CLASSIFICATION OF THE NATURAL MODES OF A COUPLED CHANNEL MODEL IN THE FRAMEWORK OF THE RIEMANN SURFACE APPROACH

N. GRAMA, C. GRAMA, I. ZAMFIRESCU
National Institute for Nuclear Physics and Engineering, P.O. Box MG-6, RO-077125 Bucharest-Magurele, Romania, E-mail: grama@theory.nipne.ro
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By using the Riemann surface approach to the natural modes of a coupled channel model a S-matrix pole and the associated mode are classified by introducing new quantum numbers with topological meaning, namely the labels of the Riemann sheet and of its k-plane image to which the pole belongs. The Riemann surface approach allows not only studying each state separately, but also understanding the jump from a state to another state. As a result a new insight into the nature of the Feshbach and Fonda-Newton resonant states is obtained.

1. INTRODUCTION

The natural modes of the particle + potential system are solutions $\phi_m(\vec{r})$ of the Schrödinger equation

$$\Delta \psi + \left( k^2 - V(\vec{r}) \right) \psi = 0,$$

that are regular functions in $\mathbb{R}^3$ and satisfy the radiation condition for $r \to \infty$ [1]. By incorporating the radiation condition into the Schrödinger equation the homogeneous Lippmann-Schwinger equation is obtained

$$\phi_m(\vec{r}) = \int d^3 r' G^{(+)}(\vec{r}, r') V(r') \phi_m(r'),$$

whose solutions $\phi_m$ which are well behaved throughout the whole space and are outgoing to infinity define the natural modes. The natural modes may be either radiative (in which case the energy $E$ is complex) or nonradiative (in which case $E$ is real). We identify the radiative natural modes with resonant states and the nonradiative ones with bound states. For a central potential of strength $g$, i.e. $gV(r)$, by using the partial wave expansions of the wave and Green functions we obtain from Eq. 2

$$\mathcal{F}_l(g, k) = W(\phi_l(r), f_l^{(+)}(r)) = 0,$$

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where $W$ is the Wronskian of the regular solution $\phi_l(r)$ and of the irregular solution $f_l^{(-)}(r)$ of the radial Schrödinger equation which satisfies the radiation condition. Eq. 3 defines the natural modes $k_l(g)$. Taking into account the definition of the $S$-matrix element $S_l$

$$S_l = \frac{W(\phi_l(r), f_l^{(-)}(r))}{W(\phi_l(r), f_l^{(+)}(r))}$$

it results that the natural modes correspond to the $S$-matrix poles. In a series of papers [2–4] the Riemann surface approach to bound and resonant states in the scattering by a central potential $gV(r)$, based on the global analysis of the function $k_l(g)$, has been developed. This approach consists in the construction of the Riemann surface $R_g$ over the $g$-plane, on which the pole function $k = k_l(g)$ is single-valued and analytic. By keeping the Riemann sheets apart the single pole laying on each $k$-plane sheet image can be identified. In other words, if the potential strength $g$ takes a value on a given sheet then the function $k = k_l(g)$ takes only one value on the $k$-plane image of the sheet, i.e. there is only a single pole on each sheet image. According to the Riemann surface approach to a given mode of the quantum system a sheet of the Riemann surface $R_g$ is associated. In this way no pole can be lost and the involuntary jump from a pole to another during the potential strength variation is avoided. A $S$-matrix pole and the associated mode are classified by introducing a new quantum number $n$ with a precise topological meaning, namely the label of the Riemann sheet $\Sigma_n(l)$ and of its $k$-plane image $\Sigma_n'(l)$ to which the pole belongs. This quantum number is completely different in kind from $l$, which is connected to the rotational invariance, while $n$ has a topological meaning. In [2–4] the classifications of the natural modes, based on the use of the new quantum number $n$ which labels the Riemann sheets $\Sigma_n(l)$, their $k$-plane images $\Sigma_n'(l)$ and the corresponding $S$-matrix pole have been discussed in the case of a single channel scattering. In [5] the conditions for the system to undergo a jump from one state $(l, n)$ to other state $(l, m)$ as a consequence of a small potential strength variation around a branch-point (BP) on $\Sigma_n(l)$ have been established.

In the present paper the classification of the modes of a solvable two-channel model (CC) is considered. Each channel contains a square potential of strength $V_{11}$ and $V_{22}$, respectively, and they are coupled by a square potential $V_{12}$. All three potentials have the same radius $R$ and for convenience the choice $V_{11} = V_{22}$ is made. The energy scale is chosen so that the eigenvalues of the target system are $\omega_1 = 0$, $\omega_2 > \omega_1 = \omega$. The case $l = 0$ is considered. The notation
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\[ k^2 = k_1^2 = \frac{2m}{\hbar^2} (E - \omega_1) = \frac{2m}{\hbar^2} E, \]
\[ k_2^2 = \frac{2m}{\hbar^2} (E - \omega_2) = \frac{2m}{\hbar^2} (E - \omega), \]  

is used, where \( E \) is the center of mass energy of the particle-target motion.

The crucial point in the construction of the Riemann surface is to find the branch-points (BP) of the pole function.

**2. BRANCH-POINTS FOR TWO COUPLED CHANNELS WITH SQUARE POTENTIALS**

In [6] the BP for two coupled channels (CC) with square potentials have been deduced. According to the implicit function theory [7, 8] the singular points \( g_i \) of the function \( k_i(g) \) are the solutions of the system

\[ \mathcal{F}_i(g, k) = 0, \]  
\[ \frac{\partial \mathcal{F}_i(g, k)}{\partial k} = 0. \n\]

From among these singular points \( g_i \), those that are branch points may be found by permitting the variable \( g \) to describe successive small circuits round each singular point \( g_i \) and by observing whether the function \( k_i(g) \) returns to its initial value. For a zero coupling constant there were four series of singular points denoted by \( g_{\alpha,s} \), \( g_{\beta,s} \), \( g_{\gamma,s} \), \( g_{\tau,s} \), \( g_{\alpha,s} \) are the BP for the first isolated channel. The \( k \)-plane image of these BP is \( k = -i \). \( g_{\beta,s} \) and \( g_{\gamma,s} \) are the BP for the second channel, with the \( k \)-plane images \( k = \sqrt{\omega - 1} \) and \( k = 0 \), respectively. The singular points \( g_{\tau,s} \) with the \( k \)-plane image \( k = i(\omega/4 - 1) \) correspond to the situation when there is a pole in each of the uncoupled channel. It has been shown that the introduction of the coupling has as an effect not only the perturbation of the BP for the uncoupled channels, but also the occurrence of new BP. There are four series of BP intrinsic to the coupling, which are denoted by \( g_{\delta,s} \), \( g_{\mu,s} \), \( g_{\nu,s} \), \( g_{\phi,s} \), respectively. In other words the BP in the CC case could be grouped into two classes: i) BP that originate from those of the uncoupled channels and ii) BP that are characteristic to the coupling. Approximate expressions of the BP and of their \( k \)-plane images have been obtained for small coupling potentials. These approximate values have been then used as starting points for the exact numerical solution of the system of equations made of the pole equation and its derivative with respect to \( k \), that give the BP. In Fig. 1 the BP that occur due to the coupling for a coupling potential \( V_{12} = 0.01 \), \( \omega = 3 \) and their \( k \)-plane images are shown.
3. Riemann Surface of \( k = k(g) \) for Two Coupled Channels with Square Potentials

Once the BP have been determined, the Riemann surface of the pole function \( k = k(g) \) over the \( g \)-plane for two coupled channels has been constructed for a chosen value of the coupling potential. The Riemann sheets are separated by taking cuts from each BP of that sheet to infinity. The border of each sheet is made of the two edges of each cut on the sheet and by a large radius circle. When the strength of the potential \( g \) is varied along the border of each sheet the image of the sheet in the \( k \)-plane is obtained by solving the pole equation (6). A general construction of the Riemann surface of \( k = k(g) \) cannot be done, as the BP behave differently with respect to the strength of the coupling potential \( V_{12} \), which means that the configuration of cuts is different for different values of \( V_{12} \).

In the single channel case the Riemann sheets and their \( k \)-plane images have been labeled according to the behaviour of the pole for \( g \rightarrow 0 \) (see [4]). For the two channel problem the classification of the \( S \)-matrix poles becomes more complicated. By analysing the Riemann sheets for the CC model it results that the sheets can be grouped into four classes according to the number and nature of cuts that separate the sheets, in contrast to the single channel case, where all the Riemann sheets had the same number of cuts. There is a series of Riemann sheets separated by a single cut determined by \( \delta_{\alpha,s} \). These Riemann sheets for the CC problem reduce to the sheets for the first isolated channel when \( V_{12} \rightarrow 0 \). Their borders are very similar to those of the Riemann sheets for the single channel case. The Riemann sheets for the two channel model which reduce to the single channel Riemann sheets when \( V_{12} = 0 \) will be denoted by the label I.
There is also a series of Riemann sheets separated by two cuts, determined by $g_{\beta,s}$ and $g_{\gamma,s}$. These Riemann sheets for the CC problem (denoted by the label II) reduce to the sheets for the second isolated channel when $V_{12} \to 0$. However the borders of these sheets are more involved than for $V_{12} = 0$. Indeed both $g_{\beta,s}$ and $g_{\gamma,s}$ are real for $V_{12} = 0$, and consequently there is a single cut on each Riemann sheet, while in the case $V_{12} \neq 0$ the BP $g_{\beta,s}$ become complex, so that there are two cuts on each Riemann sheet. The Riemann sheets of the classes I and II are separated by the BP which exist, either or not there is a coupling between the two channels.

However in the case $V_{12} \neq 0$ there are some new BP ($g_{\delta,s}$, $g_{\nu,s}$, $g_{\mu,s}$, and $g_{\phi,s}$), which are due to the coupling, i.e. they do not exist in the case $V_{12} = 0$. Consequently, there are Riemann sheets for the CC which do not reduce to the sheets for the uncoupled channels when $V_{12} \to 0$ ($V_{12} \neq 0$). By analysing these sheets it results that there is a series of Riemann sheets that are separated by a cut determined by the branch points $g_{\alpha,s}$, originating from the first channel, and other cuts determined by some BP due to the coupling. The Riemann sheets of this class will be denoted by the III. There is also another series of Riemann sheets that are separated by a cut determined by the branch points $g_{\beta,s}$ and $g_{\gamma,s}$, originating from the second channel, and other cuts determined by some BP due to the coupling. These Riemann sheets belong to the class IV.

In this way, in order to label the Riemann sheets, their $k$-plane images, the corresponding poles and the corresponding bound or resonant states, two new quantum numbers are introduced. These new quantum numbers $(m, n)$ have a topological meaning: $m = 1, \ II, \ III, \ and \ IV$ indicates the class of the Riemann sheet, as discussed above, while $n = 1, 2 \ldots$ labels the sheet for each class. In other words the Riemann sheets are denoted by $\Sigma_{m,n}$, and their $k$-plane images are denoted by $\Sigma'_{m,n}$. The pole situated on $\Sigma'_{m,n}$ and the corresponding state is labeled by $(m, n)$.

In Fig. 2, the Riemann sheets for the four classes are shown. The borders of the sheets, made of the edges of the cuts and a circle of large radius are shown. The upper edge of each cut is represented by continuous line, while the lower edge of each cut is represented by dashed line. The cuts on the given sheet are labeled by the same Greek letters as the BP which determine the cuts that separate the sheet under discussion. The sheets of the classes III and IV are more involved than the sheets that reduce to the uncoupled channels. One can see that each class of Riemann sheets is characterised by some cuts which differ from the cuts for another class of sheets.
It is interesting to understand which is the nature of the resonant states corresponding to poles situated on the images of each class of Riemann sheets. In order to do this let us see what happens when $V_{12} \to 0$. The bound or resonant state poles on the images of the sheets $\Sigma_{I,n}$ and $\Sigma_{II,n}$ can exist either there is or there is not a coupling between the two channels. The sheets $\Sigma_{I,n}$ reduce to the sheets of the first isolated channel, for which the threshold $\omega = 0$, when $V_{12} \to 0$. The poles on the images of these sheets represent shape (potential) resonant states or bound states. When $V_{12} \to 0$, ($V_{12} \neq 0$), the sheets $\Sigma_{II,n}$ reduce to the sheets of the second isolated channel, for which the threshold is $\omega \neq 0$. The resonant state poles on the Riemann sheet images $\Sigma_{II,n}$ are of Feshbach type [10]. In other words, by introducing a weak channel coupling a bound state pole in an isolated ($V_{12} = 0$) closed channel on $\Sigma_{II,n}$ becomes a resonant state pole for $V_{12} \neq 0$. As concerns the Riemann sheets $\Sigma_{III,n}$ and $\Sigma_{IV,n}$ they are determined by the BP intrinsic to the coupling. For $V_{12} = 0$ there is no $\Sigma_{III,n}$ and $\Sigma_{IV,n}$. It is known [10] that there is a kind of multichannel resonant state which exists only in the presence of a strong coupling. This is a Fonda-Newton resonant state. Moreover, there can exist Fonda-Newton resonant states even though the isolated closed channels have no bound states [9, 10]. It results that a resonant state pole on a $k$-plane image of the Riemann sheets $\Sigma_{III,n}$ and $\Sigma_{IV,n}$, intrinsic to the coupling, corresponds to a Fonda-Newton resonant state.

There was for a long time an unsolved problem: what happens with the Fonda-Newton resonant state, characteristic to the strong coupling when the coupling is decreased $V_{12} \to 0$, ($V_{12} \neq 0$)? Does it become a Feshbach resonant state, characteristic to the weak coupling?
Taking into account that a pole of the function \( k = k(g) \) cannot disappear by varying \( V_{12} \) in the region of analyticity of the function \( k(g) \) it results that a Fonda-Newton resonant state pole cannot be transformed into a Feshbach resonant state pole by varying the strength of the coupling. However it is possible to jump from a Riemann sheet image containing Fonda-Newton resonant state pole to a Riemann sheet image containing Feshbach resonant state pole by a small variation of the potential strength \( g \). Let the parameter \( g \) follow a prescribed contour starting from a value \( g \in \Sigma_{m,n} \). If \( g \) describes a closed contour which starts from a point on the sheet \( \Sigma_{m,n} \) and encloses the branch point joining the sheets \( \Sigma_{m,n} \) and \( \Sigma_{p,q} \) and no other branch points, then the variable \( g \) is transferred from the sheet \( \Sigma_{m,n} \) to the sheet \( \Sigma_{m,p} \). As a consequence the pole \( k = k^{(i)}(g) \) is transferred from the sheet image \( \Sigma'_{n,m} \) to the sheet image \( \Sigma'_{p,q} \). In other words the potential strength variation induces a jump from the state \((m, n)\) to the state \((p, q)\). In order to establish the transition rules between various states it is necessary to see what are the junctions of the Riemann sheets at the various branch-points. The analysis of the joinings of the sheets at the BP has shown that the sheets of the class II have junctions to the sheets of the class IV.

REFERENCES