analysis goes as follows. The dominant contribution to the anomaly considered in this paper was obtained, essentially, by replacing everywhere the Pauli-Jordan distributions $D_{m_j}(x)$ of various masses by $D_M(x)$ where M is some fixed positive mass. This substitution implies a corresponding splitting of the chronological products. The dominant contribution to the chronological products gives the dominant contribution to the anomaly and we have showed how this anomaly can be eliminated. It follows that we still have potential anomalies of canonical dimension with 2 units lower *i.e.* of maximal canonical dimension 3. So, *a priori*, we still might have anomalies of the type

$$\delta(x-z)\delta(y-z) w(z) \quad (211)$$

with w a Wick polynomial of canonical dimension 3. But in [9] we have proved that such anomalies are null due to the Wess-Zumino consistency relations (33). We still have to investigate the anomalies associated to two-loops graphs. The analysis is also cohomological [9]. The anomalies must be of the form

$$\mathcal{A}^{IJK}(x, y, z) = p(\partial)\delta(x, y, z) w(z) \quad (212)$$

where w is linear in the fields. Because of the condition $gh(\mathcal{A}^{IJK}) = |I| + |J| + |K| + 1$ only the case $I = J = K = \emptyset$ can produce anomalies. If the polynomial p is non-trivial one can easily exhibit the anomaly in the form of a coboundary. So we are left with

$$\mathcal{A}^{\emptyset\emptyset\emptyset}(x, y, z) = \delta(x, y, z) w(z) \quad (213)$$

with $gh(w) = 1$ *i.e.* $w = \sum_a f_a u_a$. We can write the contributions corresponding to $a \in I_2$ as $d_Q b$ so we are left with the case $a \in I_1$. For a corresponding to gluons we must use the fact that there is no vector f_a invariant with respect to $SU(3)$ and for a corresponding to the photon we can use charge invariance. So there are no anomalies for two-loops graphs in the third order.

The generalization of the preceding analysis to multi-loop contributions is not obvious and it is a subject of further investigation.

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