

ON GENERAL HIGH-ORDER SOLITONS AND BREATHERS TO A NONLOCAL SCHRÖDINGER-BOUSSINESQ EQUATION WITH A PERIODIC LINE WAVES BACKGROUND

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Abstract. General high-order soliton and breather solutions to a non-local Schrödinger-Boussinesq (NSB) equation with a periodic line waves (PLWs) background are studied *via* the bilinear KP-reduction method. We construct tau functions to the NSB equation by restricting tau functions of bilinear equations in the KP hierarchy, which generate arbitrary $2N$ -soliton and N -breather solutions on a PLWs background in the NSB equation. Based on the asymptotic analysis, the two-soliton solutions are classified into non-degenerate and degenerate two-soliton solutions. The non-degenerate two-soliton has three patterns: two-dark-antidark soliton, two-dark-dark soliton, and two-antidark-antidark soliton. The degenerate two-soliton solution possesses two distinct patterns: the degenerate-antidark soliton and the degenerate-dark soliton. The four-soliton solution on the PLWs background exhibits the superpositions of two two-soliton solutions, and admits three distinct patterns: the non-degenerate four-soliton solution, the two degenerate two-soliton solutions, and the mixture of a degenerate two-soliton solution and a non-degenerate two-soliton solution. The typical dynamical scenarios of one- and two-breather solutions on a PLWs background are analyzed in detail.

Key words: Nonlocal Schrödinger-Boussinesq equation, Solitons, Breathers,
KP hierarchy reduction method.

1. INTRODUCTION

Nonlinear evolution equations are used to describe the various nonlinear phenomena in plasma physics, optical fibers, lasers, and so on. A lot of studies of these integrable or non-integrable equations have been reported in recent years [1–4]. The nonlinearity and dispersion effects, the interaction between various solitons, the bi-Hamiltonian structure, and other physical and mathematical concepts have attracted a lot of attention. The analytic solutions, including line soliton, positon [5], lump [6], rogue wave [7–14], and semi-rational solution [15, 16] of many nonlinear evolution equations have been paid much attention.

Since Bender and Boettcher [17, 18] showed that a wide class of non-Hermitian Hamiltonians possesses real eigen-spectra when the system obeys the parity-time symmetry (PT), this concept has become the focus of many fields. If we have $V(x) = V_R(x) + iV_I(x)$, it is said that a non-Hermitian Hamiltonian $H = \partial_{xx} + V(x)$ is called PT -symmetric when $V(x)$ satisfies $V(x) = V^*(-x)$. The Schrödinger equation in quantum mechanics and the paraxial equation of diffraction in optics provide the platform for studying the PT -symmetric physical systems. The study of PT -symmetric optics can lead to novel optical components and diverse applications in optics and photonics. In recent years, many works on PT -symmetric physical systems have published [19–23]. PT symmetry has been widely applied to many fields [24–28], such as hydrodynamical systems, fiber optics, Bose-Einstein condensates, quantum chromodynamics, and so on. Due to their physical applications, various types of soliton equations with PT symmetry have been proposed and investigated [20–23, 29–47].

Very recently, the studies of several types of soliton solutions on the periodic line waves (PLWs) background have got a lot of attention [48–52]. These PLWs backgrounds are described by different type of functions, such as Jacobi elliptic functions, quasi-periodic functions, and Riemann theta functions. Kedziora *et al.* [48], Chen and Pelinovsky [49], and Feng *et al.* [50] have discussed the rogue waves and solitons on the Jacobian elliptic functions background in the nonlinear Schrödinger (NLS) equation. Tian and his group [51] have studied the solitons on the Riemann theta function background. Liu *et al.* [52] have investigated rogue waves on a PLWs background in the Kaup-Newll equation. Rao *et al.* [53–55] have investigated various kinds of solitons including lumps, rogue waves, and line solitons on the PLWs background in several nonlocal Davey-Stewartson equations.

In this paper, we study the $2N$ -soliton and N -breather solutions to the nonlocal Schrödinger-Boussinesq (NSB) equation with a PLWs background. The NSB equation takes the following explicit form in scaled units [56]:

$$\begin{aligned} 3u_{tt}(x, t) - u_{xx}(x, t) - [3u^2(x, t) + u_{xx}(x, t) + \Psi(x, t)\Psi^*(-x, t)]_{xx} &= 0, \\ i\Psi_t(x, t) - \Psi_{xx}(x, t) - 2u(x, t)\Psi(x, t) &= 0. \end{aligned} \quad (1)$$

Similar to the nonlocal NLS equation, equation (1) also possesses a PT -symmetric potential $V(x, t) = \Psi(x, t)\Psi^*(-x, t)$, and it is invariant under the complex conjugation and transformation $x \rightarrow -x, t \rightarrow -t$. By replacing the nonlocal field $\Psi^*(-x, t)$ by $\Psi^*(x, t)$ in equation (1), the NSB equation (1) becomes the local Schrödinger-Boussinesq (LSB) equation [57–59], namely,

$$\begin{aligned} 3u_{tt}(x, t) - u_{xx}(x, t) - [3u^2(x, t) + u_{xx}(x, y, t) + \Psi(x, t)\Psi^*(x, t)]_{xx} &= 0, \\ i\Psi_t(x, t) - \Psi_{xx}(x, t) - 2u(x, t)\Psi(x, t) &= 0. \end{aligned} \quad (2)$$

The LSB equation (2) is thought to be integrable as it can pass the Painlevé test

[60] and admits different kinds of exact solutions, such as order- N solitons [59], homoclinic-orbit solutions [61], and rogue waves [62, 63]. The NSB equation (1) also admits solitons on a nonzero continuous waves (CWs) background [56]. However, the key features of the solitons in the NSB equation (1) have not been studied before, to the best of our knowledge, including the classifications of soliton patterns under different parametric conditions and the degenerated solitons. Additionally, $2N$ -soliton and N -breather solutions with a periodic line waves (PLWs) background in the NSB equation have not been studied before, to the best of our knowledge. As discussed in [53–55], the collisions of solitons and PLWs are elastic, hence the classifications for patterns of solitons are not related to the background of CWs or PLWs, namely, the patterns of solitons on a CWs background or on a PLWs background should be the same. Naturally, if we investigate the patterns of solitons on a PLWs background, then we can also get the classification for patterns of the solitons on a CWs background. A natural motivation is to consider classification of patterns for the solitons on a PLWs background. Solitons, breathers, and rogue waves are localized structures that occur in many nonlinear physical systems [64–77].

The paper is organized as follows. In Sect. 2, $2N$ -soliton solutions to the NSB equation with PLWs background are expressed by a Theorem (Theorem 2.1), and their unique dynamics are analyzed in detail. In Sect. 3, N -breather solutions to the NSB equation with a PLWs background are presented by Theorem 3.1, and both one- and two-breather solutions are discussed. Finally, we present the conclusions of the present paper in Sect. 4.

2. GENERAL HIGH-ORDER SOLITONS TO THE NSB EQUATION WITH A PLWS BACKGROUND

In this Section, we study the general solitons to the NSB equation (1) with a PLWs background. To this end, we first present the general $2N$ -soliton solutions on a PLWs background in the NSB equation (1) by the following Theorem. The proof of this Theorem is given in the Appendix.

Theorem 2.1 *The general soliton solutions to the NSB equation (1) with a PLWs background are*

$$\Psi = \sqrt{2} \frac{g}{f}, u = (2 \log f)_{xx} \quad (3)$$

with functions f and g given by the following $(2N + 1) \times (2N + 1)$ determinants

$$f(x, t) = \begin{vmatrix} b_s \delta_{sj} e^{-\zeta_s + \frac{1}{p_s + p_j^*}} & \frac{1}{p_s - p_j^*} & \bar{f}_s \\ -\frac{1}{p_s - p_j^*} & -b_s^* \delta_{sj} e^{-\zeta_s^*(-x, t) - \frac{1}{p_s + p_j^*}} & \bar{f}_{N+s} \\ \frac{1}{p_{2N+1} + p_j^*} & \frac{1}{p_{2N+1} - p_j^*} & \bar{f}_{2N+1} \end{vmatrix}_{0 \leq s, j \leq N}, \quad (4)$$

$$g(x, t) = \begin{vmatrix} b_s \delta_{sj} e^{-\zeta_s - \frac{p_s}{p_j^*} \frac{1}{p_s + p_j^*}} & \frac{p_s}{p_j^*} \frac{1}{p_s - p_j^*} & \bar{g}_s \\ -\frac{p_s}{p_j^*} \frac{1}{p_s - p_j^*} & -b_s^* \delta_{sj} e^{-\zeta_s^*(-x, t) + \frac{p_s}{p_j^*} \frac{1}{p_s + p_j^*}} & \bar{g}_{N+s} \\ -\frac{p_{2N+1}}{p_j^*} \frac{1}{p_{2N+1} + p_j^*} & \frac{p_{2N+1}}{p_j^*} \frac{1}{p_{2N+1} - p_j^*} & \bar{g}_{2N+1} \end{vmatrix}_{0 \leq s, j \leq N},$$

and

$$\begin{aligned} \bar{f}_s &= \frac{1}{p_s - p_{2N+1}^*}, \bar{f}_{N+s} = -\frac{1}{p_s + p_{2N+1}^*}, \\ \bar{f}_{2N+1} &= ib_{2N+1} e^{-\zeta_{2N+1}} + \frac{1}{p_{2N+1} - p_{2N+1}^*}, \\ \bar{g}_s &= \frac{p_s}{p_j^*} \frac{1}{p_s - p_{2N+1}^*}, \bar{g}_{N+s} = \frac{p_s}{p_{2N+1}^*} \frac{1}{p_s + p_{2N+1}^*}, \\ \bar{g}_{2N+1} &= ib_{2N+1} e^{-\zeta_{2N+1}} + \frac{p_{2N+1}}{p_{2N+1}^*} \frac{1}{p_{2N+1} - p_{2N+1}^*}, \\ \zeta_s &= 2p_{s,R}x + 4p_{s,R}p_{s,I}t + \zeta_s^0, \\ \zeta_{2N+1} &= 2ip_{2N+1,I}x + 4p_{2N+1,R}p_{2N+1,I}t + \zeta_{2N+1}^0, \end{aligned} \quad (5)$$

with the following parametric restrictions

$$\begin{aligned} 1 + p_{s,R}^2 - 3p_{s,I}^2 - \frac{1}{p_{s,R}^2 + p_{s,I}^2} &= 0, \\ 1 + 3p_{2N+1,R}^2 - p_{2N+1,I}^2 - \frac{1}{p_{2N+1,R}^2 + p_{2N+1,I}^2} &= 0, \end{aligned} \quad (6)$$

where b_s, p_s are complex and $\zeta_s^0, \zeta_{2N+1}^0, b_{2N+1}$ are real, $p_{s,R}$ and $p_{s,I}$ are the real and imaginary parts of p_s (i.e., $p_s = p_{s,R} + ip_{s,I}$) for $s = 0, 1, \dots, N$.

Remark 1. Deleting the $(2N + 1)$ th row and the $(2N + 1)$ th column in the determinant expressions of f and g defined in (4), the solutions (3) correspond to soliton solutions to the NSB equation with a CWs background, which have been reported by Shi *et al.* in Ref. [56].

In what follows we present the analysis of dynamical behaviors for the soliton solutions of the NSB equation (1) with PLWs background. To this end, we have to discuss the PLWs background. Taking $N = 0$ in (4), the following solution to the NSB equation (1) is obtained with the results of Theorem 2.1:

$$\Psi = \sqrt{2} \frac{g_0}{f_0}, u = (2 \log f_0)_{xx}, \quad (7)$$

with

$$\begin{aligned} f_0 &= ib_1 e^{-\zeta_1} + \frac{1}{p_1 - p_1^*}, \\ g_0 &= ib_1 e^{-\zeta_1} + \frac{1}{p_1 - p_1^*} \frac{p_1}{p_1^*}. \end{aligned} \quad (8)$$

To consider the singularity of the solution (7), we have to rewrite the function f_0 in the following form:

$$f_0 = b_1 e^{-4p_{1,R}p_{1,I}t - \zeta_1^0} \sin(2p_{1,I}x) - ib_1 e^{-4p_{1,R}p_{1,I}t - \zeta_1^0} \left[\frac{e^{4p_{1,R}p_{1,I}t + \zeta_1^0}}{2b_1 p_{1,I}} - \cos(2p_{1,I}x) \right]. \quad (9)$$

The solution is singular when $p_{1,R} \neq 0$, hence we take $p_{1,R} = 0$ to avoid the singular solution. Then one can derive $f_{0,I} \neq 0$ when $\frac{2b_1^2 p_{1,I}^2}{e^{2\zeta_1^0}} > 1$, where $f_{0,I} = b_1 e^{-\zeta_1^0} \left[\frac{e^{\zeta_1^0}}{2b_1 p_{1,I}} - \cos(2p_{1,I}x) \right]$. Hereafter we take $p_{1,R} = 0$ and $\frac{e^{2\zeta_1^0}}{4b_1^2 p_{1,I}^2} > 1$ to keep the solution regular. Since $\frac{\partial \zeta_1}{\partial x} = 2ip_{1,I}$, $\frac{\partial \zeta_1}{\partial t} = 0$, thus the solution is PLWs and its period is $\frac{\pi}{p_{1,I}}$, $|\Phi|$ attains the maximum value $|\Phi|_{max} = \left| \frac{e^{2b_1 p_{1,I} \zeta_1^0} + 1}{2b_1 p_{1,I} e^{\zeta_1^0} - 1} \right|$ along the line $x = \frac{n\pi}{p_{1,I}}$ for $n = 0, \pm 1, \pm 2, \dots$. If that periodic solution coexists with soliton solution in the NSB equation (1), it behaves as the PLWs background. Figure 1 shows the PLW background with $p_1 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}}i$, $b_1 = 10$, $\zeta_1^0 = 0$.

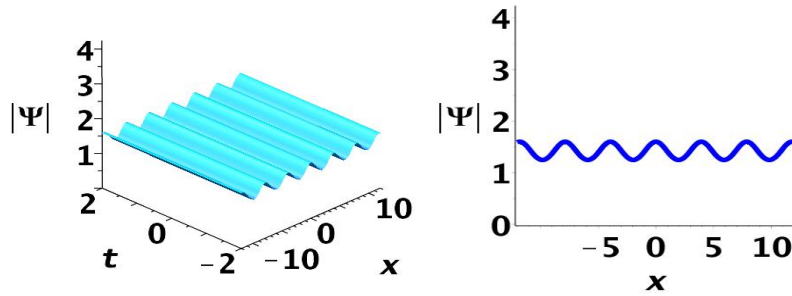


Fig. 1 – (Colour online) The PLWs background in the NSB equation (1) described by solution (7) with $p_1 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}}i$, $b_1 = 10$, $\zeta_1^0 = 0$. The left panel is a section diagram along $t = -5$ of the right panel.

The two-soliton solution on a PLWs background is obtained from Theorem 2.1 with $N = 1$ in formula (4):

$$\Psi = \sqrt{2} \frac{g_1}{f_1}, u = (2 \log f_1)_{xx}, \quad (10)$$

with

$$f_1 = \begin{vmatrix} b_1 e^{-\zeta_1 + \frac{1}{p_1 + p_1^*}} & \frac{1}{p_1 - p_1^*} & \frac{1}{p_1 - p_3^*} \\ -\frac{1}{p_1 - p_1^*} & -b_1^* e^{-\zeta_1^*(-x,t)} - \frac{1}{p_1 + p_1^*} & -\frac{1}{p_1 + p_3^*} \\ \frac{1}{p_3 + p_1^*} & \frac{1}{p_3 - p_1^*} & ib_3 e^{-\zeta_3 + \frac{1}{p_3 - p_3^*}} \end{vmatrix}, \quad (11)$$

$$g_1 = \begin{vmatrix} b_1 e^{-\zeta_1 - \frac{p_1}{p_1^*} \frac{1}{p_1 + p_1^*}} & \frac{p_1}{p_1^*} \frac{1}{p_1 - p_1^*} & \frac{p_1}{p_1^*} \frac{1}{p_1 - p_3^*} \\ -\frac{p_1}{p_1^*} \frac{1}{p_1 - p_1^*} & -b_1^* e^{-\zeta_1^*(-x,t)} + \frac{p_1}{p_1^*} \frac{1}{p_1 + p_1^*} & \frac{p_1}{p_1^*} \frac{1}{p_1 + p_3^*} \\ -\frac{p_3}{p_1^*} \frac{1}{p_3 + p_1^*} & \frac{p_3}{p_1^*} \frac{1}{p_3 - p_1^*} & ib_3 e^{-\zeta_3 + \frac{p_3}{p_1^*} \frac{1}{p_3 - p_3^*}} \end{vmatrix},$$

and

$$\zeta_1 = 2p_{1,R}x + 4p_{1,R}p_{1,I}t + \zeta_1^0, \zeta_3 = 2ip_{3,I}x + 4p_{3,R}p_{3,I}t + \zeta_3^0, \quad (12)$$

and the parameters satisfying

$$1 + p_{1,R}^2 - 3p_{1,I}^2 - \frac{1}{p_{1,R}^2 + p_{1,I}^2} = 0, \quad (13)$$

$$1 + 3p_{3,R}^2 - p_{3,I}^2 - \frac{1}{p_{3,R}^2 + p_{3,I}^2} = 0,$$

where $\zeta_1^0, \zeta_3^0, b_3$ are real, b_1 is complex, and $p_{3,R} = 0$.

Below we classify the patterns of the two-soliton solutions on the PLWs background. Since the soliton interacts elastically with the periodic line waves, thus the patterns of the two-soliton solutions on a PLWs background are the same as those on a constant background. We note that the two-soliton solution on a constant background is obtained by deleting the third row and the third line of determinant in (11), and the asymptotic expressions of the two-soliton for $p_{1,R}, p_{1,I} > 0$ are:

(i) Before collision ($t \rightarrow -\infty$)

Soliton 1 ($\zeta_1 \approx 0, \zeta_2 \approx -\infty$)

$$\Psi_1^- \simeq \sqrt{2} \frac{c_1 e^{-\zeta_1} - \frac{1}{p_1 + p_1^*} \frac{p_1}{p_1^*}}{c_1 e^{-\zeta_1} + \frac{1}{p_1 + p_1^*}}, \quad (14)$$

Soliton 2 ($\zeta_2 \approx 0, \zeta_1 \approx -\infty$)

$$\Psi_2^- \simeq \sqrt{2} \frac{c_1^* e^{-\zeta_2} - \frac{1}{p_1 + p_1^*} \frac{p_1}{p_1^*}}{c_1^* e^{-\zeta_2} + \frac{1}{p_1 + p_1^*}}, \quad (15)$$

where $e^{\tilde{\zeta}} = -\frac{(p_1 - p_1^*)^2}{4p_1 p_1^*}$.

(ii) After collision ($t \rightarrow +\infty$)

Soliton 1 ($\zeta_1 \approx 0, \zeta_2 \approx +\infty$)

$$\Psi_1^+ \simeq -\sqrt{2} \frac{p_1}{p_1^*} \frac{c_1 e^{-\zeta_1 + \tilde{\zeta}} - \frac{1}{p_1 + p_1^*} \frac{p_1}{p_1^*}}{c_1 e^{-\zeta_1 + \tilde{\zeta}} + \frac{1}{p_1 + p_1^*}}, \quad (16)$$

Soliton 2 ($\zeta_2 \approx 0, \zeta_1 \approx +\infty$)

$$\Psi_2^+ \simeq -\sqrt{2} \frac{p_1}{p_1^*} \frac{c_1^* e^{-\zeta_2 + \tilde{\zeta}} - \frac{1}{p_1 + p_1^*} \frac{p_1}{p_1^*}}{c_1^* e^{-\zeta_2 + \tilde{\zeta}} + \frac{1}{p_1 + p_1^*}}. \quad (17)$$

Here we have defined the soliton along the line $\zeta_1 = 0$ as soliton 1, and the soliton along the line $\zeta_2 = 0$ as soliton 2, where $\zeta_2 = \zeta_1^*(-x, t)$. Since $\Psi_j^+(\zeta_j) = (-\frac{p_j}{p_j^*}) \Psi_j^-(\zeta_j - \tilde{\zeta})$ and $|\frac{p_j}{p_j^*}| = 1$ for $j = 1, 2$, thus the amplitudes of the two-soliton solution after collision remain the same as before collision. $|\Psi_j|$ attains the maximum value denoted as $|\Psi_j|_{max}$ along the line $\zeta_j - \frac{1}{2} \sqrt{\frac{1}{4p_{1,R}^2(c_{1,R}^2 + c_{1,I}^2)}} = 0$:

$$\begin{aligned} |\Psi_j|_{max}^2 &= \sqrt{2}(1 - \Gamma_j), \\ \Gamma_j &= \frac{2(c_{1,R} p_{1,R} + (-1)^{j-1} c_{1,I} p_{1,I}) p_{1,R}^2 \sqrt{\frac{1}{(c_{1,R}^2 + c_{1,I}^2) p_{1,R}^2}}}{(p_{1,R}^2 + p_{1,I}^2) (1 + c_{1,R} p_{1,R} \sqrt{\frac{1}{(c_{1,R}^2 + c_{1,I}^2) p_{1,R}^2}})}. \end{aligned} \quad (18)$$

The j th soliton is a dark soliton for $\Gamma_j > 0$, an antidark soliton for $\Gamma_j < 0$, and a vanishing one for $\Gamma_j = 0$. The patterns of two-soliton solution on the PLWs background are the same as on the CWs background, then we can list the patterns of two-soliton on a PLWs background by Table 1, here we have assumed $c_{1,R} p_{1,R} \geq 0$ to keep the denominator of Γ_j positive. As seen in Table 1, the two-antidark-antidark soliton is not found in Table 1 for the reason that we constrain the parameter condition $c_{1,R} p_{1,R} \geq 0$. With the parameter restriction $c_{1,R} p_{1,R} < 0$ and other parameter choices, one can obtain $\Gamma_1 < 0, \Gamma_2 < 0$, which corresponds to antidark-antidark soliton in the NSB equation with a PLWs background. The two-soliton solution admits five distinct patterns: dark-dark soliton, dark-antidark soliton, antidark-antidark soliton, degenerate antidark-soliton, and degenerate dark-soliton. Figure 2 displays three patterns of the non-degenerate two-soliton solution, namely, two-dark-dark soliton (see Fig. 2(a)), two-dark-antidark soliton (see Fig. 2(b)), and two-antidark-antidark soliton (see Fig. 2(c)). Figure 4 shows two patterns of degenerate two-soliton solution, namely, degenerate dark-soliton solution (see Fig. 4 (a)) and degenerate antidark-soliton solution (see Fig. 4(b)). We emphasize that, the classification of patterns corresponding to the two-soliton solution and the degenerate solitons, were not discussed in Ref. [56], for the two-soliton solution on a CWs background.

With $N = 2$ in (4), the four-soliton solution to the NSB equation (1) with a PLWs background can be generated from Theorem 2.1

$$\Psi = \sqrt{2} \frac{g_2}{f_2}, u = (2 \log f_2)_{xx}, \quad (19)$$

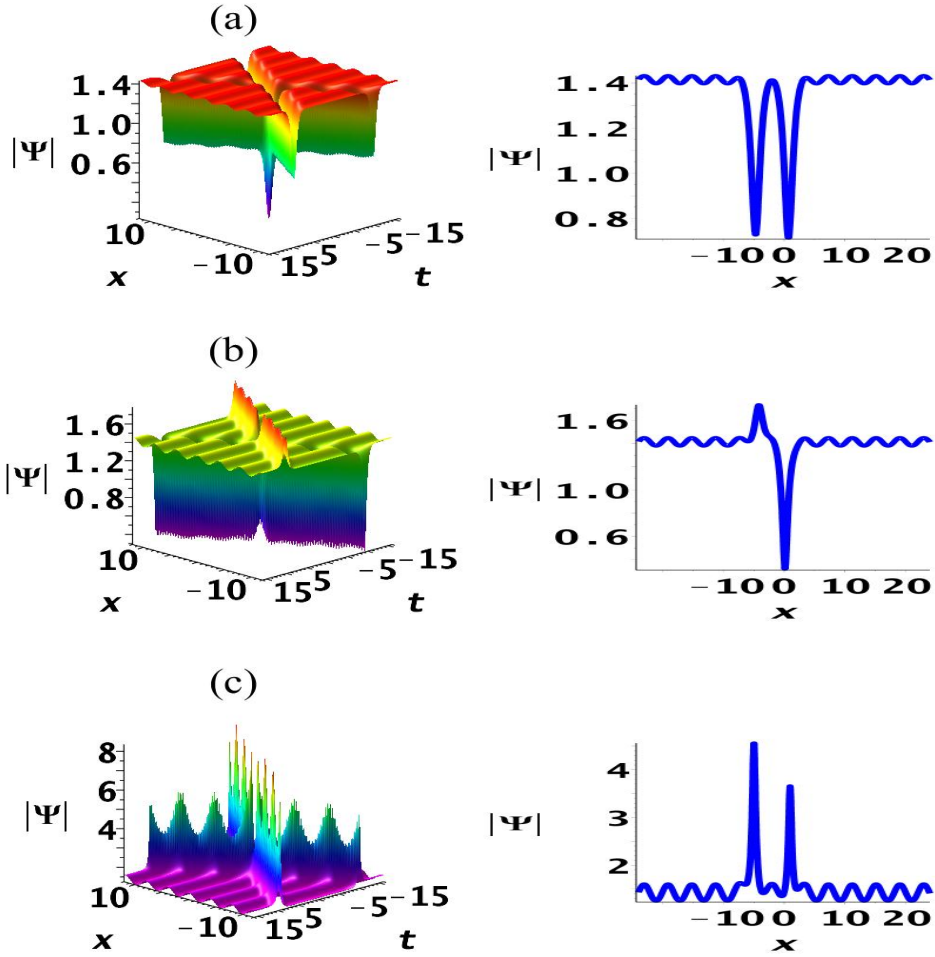


Fig. 2 – (Colour online) The two-soliton solutions of the NSB equation (1) with a PLWs background: (a) A non-degenerated two-dark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}i}$, $p_3 = \frac{1}{4}\sqrt{2 + 2\sqrt{27}}$, $b_1 = 1$, $b_3 = 100$, $\zeta_1^0 = 0$, $\zeta_3^0 = 0$; (b) A non-degenerated two-antidark-dark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}i}$, $p_3 = \frac{1}{4}\sqrt{2 + 2\sqrt{27}}$, $b_1 = \frac{1}{2} - 3i$, $b_3 = 50$, $\zeta_1^0 = 0$, $\zeta_3^0 = 0$; (c) A non-degenerated two-antidark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}i}$, $p_3 = \frac{1}{4}\sqrt{2 + 2\sqrt{27}}$, $b_1 = -\frac{1}{2} - \frac{1}{4}i$, $b_3 = 10$, $\zeta_1^0 = 0$, $\zeta_3^0 = 0$. The right panels are the corresponding section diagrams along $t = -5$ of the left panels.

Table 1

The classification of the two-soliton solution under different parametric conditions

parametric condition	Soliton 1	Soliton 2
$c_{1,IP1,I} < -c_{1,RP1,R}$	Antidark	Dark
$c_{1,IP1,I} = -c_{1,RP1,R}$	Vanishing	Dark
$c_{1,RP1,R} > c_{1,IP1,I} > -c_{1,RP1,R}$	Dark	Dark
$c_{1,IP1,I} = c_{1,RP1,R}$	Dark	Vanishing
$c_{1,IP1,I} > c_{1,RP1,R}$	Dark	Antidark

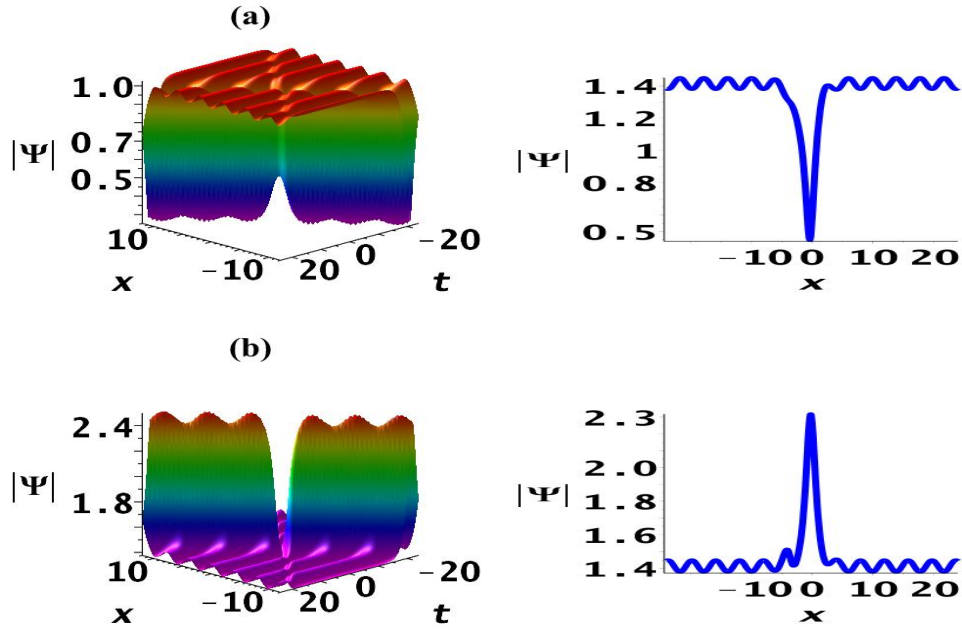


Fig. 3 – (Colour online) The two-soliton solutions of the NSB equation (1) with a PLWs background: (a) A degenerated two-dark-soliton solution with $p_1 = \frac{2}{3} + \frac{1}{36}\sqrt{-138+6\sqrt{1441}}i$, $p_3 = \frac{1}{4}\sqrt{2+2\sqrt{17}}i$, $b_1 = 1 - \frac{24}{\sqrt{-138+6\sqrt{1441}}}i$, $b_3 = -50$, $\zeta_1^0 = 0$, $\zeta_3^0 = 0$. (b) A degenerated two-antidark-soliton solution with $p_1 = \frac{2}{3} + \frac{1}{36}\sqrt{-138+6\sqrt{1441}}i$, $p_3 = \frac{1}{4}\sqrt{2+2\sqrt{17}}i$, $b_1 = -1 + \frac{24}{\sqrt{-138+6\sqrt{1441}}}i$, $b_3 = 50$, $\zeta_1^0 = 0$, $\zeta_3^0 = 0$. The right panels are corresponding section diagrams along $t = -5$ of the left panels.

with

$$\begin{aligned}
 f_2 &= \begin{vmatrix} b_1 e^{-\zeta_1} + \frac{1}{p_1 + p_1^*} & \frac{1}{p_1 + p_2^*} & \frac{1}{p_1 - p_1^*} & f_{14} & \bar{f}_1 \\ \frac{1}{p_2 + p_1^*} & b_2 e^{-\zeta_2} + \frac{1}{p_2 + p_2^*} & \frac{1}{p_2 - p_1^*} & f_{24} & \bar{f}_2 \\ -\frac{1}{p_1 - p_1^*} & \frac{1}{-p_1 + p_2^*} & -b_1^* e^{-\zeta_1^*(-x,t)} - \frac{1}{p_1 + p_1^*} & f_{34} & \bar{f}_3 \\ -\frac{1}{p_2 - p_1^*} & \frac{1}{-p_2 + p_2^*} & -\frac{1}{p_2 + p_1^*} & f_{44} & \bar{f}_4 \\ \frac{1}{p_5 + p_1^*} & \frac{1}{p_5 + p_2^*} & \frac{1}{p_5 - p_1^*} & f_{54} & \bar{f}_5 \end{vmatrix}, \\
 g_2 &= \begin{vmatrix} b_1 e^{-\zeta_1} - \frac{p_1}{p_1^*} \frac{1}{p_1 + p_1^*} & -\frac{p_1}{p_2^*} \frac{1}{p_1 + p_2^*} & \frac{p_1}{p_3^*} \frac{1}{p_1 - p_1^*} & \bar{g}_{14} & \bar{g}_1 \\ -\frac{p_2}{p_1^*} \frac{1}{p_2 + p_1^*} & b_2 e^{-\zeta_2} - \frac{p_2}{p_2^*} \frac{1}{p_2 + p_2^*} & \frac{p_2}{p_1^*} \frac{1}{p_2 - p_1^*} & \bar{g}_{24} & \bar{g}_2 \\ -\frac{p_1}{p_1^*} \frac{1}{p_1 - p_1^*} & -\frac{p_1}{p_2^*} \frac{1}{p_1 - p_2^*} & -b_1^* e^{-\zeta_1^*(-x,t)} + \frac{p_1}{p_1^*} \frac{1}{p_1 + p_1^*} & \bar{g}_{34} & \bar{g}_3 \\ -\frac{p_2}{p_1^*} \frac{1}{p_2 - p_1^*} & -\frac{p_2}{p_2^*} \frac{1}{p_2 - p_2^*} & \frac{p_2}{p_1^*} \frac{1}{p_2 + p_1^*} & \bar{g}_{44} & \bar{g}_4 \\ \frac{p_5}{p_1^*} \frac{1}{p_5 + p_1^*} & \frac{p_5}{p_2^*} \frac{1}{p_5 + p_2^*} & -\frac{p_5}{p_1^*} \frac{1}{p_5 - p_1^*} & \bar{g}_{54} & \bar{g}_5 \end{vmatrix}, \tag{20}
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{f}_1 &= \frac{1}{p_1 - p_5^*}, \bar{f}_2 = \frac{1}{p_2 - p_5^*}, \bar{f}_3 = -\frac{1}{p_1 + p_5^*}, \bar{f}_4 = -\frac{1}{p_2 + p_5^*}, \\
 \bar{f}_5 &= ib_5 e^{-\zeta_5} + \frac{1}{p_5 - p_5^*}, \bar{g}_1 = \frac{p_1}{p_5^*} \frac{1}{p_1 - p_5^*}, \bar{g}_2 = \frac{p_2}{p_5^*} \frac{1}{p_2 - p_5^*}, \\
 \bar{g}_3 &= \frac{p_1}{p_5^*} \frac{1}{p_1 + p_5^*}, \bar{g}_4 = \frac{p_2}{p_5^*} \frac{1}{p_2 + p_5^*}, \bar{g}_5 = ib_5 e^{-\zeta_5} + \frac{p_5}{p_5^*} \frac{1}{p_5 - p_5^*}, \\
 \bar{f}_{14} &= \frac{1}{p_1 - p_2^*}, \bar{f}_{24} = \frac{1}{p_2 - p_2^*}, \bar{f}_{34} = -\frac{1}{p_1 + p_2^*}, \\
 \bar{f}_{44} &= -b_2^* e^{-\zeta_2^*(-x,t)} - \frac{1}{p_2 + p_2^*}, \bar{f}_{54} = \frac{1}{p_5 - p_2^*}, \\
 \bar{g}_{14} &= \frac{p_1}{p_2^*} \frac{1}{p_1 - p_2^*}, \bar{g}_{24} = \frac{p_2}{p_2^*} \frac{1}{p_2 - p_2^*}, \bar{g}_{34} = \frac{p_1}{p_2^*} \frac{1}{p_1 + p_2^*}, \\
 \bar{g}_{44} &= -b_2^* e^{-\zeta_2^*(-x,t)} + \frac{p_2}{p_2^*} \frac{1}{p_2 + p_2^*}, \bar{g}_{54} = -\frac{p_5}{p_2^*} \frac{1}{p_5 - p_2^*}, \\
 \zeta_j &= 2p_{j,R}x + 4p_{j,RP}j_I t + \zeta_j^0, \\
 \zeta_5 &= 2ip_{5,I}x + 4p_{5,RP}p_{5,I}t + \zeta_5^0,
 \end{aligned} \tag{21}$$

and the parameters satisfy

$$\begin{aligned}
 1 + p_{j,R}^2 - 3p_{j,I}^2 - \frac{1}{p_{j,R}^2 + p_{j,I}^2} &= 0, \\
 1 + 3p_{5,R}^2 - p_{5,I}^2 - \frac{1}{p_{5,R}^2 + p_{5,I}^2} &= 0,
 \end{aligned} \tag{22}$$

for $j = 1, 3$, where $\zeta_j^0, \zeta_5^0, b_5$ are real, b_j is complex, and $p_{5,R} = 0$.

Since the four-soliton solution is a superposition of two two-soliton solutions, according to the analysis for the two-soliton solution on a PLWs background, thus

the four-soliton solution on a PLWs background has three different types: (i) Two degenerate two-soliton solutions for $(c_{1,RP1,R} + c_{1,IP1,I})(c_{1,RP1,R} - c_{1,IP1,I}) = 0$ and $(c_{3,RP3,R} + c_{3,IP3,I})(c_{3,RP3,R} - c_{3,IP3,I}) = 0$. There are three different kinds of two degenerate two-soliton solutions in the four-soliton solution (19): two degenerate two-dark-soliton solutions, two degenerate two-antidark-soliton solutions, a mixture of one degenerate two-dark-soliton solution and one degenerate two-antidark-soliton solution. These three kinds of two generate two-soliton solutions are shown in Fig. 4; (ii) A mixture of one degenerate two-soliton solution and non-degenerate two-soliton solution for $(c_{1,RP1,R} + c_{1,IP1,I})(c_{1,RP1,R} - c_{1,IP1,I}) = 0$ and $(c_{3,RP3,R} + c_{3,IP3,I})(c_{3,RP3,R} - c_{3,IP3,I}) \neq 0$. Since the degenerate two-soliton solution possesses two patterns, and the non-degenerate two-soliton solution has three patterns, thus there are six patterns in such mixed soliton solution. Three patterns of such mixed solution are displayed in Fig. 5; (iii) A non-degenerate four-soliton solution for $(c_{1,RP1,R} + c_{1,IP1,I})(c_{1,RP1,R} - c_{1,IP1,I}) \neq 0$ and $(c_{3,RP3,R} + c_{3,IP3,I})(c_{3,RP3,R} - c_{3,IP3,I}) \neq 0$. There are five patterns of the non-degenerate four-soliton solutions, and three of the them are shown in Fig. 6.

3. GENERAL HIGH-ORDER BREATHERS TO THE NSB EQUATION WITH A PLWS BACKGROUND

In this Section, we study the general breathers to the NSB equation (1) with a PLWs background. For this purpose, we present the N -breather solution on a PLWs background to the NSB equation (1) by the following Theorem, and the proof of this Theorem is given in the Appendix.

Theorem 3.1 *The general breather solutions to the NSB equation (1) with a PLWs background are*

$$\Psi = \sqrt{2} \frac{g}{f}, u = (2 \log f)_{xx} \quad (23)$$

where

$$\begin{aligned} f(x, t) &= \left| \frac{\delta_{sj}}{\omega_s e^{\zeta_s}} + \frac{1}{\frac{1}{2}(\omega_s + \omega_j) + i(\lambda_s - \lambda_j)} \right|_{1 \leq s, j \leq N}, \\ g(x, t) &= \left| \frac{\delta_{sj}}{\omega_s e^{\zeta_s}} - \frac{\omega_s + 2i\lambda_s}{\omega_j - 2i\lambda_j} \frac{1}{\frac{1}{2}(\omega_s + \omega_j) + i(\lambda_s - \lambda_j)} \right|_{1 \leq s, j \leq N}, \end{aligned} \quad (24)$$

$$\zeta_s = i\omega_s x - 2\lambda_s \omega_s t + \zeta_s^0,$$

and the parameters meet two restrictions: (i)

$$48\lambda_s^4 + 8\lambda_s^2 \omega_s^2 - \omega_s^4 + 4\lambda_s^2 + \omega_s^2 + 4 = 0, \quad (25)$$

and (ii)

$$N = 2\tilde{N} + 1, \omega_{\tilde{N}+s} = -\omega_s, \lambda_{\tilde{N}+s} = -\lambda_s, \zeta_{\tilde{N}+s}^0 = -\zeta_s^0, \lambda_{2\tilde{N}+1} = 0, \quad (26)$$

where $\omega_s, \lambda_s, \zeta_s^0$ are real parameters.

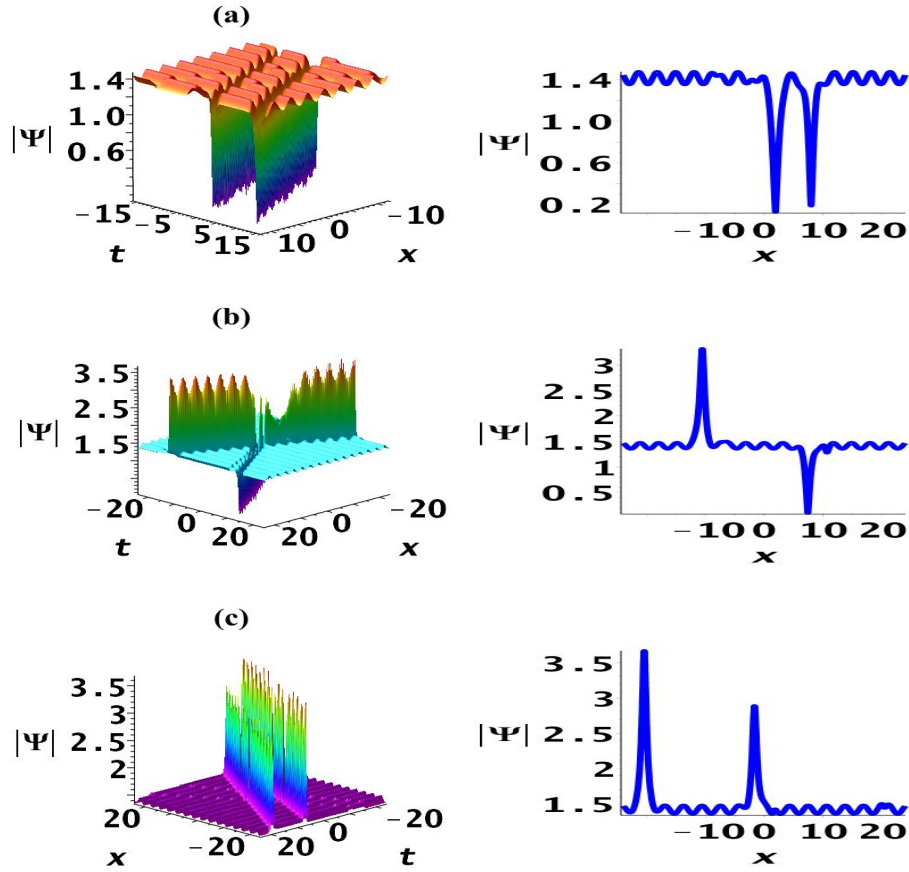


Fig. 4 – (Colour online) The four-soliton solutions (19) to the NSB equation (1) with a PLWs background, which consist of three types of two degenerated two-soliton solutions: (a) Two degenerate two-dark-soliton solutions with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}}$, $p_3 = \frac{7}{6} + \frac{1}{36}\sqrt{-534 + 6\sqrt{38137}i}$, $p_5 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}i}$, $c_1 = 1 - \frac{12}{\sqrt{-42 + 6\sqrt{241}}}i$, $c_3 = 1 - \frac{42}{\sqrt{-534 + 6\sqrt{38137}i}}i$, $c_5 = 1$, $\zeta_1^0 = 4\pi$, $\zeta_3^0 = -4\pi$, $\zeta_5^0 = -\pi$; (b) A mixture of one degenerate two-dark-soliton solution and one degenerate two-antidark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}}$, $p_3 = -\frac{5}{6} + \frac{1}{36}\sqrt{-246 + 6\sqrt{7993}i}$, $p_5 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}i}$, $c_1 = 1 - \frac{12}{\sqrt{-42 + 6\sqrt{241}}}i$, $c_3 = 2 + \frac{60}{\sqrt{-246 + 6\sqrt{7993}i}}i$, $c_5 = 1$, $\zeta_1^0 = 4\pi$, $\zeta_3^0 = -4\pi$, $\zeta_5^0 = -\pi$; (c) Two degenerate two-antidark-soliton solutions with $p_1 = -1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}}$, $p_3 = -\frac{5}{6} + \frac{1}{36}\sqrt{-246 + 6\sqrt{7993}i}$, $p_5 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}i}$, $c_1 = 1 - \frac{12}{\sqrt{-42 + 6\sqrt{241}}}i$, $c_3 = 2 + \frac{60}{\sqrt{-246 + 6\sqrt{7993}i}}i$, $c_5 = 1$, $\zeta_1^0 = 4\pi$, $\zeta_3^0 = -4\pi$, $\zeta_5^0 = -\pi$. The right panels are the corresponding section diagrams along $t = -5$ of the left panels.

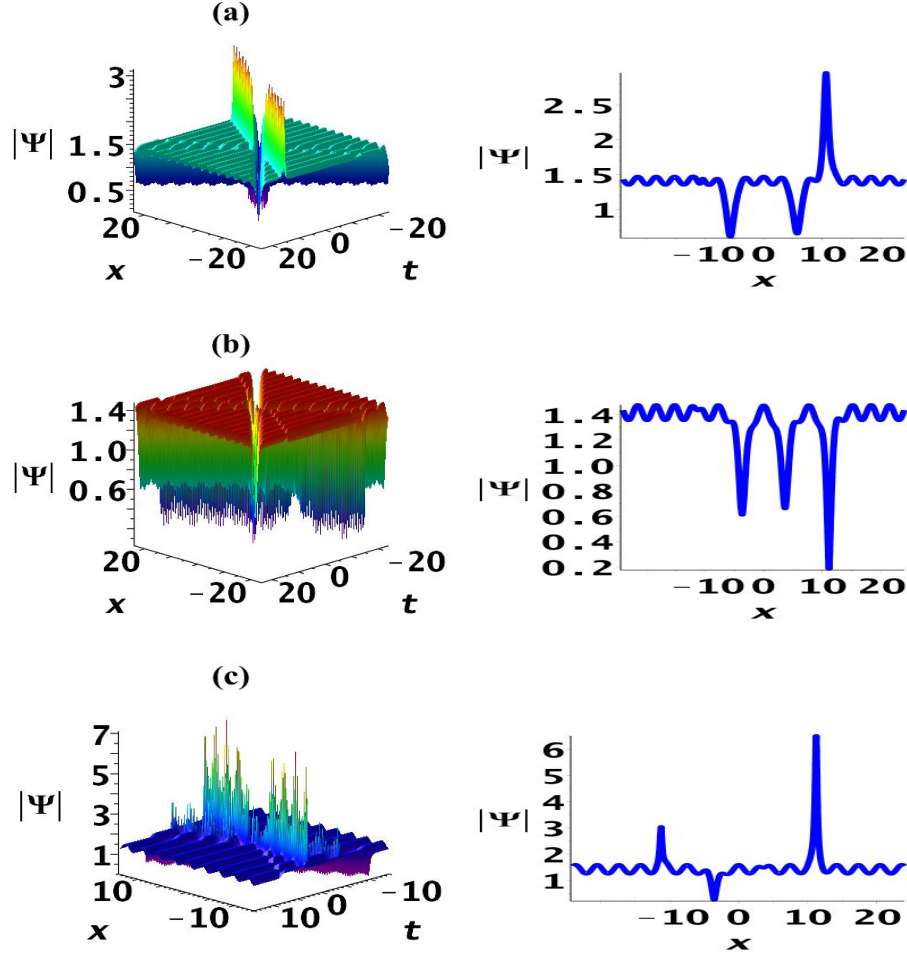


Fig. 5 – (Colour online) The four-soliton solution (19) to the NSB equation (1) with a PLWs background: (a) A non-degenerate two-dark-soliton solution and a degenerate two-antidark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42+6\sqrt{241}}i$, $p_3 = \frac{2}{3} + \frac{1}{36}\sqrt{-138+6\sqrt{1441}}i$, $c_1 = -1 + \frac{24}{\sqrt{-138+6\sqrt{1441}}}i$, $c_3 = 1$, $c_5 = 1$, $p_5 = \frac{1}{4}\sqrt{2+2\sqrt{17}}i$, $\zeta_1^0 = -\pi$, $\zeta_3^0 = \pi$, $\zeta_5^0 = -\pi$; (b) A non-degenerate two-dark-soliton solution and a degenerate two-dark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42+6\sqrt{241}}i$, $p_3 = \frac{2}{3} + \frac{1}{36}\sqrt{-138+6\sqrt{1441}}i$, $c_1 = -1 - \frac{24}{\sqrt{-138+6\sqrt{1441}}}i$, $c_3 = 1$, $c_5 = 1$, $p_5 = \frac{1}{4}\sqrt{2+2\sqrt{17}}i$, $\zeta_1^0 = -\pi$, $\zeta_3^0 = \pi$, $\zeta_5^0 = -\pi$; (c) A non-degenerate two-antidark-soliton solution and a degenerate two-dark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42+6\sqrt{241}}i$, $p_3 = \frac{2}{3} + \frac{1}{36}\sqrt{-138+6\sqrt{1441}}i$, $c_1 = -1 - \frac{24}{\sqrt{-138+6\sqrt{1441}}}i$, $c_3 = -\frac{1}{2} - \frac{1}{6}i$, $c_5 = 1$, $p_5 = \frac{1}{4}\sqrt{2+2\sqrt{17}}i$, $\zeta_1^0 = -\pi$, $\zeta_3^0 = \pi$, $\zeta_5^0 = -\pi$. The right panels are the corresponding section diagrams along $t = -5$ of the left panels.

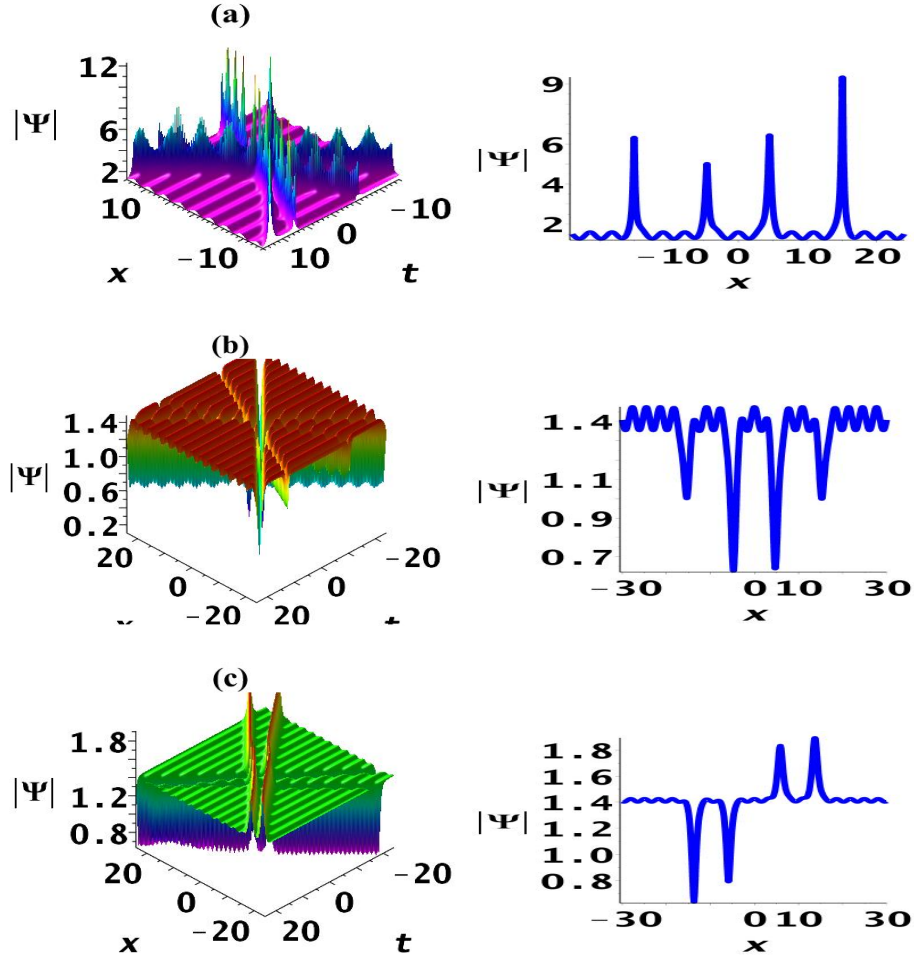


Fig. 6 – (Colour online) The four-soliton solution (19) to the NSB equation (1) with a PLWs background: (a) A non-degenerate four-antidark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}i}$, $p_3 = \frac{2}{3} + \frac{1}{36}\sqrt{-138 + 6\sqrt{1441}i}$, $p_5 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}i}$, $c_1 = -\frac{1}{2} - \frac{1}{6}i$, $c_3 = -\frac{1}{2} - \frac{1}{4}i$, $c_5 = 1$, $\zeta_1^0 = -4\pi$, $\zeta_3^0 = 4\pi$, $\zeta_5^0 = -\pi$; (b) A non-degenerate four-dark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}i}$, $p_3 = \frac{2}{3} + \frac{1}{36}\sqrt{-138 + 6\sqrt{1441}i}$, $p_5 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}i}$, $c_1 = 1$, $c_3 = 1$, $c_5 = 1$, $\zeta_1^0 = -4\pi$, $\zeta_3^0 = 4\pi$, $\zeta_5^0 = -\pi$; (c) A non-degenerate two-antidark-two-dark-soliton solution with $p_1 = 1 + \frac{1}{12}\sqrt{-42 + 6\sqrt{241}i}$, $p_3 = \frac{2}{3} + \frac{1}{36}\sqrt{-138 + 6\sqrt{1441}i}$, $p_5 = \frac{1}{4}\sqrt{2 + 2\sqrt{17}i}$, $c_1 = -4i$, $c_3 = -6i$, $c_5 = 1$, $\zeta_1^0 = -4\pi$, $\zeta_3^0 = 4\pi$, $\zeta_5^0 = -\pi$. The right panels are the corresponding section diagrams along $t = -5$ of the left panels.

Remark 2. Replacing the parametric restriction (ii) in (26) by the following restriction

$$N = 2\tilde{N}, \omega_{\tilde{N}+s} = -\omega_s, \lambda_{\tilde{N}+s} = -\lambda_s, \zeta_{\tilde{N}+s}^0 = -\zeta_s^0, \quad (27)$$

the breather solution corresponds to the breather to the NSB equation with a CWs background. In particular, the breather solution also satisfies the LSB equation, and we comprehensively focus on dynamics of the breathers on a PLWs background in the NSB equation, hence we do not consider this subclass of breather solution under parametric restrictions (25) and (27) in this paper.

By taking the following parameter conditions in formula (24)

$$N = 3, \omega_2 = -\omega_1, \lambda_2 = -\lambda_1, \zeta_2^0 = -\zeta_s^0, \lambda_3 = 0, \quad (28)$$

Theorem 3.1 yields the one-breather solution on a PLWs background to the NSB equation:

$$\Phi = \sqrt{2} \frac{\tilde{g}_1}{\tilde{f}_1}, u = (2 \log \tilde{f}_1)_{xx}, \quad (29)$$

where functions \tilde{f}_1 and \tilde{g}_1 are

$$\tilde{f}_1 = \begin{vmatrix} \frac{1}{\omega_1 e^{\zeta_1}} + \frac{1}{\omega_1} & \frac{1}{2i\lambda_1} & \frac{1}{\frac{1}{2}(\omega_1 + \omega_3) + i\lambda_1} \\ \frac{1}{-2i\lambda_1} & -\frac{1}{\omega_1 e^{\zeta_1^*}} - \frac{1}{\omega_1} & \frac{1}{\frac{1}{2}(-\omega_1 + \omega_3) - i\lambda_1} \\ \frac{1}{\frac{1}{2}(\omega_3 + \omega_1) - i\lambda_1} & \frac{1}{\frac{1}{2}(\omega_3 - \omega_1) + i\lambda_1} & \frac{1}{\omega_3 e^{\zeta_3}} + \frac{1}{\omega_3} \end{vmatrix},$$

$$\tilde{g}_1 = \begin{vmatrix} \frac{1}{\omega_1 e^{\zeta_1}} - \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{\omega_1} & \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{2i\lambda_1} & -\frac{\omega_1 + 2i\lambda_1}{\omega_3} \frac{1}{\frac{1}{2}(\omega_1 + \omega_3) + i\lambda_1} \\ -\frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{2i\lambda_1} & -\frac{1}{\omega_1 e^{\zeta_1^*}} + \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{\omega_1} & \frac{\omega_1 + 2i\lambda_1}{\omega_3} \frac{1}{\frac{1}{2}(-\omega_1 + \omega_3) - i\lambda_1} \\ -\frac{\omega_3}{\omega_1 - 2i\lambda_1} \frac{1}{\frac{1}{2}(\omega_3 + \omega_1) - i\lambda_1} & -\frac{\omega_3}{\omega_1 - 2i\lambda_1} \frac{1}{\frac{1}{2}(\omega_3 - \omega_1) + i\lambda_1} & \frac{1}{\omega_3 e^{\zeta_3}} - \frac{1}{\omega_3} \end{vmatrix}, \quad (30)$$

and

$$\begin{aligned} \zeta_1 &= i\omega_1 x - 2\lambda_1 \omega_1 t + \zeta_1^0, \\ \zeta_3 &= i\omega_3 x + \zeta_3^0. \end{aligned} \quad (31)$$

Here the real parameters $\omega_1, \omega_3, \lambda_1, \zeta_1^0, \zeta_3^0$ follow the following restrictions:

$$\begin{aligned} 48\lambda_1^4 + 8\lambda_1^2\omega_1^2 - \omega_1^4 + 4\lambda_1^2 + \omega_1^2 + 4 &= 0, \\ -\omega_3^4 + \omega_3^2 + 4 &= 0. \end{aligned} \quad (32)$$

The period and the direction of propagation of the one-breather solution is determined by ζ_1 , while the PLWs background is controlled by ζ_3 . Since $Im(\zeta_1) = \omega_1 x$ and $Re(\zeta_1) = -2\lambda_1 \omega_1 t + \zeta_1^0$, thus the one-breather solution is only periodic along x direction and localized along t direction, and the period is $\frac{2\pi}{\omega_1}$. From the characteristic $\partial\zeta_3/\partial t = 0$ and $\partial\zeta_3/\partial x = i\omega_3$, the PLWs background is only periodic along x with the period $\frac{2\pi}{|\omega_3|}$. The same as the solitons on the PLWs background, the amplitudes

of the PLWs background in (29) are also determined by the parameters ζ_3^0 . Figure 7 displays the one-breather solution (29) to the NSB equation (1) with two different values of ζ_3 for $\zeta_3^0 = -\pi$ and $\zeta_3^0 = -\frac{\pi}{5}$. In panel (b), the peaks of the PLWs are higher than in panel (a), and the wave structures of the one-breather solution are swallowed by the PLWs.

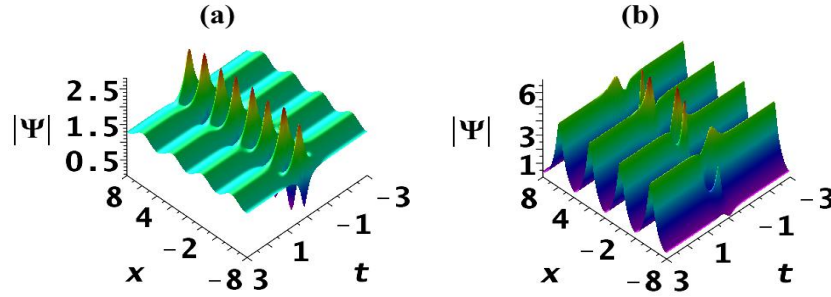


Fig. 7 – (Colour online) The one-breather solution (19) to the NSB equation (1) with a PLWs background with parameters $\omega_1 = 2, \lambda_1 = \frac{1}{12}\sqrt{-54 + 6\sqrt{177}}, \zeta_1^0 = 0, \omega_3 = \frac{1}{2}\sqrt{2 + 2\sqrt{17}}$: (a) $\zeta_3^0 = -\pi$; (b) $\zeta_3^0 = -\frac{\pi}{5}$.

With larger $N = 2\tilde{N} + 1$ ($\tilde{N} \geq 2$) in (24) and the input parameters following the restrictions in (25) and (26), \tilde{N} -breather solutions to the NSB equation (1) with PLWs background would be obtained. The \tilde{N} individual breathers interact with each other on the PLWs background, and the wavefronts of breathers in the near field where the breathers interact would change, and some novel wave patterns would form. For instance, we take $N = 5$ (*i.e.*, $\tilde{N} = 2$) in (24) and parameters satisfying

$$\begin{aligned} 48\lambda_j^4 + 8\lambda_j^2\omega_j^2 - \omega_j^4 + 4\lambda_j^2 + \omega_j^2 + 4 &= 0, \\ -\omega_5^4 + \omega_5^2 + 4 &= 0. \end{aligned} \quad (33)$$

for $j = 1, 2$. The two-breather solution on a PLWs background is derived. Figure 8 shows two types of the two-breather solution on a PLWs background for parameters

$$\begin{aligned} \omega_1 = 4, \lambda_1 = \frac{1}{12}\sqrt{-198 + 6\sqrt{3921}}, \omega_2 = 2, \lambda_2 = \frac{1}{12}\sqrt{-54 + 6\sqrt{177}}, \\ \omega_3 = \frac{1}{2}\sqrt{2 + 2\sqrt{17}}, \zeta_1^0 = 0, \zeta_5^0 = -\frac{2}{3}\pi, \end{aligned} \quad (34)$$

and two different values of ζ_2^0 . Here we take different values of ζ_2^0 to move the location of one breather determined by ζ_2 to observe the wave structures of the two breathers. In panel (a), the two breathers combine into one breather, which has a similar wave structure as shown in Fig. 7(a). However, the maximum value of the one-breather solution $|\Psi|$ shown in Fig. 8(a) can exceed 3.5, while the maximum

value of the one-breather solution illustrated in Fig. 7(a) stays below 3. In panel (b), the two breathers completely separate from each other, and more periodic wave structures appear in the PLWs background due to the interaction between the two breathers and the background PLWs.

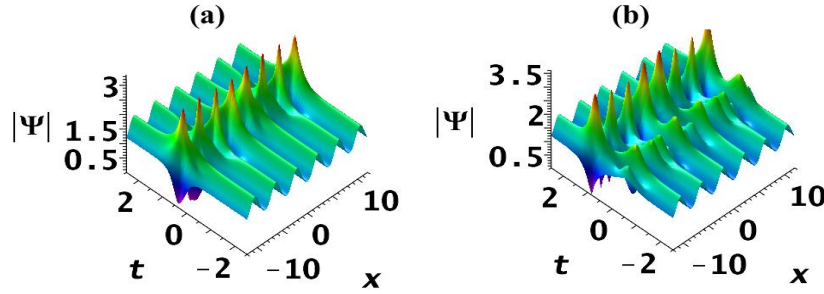


Fig. 8 – (Colour online) The two-breather solution (19) to the NSB equation (1) with a PLWs background with parameters given by (34) and (a) $\zeta_2^0 = -\pi$, (b) $\zeta_2^0 = -\frac{\pi}{6}$.

4. CONCLUSION

In this paper, we have studied the general solitons and breathers to the NSB equation (1) with a PLWs background *via* the bilinear KP-reduction method. Both of soliton and breather solutions on the PLWs background are expressed by $(2N + 1) \times (2N + 1)$ determinants. By deleting the $(2N + 1)$ th row and the $(2N + 1)$ th column in the determinant expressions of these solutions, the solutions reduce to soliton or breather solutions on a CW background. First, we have investigated the solitons in the NSB equation (1) with a PLWs background. The first-order soliton solutions are classified into non-degenerate two-soliton solutions and degenerate two-soliton solutions. The non-degenerate two-soliton solution has three distinct patterns: dark-dark soliton, antidark-antidark soliton, and dark-antidark soliton, while the degenerate case has two different patterns: degenerate two-dark-soliton solution and degenerate two-antidark-soliton solution. The three patterns of the non-degenerate two-soliton solution are displayed in Fig. 2, and the two patterns of degenerate cases are shown in Fig. 3. We have further investigated the four-soliton on a PLWs background, which have four distinct patterns: two degenerate two-soliton solutions (see Fig. 4), a non-degenerate four-soliton solution (see Fig. 5), and a mixture of a degenerate two-soliton solution and a non-degenerate two-soliton solution (see Fig. 6). Second, we have discussed the general breathers of the NSB equation (1) with a PLWs background. The first- and second-order breathers have been investigated, see Figs. 7 and 8.

The rational solutions to the NSB equation (1), including rational solitons and rational rogue waves, have not been reported for the NSB equation (1), to the best of our knowledge. A natural extension of the present paper might be the construction of rational solitons with either a CW background or a PLWs background, to the nonlocal NSB equation (1) *via* the bilinear KP-reduction method. We plan to report those results elsewhere.

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Appendix A

In this Appendix, we present the proofs of Theorem 2.1 and Theorem 3.1 based on the bilinear KP-reduction method. First, the NSB equation (1) is transformed into the following bilinear forms

$$\begin{aligned} (D_x^2 - iD_t)g \cdot f &= 0, \\ (D_x^4 + D_x^2 - 3D_t^2 - 2)f \cdot f &= -2gg^*(-x, t), \end{aligned} \quad (35)$$

via the variable transformation

$$\Psi = \sqrt{2} \frac{g(x, t)}{f(x, t)}, u = [2 \log f(x, t)]_{xx}, \quad (36)$$

where the function f in (36) is subject to the conjugate and nonlocal symmetry condition

$$f^*(-x, t) = cf(x, t), \quad (37)$$

where c is a constant.

For the purpose of deriving solutions to the bilinear equation of NSB equation (35), we start with the following three coupled $(3+1)$ -dimensional bilinear equations in the KP hierarchy

$$\begin{aligned} (D_{x_1}^2 - D_{x_2})\tau_{n+1} \cdot \tau_n &= 0, \\ (D_{x_1} D_{x_{-1}} - 2)\tau_n \cdot \tau_n &= -2\tau_{n+1}\tau_{n-1}, \\ (D_{x_1}^4 - 4D_{x_1} D_{x_3} + 3D_{x_2}^2)\tau_n \cdot \tau_n &= 0, \end{aligned} \quad (38)$$

which have following tau functions [56, 78–81]

$$\tau_n = \det_{1 \leq s, j \leq M} (m_{s, j}^{(n)}), \quad (39)$$

where

$$\begin{aligned} m_{s,j}^{(n)} &= a_{s,j} e^{-\xi_s - \eta_j} + \frac{p_s + r_s}{p_s + q_j} \left(-\frac{p_s}{q_j} \right), \\ \xi_s &= \frac{1}{p_s} x_{-1} + p_s x_1 + p_s^2 x_2 + p_s^3 x_3 + \xi_s^0, \\ \eta_j &= \frac{1}{q_j} x_{-1} + q_j x_1 - q_j^2 x_2 + q_j^3 x_3 + \eta_j^0, \end{aligned} \quad (40)$$

and p_s, q_j, r_s, ξ_s^0 and η_j^0 are complex numbers.

First, to construct soliton solutions given in Theorem 2.1, we take $M = 2N + 1$ in (39), and $p_s + r_s = 1$ and $\alpha_{s,j} = \delta_{s,j} b_s$ in (40) for $s, j = 0, 1, 2, \dots, 2N + 1$, where $\delta_{s,j}$ is the Kronecker delta ($\delta_{s,j} = 1$ when $s = j$ and $\delta_{s,j} = 0$ elsewhere). To restrict τ_n following the dimension reduction

$$(\partial_{x_1} + 4\partial_{x_3} - \partial_{x_{-1}})\tau_n = c\tau_n, \quad (41)$$

we take p_s and q_s following the condition

$$1 + 4(p_s^2 - p_s q_s + q_s^2) - \frac{1}{p_s q_s} = 0. \quad (42)$$

By adding the second bilinear equation to the third bilinear equation in (38), and using the dimension relation (41), the function τ_n (39) under parametric restriction (42) obeys the following two coupled $(1 + 1)$ -dimensional bilinear systems:

$$\begin{aligned} (D_{x_1}^2 - D_{x_2})\tau_{n+1} \cdot \tau_n &= 0, \\ (D_{x_1}^4 + D_{x_1}^2 + 3D_{x_2}^2 - 2)\tau_n \cdot \tau_n &= -2\tau_{n+1}\tau_{n-1}. \end{aligned} \quad (43)$$

Assuming x_1 real and x_2, b_{2N+1} pure imaginary, and the parameters following the restrictions

$$\begin{aligned} p_{N+s} &= -p_s, q_{N+s} = -q_s, q_s = p_s^*, \\ q_{2N+1} &= -p_{2N+1}^*, \xi_{N+s}^0 + \eta_{N+s}^0 = -(\xi_s^0 + \eta_s^0) = -\zeta_s^0, \end{aligned} \quad (44)$$

then one can obtain

$$\begin{aligned} m_{N+s, N+j}^{(n)*}(-x_1, t) &= -m_{j,s}^{(-n)}(x_1, t), m_{N+s, j}^{(n)*}(-x_1, t) = -m_{N+j, s}^{(-n)}(x_1, t), \\ m_{s, N+j}^{(n)*}(-x_1, t) &= -m_{j, N+s}^{(-n)}(x_1, t), m_{s, 2N+1}^{(n)*}(-x_1, t) = -m_{2N+1, N+s}^{(-n)}(x_1, t), \\ m_{2N+1, j}^{(n)*}(-x_1, t) &= -m_{N+j, 2N+1}^{(-n)}(x_1, t), m_{2N+1, 2N+1}^{(n)*}(-x_1, t) \\ &= -m_{2N+1, 2N+1}^{(-n)}(x_1, t), \end{aligned} \quad (45)$$

which imply

$$\tau_n^*(-x_1, t) = (-1)^{3N} \tau_{-n}(x_1, t). \quad (46)$$

The $(1+1)$ -dimensional bilinear equations (43) reduce to bilinear equations (35) of the NSB equation under independent variables

$$x_1 = x, x_2 = -it, \quad (47)$$

for $f = \tau_0$ and $g = \tau_1$. With $b_{2N+1} \rightarrow ib_{2N+1}$, we derive soliton solutions given in Theorem 2.1 to the NSB equation (1) with a PLWs background.

Second, to construct breather solutions given in Theorem 3.1, we take $M = 2N + 1$ in (39), and $r_s = q_s$ and $\alpha_{s,j} = \delta_{s,j}$ in (40) for $s, j = 1, 2, \dots, 2N + 1$. Then we take p_s and q_s following the condition

$$4(p_s^2 - p_s q_s + q_s^2) - \frac{1}{p_s q_s} - 1 = 0, \quad (48)$$

to restrict τ_n to the following dimension reduction

$$(4\partial_{x_3} - \partial_{x_{-1}} - \partial_{x_1})\tau_n = c\tau_n. \quad (49)$$

The $(3+1)$ -dimensional bilinear equations in (38) reduce to the following coupled $(1+1)$ -dimensional bilinear system:

$$\begin{aligned} (D_{x_1}^2 + D_{x_2})\tau_{n+1} \cdot \tau_n &= 0, \\ (D_{x_1}^4 - D_{x_1}^2 + 3D_{x_2}^2 - 2)\tau_n \cdot \tau_n &= -2\tau_{n+1}\tau_{n-1}. \end{aligned} \quad (50)$$

Assuming $\xi_s^0 + \eta_s^0$ real, and taking $q_s = p_s^*$ and independent variables $x_1 = ix, x_2 = it$, then the function τ_n (39) under restriction (48) satisfies the following conjugate and nonlocal symmetry condition

$$\tau_n^*(-x, t) = \tau_{-n}(x, t). \quad (51)$$

The bilinear equations in (50) reduce to bilinear equations (35) of the NSB equation with $\tau_0 = f, \tau_1 = g$. Finally, taking $p_s = \frac{\omega_s}{2} + i\lambda_s$ and $\xi_s^0 + \eta_s^0 = \zeta_s^0$, we obtain the breather solutions given in Theorem 3.1.

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