ON THE QUANTUM-MECHANICAL SCATTERING PROBLEM
IN THE LOBACHEVSKY SPACE

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The formulation of the quantum-mechanical scattering problem in the three-dimensional Lobachevsky space is considered with a potential well as model potential. The analog of the plane wave in the three-dimensional Lobachevsky space is used for description of the incident wave. The graphical solution of the problem of bound s-states is given. The influence of curvature on the number of the bound states is investigated.

1. INTRODUCTION

Quantum-mechanical problems in spaces of constant positive and negative curvature are the object of interest of researchers since 1940, when Schrödinger [1] was the first to solve the quantum-mechanical problem of the atom on the three-dimensional sphere (Einstein’s universe). The analogous problem in the three-dimensional Lobachevsky space was first solved by Infeld and Shild [2]. In recent years the quantum-mechanical models based on the geometry of spaces of constant curvature have attracted considerable attention due to their interesting mathematical features [3, 4, 6] as well as the possibility of applications to physical problems [5]. For example, these models are used for the description of the bound states in nuclear and elementary particle physics [3]. Thus, Kepler problem on the sphere $S_3$ has been used as a model for description of quarkonium spectra [7]. Kepler–Coulomb problem on the sphere $S_3$ has been used as a model for description of excitations in quantum dots [8, 9]. Many aspects of this problem in spaces of constant curvature, in particular separation of variables and path integral formulation, have been investigated in the papers [10–12]. However, until now, the problem of scattering in spaces of a constant curvature was not formulated and solved with the use of an algebraic approach.

The important problem with the formulation of the scattering problem in the three-dimensional Lobachevsky space was the choice of an expression for the incident wave. The use of Shapiro’s plane wave related to the representations of the group of motion of Lobachevsky’s space, made it possible to formulate

and solve the scattering problem on the Coulomb center [13]. In this paper the formulation of the quantum-mechanical scattering problem in the three-dimensional Lobachevsky space is considered with potential well as model potential. The graphical solution of the problem of bound s- states is given. The influence of curvature on the number of the bound states is investigated.

2. FORMULATION OF THE PROBLEM

We use embedding of the Lobachevsky space in 4-dimensional pseudoeuclidean space with coordinates \( x_\mu, \mu = 1, 2, 3, 4 \), given by the formula

\[
x_\mu x_\mu = x_1^2 + x_2^2 = -\rho^2, \quad x = \{x_1, x_2, x_3\}, \quad x_4 = ix_0.
\]

(1)

The Schrödinger equation is (\( h = m = 1 \)):

\[
H \Psi = E \Psi, \quad H = \frac{1}{4\rho^2} M_{\mu\nu} M_{\mu\nu} + U, \quad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu,
\]

(2)

where \( U \) is the potential energy.

The scattering solution of the Schrödinger equation behaves at large distances like the superposition of an incident wave and scattered spherical waves. In the flat space the plane wave is considered as the incident wave. In the Lobachevsky space the Schrödinger equation does not have plane wave solutions. The solution of the free particle Schrödinger equation in the closest form to the plane wave is (see [14, 15])

\[
\xi(x, n) = \left( \frac{x_0 - xn}{\rho} \right)^{1-\eta}, \quad \eta = \sqrt{2E\rho^2} - 1,
\]

(3)

where \( n \) is a unit vector that defines the direction of wave propagation in the Lobachevsky space.

The spherical wave is considered as the scattered wave. In flat space this is the outgoing wave, having the form \( f(\theta) \exp(ikr)/r \) at large distances \( r \) from the center. In the Lobachevsky space the Schrödinger equation also has solutions of the form of spherical wave. These solutions can be found by using spherical coordinates

\[
x_0 = \rho \cosh \beta, \quad x_1 = \rho \sinh \beta \sin \theta \cos \varphi,
\]

\[
x_2 = \rho \sinh \beta \sin \theta \sin \varphi, \quad x_3 = \rho \sinh \beta \cos \theta,
\]

\[
0 \leq \beta < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi.
\]

(4)

Separating in the solution of the Schrödinger equation the dependence on the angles \( \theta \) and \( \varphi \) by using spherical harmonics, namely \( \Psi = R(\beta)Y^m_l(\theta, \varphi) \), we
obtain in the case of $U = 0$ the radial equation
\[
\left[ \frac{1}{2\rho^2} \left( -\frac{1}{\sinh^2 \beta} \frac{d}{d\beta} \sinh \beta \frac{d}{d\beta} + \frac{l(l+1)}{\sinh^2 \beta} \right) - E \right] R_j(\beta) = 0. \tag{5}
\]

The regular solution for $\beta = 0$ of this equation is
\[
S_j(\beta) = \frac{\pi}{\sqrt{2\sinh \beta}} \Gamma(i\eta + l + 1) \frac{1}{\Gamma(i\eta + l + 1)} P_{\frac{1}{2} + i\eta} \left( \cosh \beta \right). \tag{6}
\]

The asymptotic form of the solution $S_{j\ell}$ for $\beta \to \infty$ is given by the expression
\[
S_{j\ell}(\beta) \approx \frac{1}{2i\eta \sinh \beta} \left[ e^{i\eta \beta} - \frac{\Gamma(i\eta + l + 1) \Gamma(1 - \eta) \cosh \beta}{\Gamma(l - i\eta + 1) \Gamma(i\eta + 1)} e^{-i\eta \beta} \right]. \tag{7}
\]

The solution of equation (5) with outgoing spherical wave in the Lobachevsky space is described by
\[
C_{j\ell}(\beta) = \frac{\pi}{\sqrt{2\sinh \beta}} \frac{1}{2} \left[ \Gamma(i\eta + l + 1) \frac{1}{\Gamma(i\eta + 1)} P_{\frac{1}{2} + i\eta} \left( \cosh \beta \right) \right.
\]
\[
\left. + \frac{\Gamma(i\eta - l - 1) \Gamma(l + 1) \cosh \beta}{\Gamma(i\eta + 1) \Gamma(i\eta + 1)} \right]. \tag{8}
\]

When $\beta \to \infty$ we have
\[
C_{j\ell}(\beta) \approx \frac{1}{2i\eta \sinh \beta} e^{i\eta \beta}. \tag{9}
\]

We choose vector $\mathbf{n}$ in (3) in the form $\mathbf{n} = (0,0,1)$. Then the incident wave can be written as
\[
\xi(\beta, \theta) = (\cosh \beta - \sinh \beta \cos \theta)^{-1 - i\eta}. \tag{10}
\]

The incident wave (11) can be expressed through the spherical waves (6) (see, for example, [14, 15]) as
\[
\xi(\beta, \theta) = \sum_{l=0}^{\infty} (2l+1) S_{jl}(\beta) P_l(\cos \theta). \tag{11}
\]

The exact wave function which is the solution of the Schrödinger equation with the potential energy $U(\beta)$ takes for $\beta \to \infty$ the form
\[
\Psi \approx (\cosh \beta - \sinh \beta \cos \theta)^{-1 - i\eta} + \frac{f(\theta)}{\rho \sinh \beta} e^{i\eta \beta}. \tag{12}
\]

Here the function $f(\theta)$ plays the role of the scattering amplitude.
3. SCATTERING BY SPHERICAL POTENTIAL WELL

As an example let us consider particle scattering on the spherical symmetrical potential well. Let us assign the potential of the well as follows

\[ U = \begin{cases} 
0 & \text{for } \beta \geq \alpha, \\
-U_0 & \text{for } \beta \leq \alpha, 
\end{cases} \]

where constant \( \alpha \) is the radius of the well.

As the incident wave we will consider the wave of form (3). Let us denote by \( \varphi(\beta, \theta) \) the wave for \( \beta \leq \alpha \) and \( \chi(\vec{x}, \vec{n}) \) the scattered wave.

We use for the incident wave expansion (12). The wave inside of the potential well can be expressed as

\[ \varphi(\beta, \theta) = \sum_{l=0}^{\infty} A_l S_{\eta l}(\beta) P_l(\cos \theta), \quad (13) \]

where

\[ \eta' = \sqrt{2(E + U_0)\rho^2 - 1}. \quad (14) \]

The scattered wave is

\[ \chi(\beta, \theta) = \sum_{l=0}^{\infty} B_l C_{\eta l}(\beta) P_l(\cos \theta). \quad (15) \]

The expansion coefficients \( A_l \) and \( B_l \) can be determined from the continuity condition of wave function and its derivative at the boundary of the well \( \beta = \alpha \), which reduces to the following system of linear equations:

\[ \begin{align*}
& p_{\eta l}(\cosh \alpha) + q_{\eta l}(\cosh \alpha)B_l = p_{\eta l}(\cosh \alpha)A_l \\
& p'_{\eta l}(\cosh \alpha) + q'_{\eta l}(\cosh \alpha)B_l = p'_{\eta l}(\cosh \alpha)A_l,
\end{align*} \quad (16) \]

where we introduced the following notations

\[ p_{\eta l}(\cosh \beta) = \sqrt{2\sinh \beta \pi} S_{\eta l}, \]
\[ q_{\eta l}(\cosh \beta) = \sqrt{2\sinh \beta \pi} C_{\eta l} = \frac{1}{2}(p_{\eta l}(\cosh \beta) + p_{\eta, -l-1}(\cosh \beta)) \]
and

\[ p'_{\eta l}(\cosh \beta) = \frac{dp_{\eta l}(\cosh \beta)}{d\beta}, \quad q'_{\eta l}(\cosh \beta) = \frac{dq_{\eta l}(\cosh \beta)}{d\beta}. \]

As a result, we obtain for the coefficients \( B_l \) and \( A_l \) the following expressions.
Quantum-mechanical scattering in the Lobachevski space

\[ B_l = \frac{p_{nl}'(\cosh \alpha)p_{nl}^*(\cosh \alpha) - p_{nl}''(\cosh \alpha)p_{nl}'(\cosh \alpha)}{q_{nl}(\cosh \alpha)p_{nl}'(\cosh \alpha) - p_{nl}'(\cosh \alpha)q_{nl}'(\cosh \alpha)}, \]  
\[ (17) \]

and

\[ A_l = \frac{q_{nl}(\cosh \alpha)p_{nl}'(\cosh \alpha) - q_{nl}'(\cosh \alpha)p_{nl}(\cosh \alpha)}{q_{nl}(\cosh \alpha)p_{nl}'(\cosh \alpha) - p_{nl}'(\cosh \alpha)q_{nl}'(\cosh \alpha)}. \]  
\[ (18) \]

Thus for the scattering amplitude we have:

\[ f(\theta) = \frac{P}{2i\hbar} \sum_{l=0}^{\infty} B_l P_l(\cos \theta) \]  
\[ (19) \]

The poles of \( B_l \) in the range of negative energies determine the bound states in the well.

### 4. THE CASE OF THE s-STATES

In particular, when \( l = 0 \) we have

\[ \sqrt{k^2 - 1} \cot(\alpha \sqrt{k^2 - 1}) = -\sqrt{\kappa^2 + 1} = -\sqrt{2\rho^2 U_0 - k^2 + 1}, \]  
\[ (20) \]

where

\[ \kappa = \sqrt{2\rho^2 \epsilon}, \quad k = \sqrt{2\rho^2 (U_0 - \epsilon)}, \quad \epsilon = -E \]  
\[ (21) \]

This equation determines the energy levels of the system.

Let us introduce the variables

\[ \zeta = \alpha \sqrt{k^2 - 1} \geq 0 \]

and

\[ \gamma = \alpha \sqrt{\kappa^2 + 1} \geq 0. \]

Then we obtain

\[ \gamma = -\zeta \cot \zeta, \quad \gamma^2 + \zeta^2 = 2\rho^2 U_0 \alpha^2. \]  
\[ (22) \]

The equations (23) can be solved numerically or graphically. The values \( \zeta \) and \( \gamma \), which satisfy equations (23) are determined by the points of intersections of the curve \( \gamma = -\zeta \cot \zeta \) with the circle of radius \( \rho \alpha \sqrt{2U_0} \).

The curves are represented on the Figs. 1–3. We see that there are such values of curvature for which no stationary states exist. But with the increase of the curvature radius \( \rho \) bound states appear, the number of which rises with the increase of \( \rho \). Also from Figs. 1–3 we can observe that with the growth of \( U_0 \), the number of bound states increases.
Fig. 1. – The graphical solution of equations $\gamma = -\zeta \cot \zeta$ and $\gamma^2 + \zeta^2 = 2\rho^2 U_0 \alpha^2$, for $\alpha = 0.005$, $U_0 = 1$, $\rho = 50, 100, 250, 400, 1000$.

Fig. 2. – The graphical solution of equations $\gamma = -\zeta \cot \zeta$ and $\gamma^2 + \zeta^2 = 2\rho^2 U_0 \alpha^2$, for $\alpha = 0.005$, $U_0 = 10$, $\rho = 50, 100, 250, 400, 1000$. 
Table 1
The number of bound states dependence on the radius of space curvature for $\alpha = 0.005$, $U_0 = 1$

<table>
<thead>
<tr>
<th>Value $\rho$</th>
<th>Value $\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 50$</td>
<td>$\epsilon_1 = 0.0289$</td>
</tr>
<tr>
<td>$\rho = 100$</td>
<td>$\epsilon_1 = 0.3772$</td>
</tr>
<tr>
<td>$\rho = 250$</td>
<td>$\epsilon_1 = 0.4142, \epsilon_2 = 0.8495$</td>
</tr>
<tr>
<td>$\rho = 400$</td>
<td></td>
</tr>
<tr>
<td>$\rho = 1000$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2
The number of bound states dependence on the radius of space curvature for $\alpha = 0.005$, $U_0 = 10$

<table>
<thead>
<tr>
<th>Value $\rho$</th>
<th>Value $\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 50$</td>
<td>$\epsilon_1 = 1.8628$</td>
</tr>
<tr>
<td>$\rho = 100$</td>
<td>$\epsilon_1 = 1.5990, \epsilon_2 = 7.7560$</td>
</tr>
<tr>
<td>$\rho = 250$</td>
<td>$\epsilon_1 = 1.4912, \epsilon_2 = 6.0718, \epsilon_3 = 9.0054$</td>
</tr>
<tr>
<td>$\rho = 400$</td>
<td>$\epsilon_1 = 1.35, \epsilon_2 = 3.5775, \epsilon_3 = 5.5154, \epsilon_4 = 7.12, \epsilon_5 = 8.3763, \epsilon_6 = 9.2773, \epsilon_7 = 9.8192$</td>
</tr>
<tr>
<td>$\rho = 1000$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3
The number of bound states dependence on the radius of space curvature for $\alpha = 0.005$, $U_0 = 100$

<table>
<thead>
<tr>
<th>Value $\rho$</th>
<th>Value $\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 50$</td>
<td>$\epsilon_1 = 54.0462$</td>
</tr>
<tr>
<td>$\rho = 100$</td>
<td>$\epsilon_1 = 41.4281, \epsilon_2 = 84.9497$</td>
</tr>
<tr>
<td>$\rho = 250$</td>
<td>$\epsilon_1 = 2.6466, \epsilon_2 = 30.6285, \epsilon_3 = 55.1936, \epsilon_4 = 74.6678, \epsilon_5 = 97.1223$</td>
</tr>
<tr>
<td>$\rho = 400$</td>
<td>$\epsilon_1 = 8.8385, \epsilon_2 = 27.3263, \epsilon_3 = 44.1024, \epsilon_4 = 71.3362, \epsilon_5 = 81.6279, \epsilon_6 = 89.6548, \epsilon_7 = 95.3989, \epsilon_8 = 98.8492$</td>
</tr>
<tr>
<td>$\rho = 1000$</td>
<td>$\epsilon_1 = 7.98, \epsilon_2 = 15.983, \epsilon_3 = 23.6908, \epsilon_4 = 31.0612, \epsilon_5 = 38.0774, \epsilon_6 = 51.0143, \epsilon_7 = 9.8192, \epsilon_8 = 56.9256, \epsilon_9 = 62.4226, \epsilon_{10} = 67.6222, \epsilon_{11} = 72.4034, \epsilon_{12} = 76.8051, \epsilon_{13} = 80.8262, \epsilon_{14} = 84.4661, \epsilon_{15} = 87.8224, \epsilon_{16} = 90.5999, \epsilon_{17} = 93.093, \epsilon_{18} = 95.2029, \epsilon_{19} = 96.9264, \epsilon_{20} = 98.2728, \epsilon_{21} = 99.2323, \epsilon_{22} = 99.808$</td>
</tr>
</tbody>
</table>
Fig. 3. – The graphical solution of equations \( \gamma = -\zeta \cot \zeta \) and \( \gamma^2 + \zeta^2 = 2\rho^2 U_0 \alpha^2 \),
for \( \alpha = 0.005, U_0 = 100, \rho = 50, 100, 250, 400, 1000 \).

REFERENCES