The Zeeman effect for hydrogen atom in twist-deformed space-time

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Abstract

In this article we find the Zeeman corrections for hydrogen atom in the case of twist-deformed space-time. Particularly, we derive the corresponding orbital and spin $\hat{g}$-factors as well as we notice, that the second one of them remains undeformed.
1 Introduction

The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2], [3] and String Theory models [4], [5], indicating that space-time at the Planck scale should be noncommutative, i.e., it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. [6], [7]) as well as with field theoretical models (see e.g. [8], [9]), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic [10] and nonrelativistic [11] symmetries, one can distinguish three basic types of space-time noncommutativity (see also [12] for details):

1) Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [13]-[15]

\[
[x_\mu, x_\nu] = i\theta^{\mu\nu},
\]  

2) Lie-algebraic modification of classical space-time [15]-[18]

\[
[x_\mu, x_\nu] = i\theta_{\mu\rho}^\nu x_\rho,
\]  

and

3) Quadratic deformation of Minkowski and Galilei spaces [15], [18]-[20]

\[
[x_\mu, x_\nu] = i\theta_{\mu\rho}^{\sigma} x_\rho x_\tau,
\]  

with coefficients $\theta^{\mu\nu}$, $\theta_{\mu\rho}^\nu$ and $\theta_{\mu\rho}^{\sigma}$ being constants.

Moreover, it has been demonstrated in [12], that in the case of the so-called N-enlarged Newton-Hooke Hopf algebras $U_0^{(N)}(NH_{\pm})$ the twist deformation provides the new space-time noncommutativity of the form\textsuperscript{1,2}

\[
4) \quad [t, x_i] = 0, \quad [x_i, x_j] = i f_\pm \left( \frac{t}{\tau} \right) \theta_{ij}(x),
\]  

with time-dependent functions

\[
f_+ \left( \frac{t}{\tau} \right) = f \left( \sinh \left( \frac{t}{\tau} \right), \cosh \left( \frac{t}{\tau} \right) \right), \quad f_- \left( \frac{t}{\tau} \right) = f \left( \sin \left( \frac{t}{\tau} \right), \cos \left( \frac{t}{\tau} \right) \right),
\]

$\theta_{ij}(x) \sim \theta_{ij} = \text{const}$ or $\theta_{ij}(x) \sim \theta_{ij}^k x_k$ and $\tau$ denoting the time scale parameter - the cosmological constant. Besides, it should be noted, that the above mentioned quantum

\textsuperscript{1}x_0 = ct.

\textsuperscript{2}The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [13], [14]), for the quantum N-enlarged Newton-Hooke Hopf algebras.
spaces 1), 2) and 3) can be obtained by the proper contraction limit of the commutation relations 4)\(^3\).

In this article we investigate the impact of the twisted N-enlarged Newton-Hooke space-time [12]\(^4\)

\[
[\hat{x}_1, \hat{x}_2] = i f_{\kappa_a}(t), \tag{5}
\]
on the spectrum of hydrogen atom in a weak magnetic field, with the electron spin taken into account [21]. Particularly, we write the energy operator describing the anomalous Zeeman effect for the nonrelativistic quantum space (5) and further, assuming that the function \(f_{\kappa_a}(t)\) is small, we find the corrections to the so-called orbital and spin \(\hat{g}\)-factors for the weak external magnetic field \(\vec{B}\). In such a way we demonstrate that the second one of them remains undeformed for any value of deformation parameter \(\kappa_a\).

The paper is organized as follows. In second Section we remaind the basic facts concerning Zeeman effect for hydrogen atom with the electron spin taken into account. In Section 3 we analyze the Zeeman anomaly for quantum space-times (5). Particularly, we find the first-ordered, time-dependent corrections to the spectrum of weak magnetic field as well as, we provide the twist-deformed orbital and spin \(\hat{g}\)-factors for electron respectively. The discussion and final remarks are presented in the last Section.

2 The Zeeman effect for hydrogen atom

In this Chapter we shortly remaind the basic facts concerning Zeeman effect for hydrogen atom with spin electron taken into account, in the case of classical space-time [21]. For this purpose, we start with the proper Hamiltonian function\(^5\)

\[
H(\vec{p}, r) = \frac{\vec{p}^2}{2m} + V_C(r) + V_{LS}(r) + H_{BS} = H_0 + V_{LS}(r) + H_{BS}, \tag{6}
\]
which contains Coulomb and spin-orbit potentials as well as the spin magnetic-momenta interaction term, given by

\[
V_C(r) = -\frac{Ze^2}{r}, \tag{7}
\]
\[
V_{LS}(r) = \frac{1}{2m^2c^2} \frac{1}{r} \frac{dV_c}{dr} (\vec{L} \cdot \vec{S}), \tag{8}
\]
and

\[
H_{BS} = -\frac{e}{mc} \vec{B} \cdot \vec{S}, \tag{9}
\]

\(^3\)Such a result indicates that the twisted N-enlarged Newton-Hooke Hopf algebra plays a role of the most general type of quantum group deformation at nonrelativistic level.

\(^4\)In the formula (5) \(\kappa_a\) denotes the deformation parameter such that \(\lim_{\kappa_a \to 0} f_{\kappa_a}(t) = 0\).

\(^5\)In the above formulas \(m\) and \(e\) denote the mass and charge of the electron respectively, while \(c\) is the speed of light.
respectively. One can check that after the minimal coupling substitution\(^6\)
\[
\bar{p} \to \bar{p}' = \bar{p} - \frac{e}{c} \mathbf{A},
\]
the above energy operator takes the form\(^7\)
\[
H = \frac{\bar{p}^2}{2m} - \frac{Ze^2}{r} - \frac{e}{2mc} (\bar{p} \cdot \mathbf{A} + \mathbf{A} \cdot \bar{p}) + \frac{e^2 \mathbf{A}^2}{2mc^2} - \frac{e}{mc} \mathbf{B} \cdot \mathbf{S} + \frac{(2m^2c^2)^{-1} r^{-1} V_C'(r) (\mathbf{L} \cdot \mathbf{S})}{8mc^2},
\]
while for the magnetic induction chosen into the positive \(x_3\)-direction (\(\mathbf{B} = B \hat{e}_3\)), it looks as follows
\[
H = \frac{\bar{p}^2}{2m} + \frac{e^2 B^2}{8mc^2} (x_1^2 + x_2^2) - \frac{e}{2mc} BL_3 - \frac{e}{mc} BS_3 - \frac{Ze^2}{r} + \frac{(2m^2c^2)^{-1} r^{-1} V_C'(r) (\mathbf{L} \cdot \mathbf{S})}{8mc^2},
\]
Hence, we see, that the total Hamiltonian of the system (12) is made up of the four dynamical terms, such that
\[
H_C(r) = \frac{p_3^2}{2m} - \frac{Ze^2}{r},
\]
\[
H_O(r) = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{e^2 B}{8mc^2} (x_1^2 + x_2^2),
\]
\[
H_B = -\frac{e}{2mc} B (L_3 + 2S_3),
\]
and
\[
H_{LS}(r) = (2m^2c^2)^{-1} r^{-1} V_C'(r) (\mathbf{L} \cdot \mathbf{S}) = (2m^2c^2)^{-1} \frac{Ze^2}{r^3} (\mathbf{L} \cdot \mathbf{S}).
\]
First of them describes the electron in the presence of Coulomb field, while the second one corresponds to the electron moving perpendicularly in the standard, two-dimensional harmonic oscillator potential. Besides, the formulas (15) and (16) encode the Hamiltonian for the anomalous Zeeman effect and spin-orbit interaction terms respectively.

It is well-known, that the first-order corrections to the energy spectrum of the weak-field Zeeman anomaly can be found with use of the stationary quantum mechanical perturbation theory [21]; they are given by
\[
\Delta E_{\text{weak}} = \langle n, l, j, m_j | H_B | n, l, j, m_j \rangle = -\frac{eB}{2mc} m_j \left[ 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} \right] = -\frac{eB}{2mc} m_j g,
\]
in an appropriate basis set \(|n, l, j, m_j\rangle\) defined by the principal \((n)\), orbital \((l)\), spin \((s)\) and total \((j = l \pm \frac{1}{2})\) quantum numbers. Besides, it should be noted, that the present in the above formula symbol \(g\) denotes so-called Landé factor, which for \(j = l\) and \(s = 0\) produces the orbital \(g\)-factor \(g_l = 1\), while for \(j = s\) and \(l = 0\), it gives the spin Landé factor \(g_s = 2\).
3 The twist-deformed Zeeman effect for hydrogen atom

Let us now turn to the main aim of our investigations, i.e., to analyze of Zeeman anomaly for quantum space-times (5). In first step of our construction, we extend the twisted space to the whole algebra of momentum and position operators as follows

\[
[ \hat{x}_1, \hat{x}_2 ] = i f = 0 = [ \hat{x}_i, \hat{x}_3 ] , \quad [ \hat{x}_i, \hat{p}_j ] = i \hbar \delta_{ij} .
\] (18)

One can check that the above relations satisfy the Jacobi identity and for deformation parameter \( \kappa_a \) approaching zero become classical.

Next, we define the proper Hamiltonian operator in a standard way by

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{e^2 B^2 (\hat{x}_1^2 + \hat{x}_2^2)}{8mc^2} - \frac{e}{2mc} BL_3 - \frac{e}{mc} BS_3 - \frac{Ze^2}{\hat{r}} + \frac{1}{2m} \frac{eB^2}{2mc} (\hat{x}_1^2 + \hat{x}_2^2) + \frac{1}{2} V'(\hat{r}(\hat{x}, \hat{p})) \hat{r} + \frac{1}{2} (\hat{L} \cdot \hat{S}) ,
\] (19)

and in order to perform the basic analyze of the system, we represent the noncommutative variables \((\hat{x}_i, \hat{p}_i)\) by the classical ones \((x_i, p_i)\) as (see e.g. [22]-[24])

\[
\hat{x}_1 = x_1 - f = \frac{eB^2}{8mc^2} (\hat{x}_1^2 + \hat{x}_2^2),
\]

\[
\hat{x}_2 = x_2 + f = \frac{eB^2}{8mc^2} (\hat{x}_1^2 + \hat{x}_2^2),
\]

\[
\hat{x}_3 = x_3 , \quad \hat{p}_i = p_i ,
\]

(20)\(\hat{x}_1 \hat{p}_i - \hat{p}_i \hat{x}_1 = i \hbar \delta_{ij} .
\] (23)

Then, the Hamiltonian (19) takes the form

\[
\hat{H}(t) = \frac{1}{2m(t)} (p_1^2 + p_2^2) + \frac{p_3^2}{2m} + \hat{m}(t) \hat{\omega}^2(t)/2 (x_1^2 + x_2^2) - \left[ S(t) + \frac{eB}{2mc} \right] L_3 + \frac{1}{2} V'(\hat{r}(\hat{x}, \hat{p})) \hat{r} + (2m^2 c^2)^{-1} V_c(\hat{r}(\hat{x}, \hat{p}))\hat{r} + \frac{1}{2} (\hat{L} \cdot \hat{S}) ,
\] (24)

with time-dependent mass and frequency given by

\[
\hat{m}(t) = \frac{m}{1 + e^2 B^2 f^2(\hat{r}(\hat{x}, \hat{p})) / (64c^2 \hbar^2} ,
\]

\[
\hat{\omega}(t) = \frac{eB}{2mc} \left[ 1 + \frac{e^2 B^2 f^2(\hat{r}(\hat{x}, \hat{p}))}{64c^2 \hbar^2} \right]^{1/2} ,
\]

\[
S(t) = \frac{\hat{m}(t) \hat{\omega}^2(t) f(\hat{r}(\hat{x}, \hat{p}))}{4\hbar} L_3 ,
\]

(25)\(\hat{m}(t) \hat{\omega}^2(t) f(\hat{r}(\hat{x}, \hat{p})) = \frac{1}{4\hbar} L_3 .
\] (27)

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Note: We define the Hamiltonian operator by replacement in the formula (12) the classical operators \((x_i, p_i)\) by their noncommutative counterparts \((\hat{x}_i, \hat{p}_i)\).
respectively. Besides, by tedious calculations one can check, that for small values of deformation function \( f_{\kappa_a}(t) \), the total energy operator (24) is made up from the following four terms

\[
\hat{H}_C(r) = \frac{p_3^2}{2m} - \frac{Ze^2}{r}, \tag{28}
\]

\[
\hat{H}_O(r) = \frac{1}{2m} \left( p_1^2 + p_2^2 \right) + \frac{e^2B^2}{8mc^2} \left( x_1^2 + x_2^2 \right) + \mathcal{O}(f_{\kappa_a}^2(t)), \tag{29}
\]

\[
\hat{H}_B(t) = A(t)L_3 - eB/(mc)S_3 + \mathcal{O}(f_{\kappa_a}^2(t)), \tag{30}
\]

\[
\hat{H}_{LS}(r, t) = (2m^2c^2)^{-1}(Ze^2r^{-3}) \left[ 1 + (3f_{\kappa_a}(t)/4h)L_3r^{-2} \right] (\bar{L} \cdot \bar{S}) + \mathcal{O}(f_{\kappa_a}^2(t)), \tag{31}
\]

\[
+ \mathcal{O}(f_{\kappa_a}^2(t)) \tag{32}
\]

where symbol \( \mathcal{O}(f_{\kappa_a}^2(t)) \) denotes the elements at least quadratic in parameter \( \kappa_a \), while the time-dependent coefficient \( A(t) \) is given by

\[
A(t) = -S(t) - \frac{Zf_{\kappa_a}(t)}{4\hbar r^3} - \frac{eB}{2mc}. \tag{33}
\]

Of course, in \( f_{\kappa_a}(t) \) approaching zero limit the above formulas become the same as classical ones (13)-(16).

In the next step of our construction one should observe, that formally, the first-order Zeeman anomaly corrections to hydrogen spectrum in a week external magnetic field can be written as\(^9\)

\[
E_n^{(1)}(t) = \Delta \hat{E}_{\text{weak}}(t) = \langle n, l, j, m_j | \hat{H}_B(t) | n, l, j, m_j \rangle, \tag{34}
\]

and due to the relations [21]

\[
\langle L_3 \rangle = \hbar m_j \left[ \frac{j(j+1) - s(s+1) + l(l+1)}{2j(j+1)} \right], \tag{35}
\]

\[
\langle S_3 \rangle = \hbar m_j \left[ \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} \right], \tag{36}
\]

they take the form

\[
\Delta \hat{E}_{\text{weak}}(t) = -\frac{eB\hbar}{2mc}m_j \left[ \left\langle \frac{2mc}{eB} \left( S(t) + \frac{Zef_{\kappa_a}(t)}{4\hbar} \left\langle \frac{1}{r^3} \right\rangle + 1 \right) \right\rangle \right. \times
\]

\[
\times \left( \frac{j(j+1) - s(s+1) + l(l+1)}{2j(j+1)} \right) + \left. \right\rangle \frac{2j(j+1)}{2j(j+1)} + \left. 2 \left( \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} \right) \right) \right \} = -\frac{eB\hbar}{2mc}m_j \hat{g}(t), \tag{37}
\]

\[
E_n(t) = E_n^{(0)} + \lambda E_n^{(1)}(t) + \cdots, \quad |n, l, j, m_j \rangle(t) = |n, l, j, m_j \rangle + \lambda |n, l, j, m_j \rangle^{(1)}(t) + \cdots,
\]

where \( H_0 |n, l, j, m_j \rangle = E_n^{(0)} |n, l, j, m_j \rangle \).

\(^9\)In our calculation of formula (34) we consider the following perturbation of eigenvalues \( E_n(t) \) and eigenvectors \( |n, l, j, m_j \rangle(t) \) respectively
with $\langle \frac{1}{r^3} \rangle = \frac{Z^3}{l(l+1/2)(l+1)n^3a_0^3}$ being a Kramer’s recursive relation, $a_0 = \frac{\hbar^2}{me^2}$, and with $\hat{g}(t)$ denoting the Landé factor for twist-deformed space-time. Unfortunately, the first term in the above formula remains indeterminate due to the singularity of expectation value $\langle r^{-3} \rangle$ for $l = 0$. In order solve this problem one should exchange (in accordance with articles [25] and [26]) the Kramer’s factor to the following one: $\langle r^{-3} (1 - a_0\alpha^2/2r) \rangle$ with $\alpha = e^2/(\hbar c)$. Then, the pure orbital moment Landé factor can be get from (37) by taking $j = l$ and $s = 0$, and it looks as follows

$$\hat{g}_l(t) = 2mc \left( S(t) + \frac{Zef_{\kappa a}(t)}{4\hbar} \langle r^{-3} (1 - a_0\alpha^2/2r) \rangle \right) + 1 ,$$

(39)

while for $f_{\kappa a}(t) = 0$, it reproduces the commutative $g$-factor $g_l = 1$. Besides, by putting $j = s$ and $l = 0$ in (37) we obtain the pure spin moment of the form

$$\hat{g}_s(t) = 2 = g_s ,$$

(40)

what finish our considerations.

4 Final remarks

As it was already mentioned in Introduction, in this article, we investigate the impact of twisted space-time (5) on the energy spectrum of hydrogen atom in a weak magnetic field, with the electron spin taken into account. Particularly, we find the linear (with respect the noncommutativity function $f_{\kappa a}(t)$) corrections to the orbital and spin Zeeman $\hat{g}$-factors (39) and (40) respectively. Surprisingly, we notice that the second one of them remains undeformed.

Of course, the presented and discussed in this article results can be generalized to the case of arbitrary (for example much more strong) magnetic field $B$ as well. Unfortunately, the studies in this direction seem to be very complicated and for this reason they are postponed for further investigations.

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References


Particularly, for $f_{\kappa a}(t) = \kappa_a = \theta$ we obtain the result for canonically deformed space-time [13]-[15].
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