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DARBOUX TRANSFORMATIONS
IN NONCOMMUTATIVE MECHANICS

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We provide explicit Darboux transformations for the symplectic structures of noncommutative mechanics with arbitrary nonconstant noncommutativity parameter and magnetic field.

1. INTRODUCTION

In the last years quantum mechanics with noncommuting coordinates attracted much attention [1]. In its simplest form noncommutative (NC) mechanics follows the structure of ordinary mechanics, but allows in addition for nonzero commutators among the coordinates, and sometimes also among the momenta.

At the classical level one similarly generalizes the symplectic structure by allowing further nonvanishing Poisson brackets among the coordinates \(q\) (and sometimes among the momenta \(p\)). That means the following Poisson brackets are nonvanishing

\[
\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \theta_{ij}(q, p), \quad \{p_i, p_j\} = F_{ij}(q, p),
\]

and are providing the symplectic structure of noncommutative mechanics. Usually the Poisson brackets are taken to be constant and space is taken to be two-dimensional for notational simplicity (\(i, j = 1, 2; \ \theta_{12} = \theta; \ \theta_{ij} \neq \theta \) above). In this case Darboux coordinates \(Q_i, P_j\) (in terms of which the Poisson brackets are canonical: \(\{Q_i, P_j\} = \delta_{ij}, \ \{Q_i, Q_j\} = 0, \ \{P_i, P_j\} = 0\) are easily found [2], e.g.

\[
Q_1 = \frac{q_1 + \theta p_2}{1 - \theta^2}, \quad Q_2 = q_2, \quad P_1 = p_1, \quad P_2 = \frac{p_2 + \theta q_1}{1 - \theta^2}.
\]

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We wish to generalize the above to nonconstant Poisson brackets, a technically more difficult case. As is customary, we will stay within (2+1) dimensions, although some of our statements will be of general validity.

Our results are directly applicable at the quantum mechanical level whenever no operator ordering issues occur in the brackets. Otherwise, one has to specify an ordering, an issue not discussed here.

In section 2 we will introduce the problem in the simple case in which only one of the extra commutators, \( [q_1, q_2] \) xor \( [p_1, p_2] \) is nonvanishing.

When both of these are nonzero, two alternative formalisms can be considered. They will be discussed in turn, in sections 3 and 4.

The significance of our results is the following. Whenever one is given a particular symplectic form \( \omega(q, p) \) and a Hamiltonian \( H_0(q, p) \), one may in principle render the symplectic form canonical, \( \omega_0 \), by going to Darboux coordinates \( Q, P \). The price to be paid is that the new Hamiltonian \( H(Q, P) \) is in general more complicated. Also, the new coordinates \( Q(q, p), P(q, p) \) are rarely available explicitly. We provide them explicitly for the generic structure (1) of NC mechanics. One must then express the old phase-space coordinates \( q, p \) in terms of the new ones \( Q, P \) (i.e. invert the Darboux transformation) and replace them in the expression of \( H_0(q, p) \). Then the new \( H(Q, P) \) is found. Combined with the canonical \( \omega_0, (\omega_0, H) \), it is completely equivalent as far as dynamics is concerned to the \( (\omega, H_0) \) initial NC system. We manage to explicitly invert the Darboux transformation and find \( H(Q, P) \) in the limit in which \( \theta \) is small or constant, or when both \( F \) and \( \theta \) are variable but small.

An interesting different approach to Darboux transformations can be found in Ref. [3]

2. SIMPLE CASES

Consider first the simple case in which \( \theta F = 0 \). Take first \( \theta = 0 \). It can then be shown [2] that, due to the Jacobi identities implied by (1), \( F \) cannot depend on the momenta. Then \( [p_i, p_j] = F_{ij}(q) \) simply mimics a magnetic background.

More precisely,

\[
Q_i = q_i, \quad P_i = p_i + A_i(q),
\]

are canonical space coordinates, provided

\[
\partial_i A_j - \partial_j A_i = F_{ij},
\]

If one substitutes (3) in the Hamiltonian \( H(p, q) = \frac{p^2}{2m} + V(q) \) one obtains
Darboux transformation in noncommutative mechanics

\[ H(P, Q) = \frac{(P_i - A_i)^2}{2m} + V(Q), \]  

the Hamiltonian in presence of a magnetic background (4), expressed now in terms of canonical coordinates.

If on the other hand \( F = 0 \) in (1), the Jacobi identities imply [2] that \( \theta \) cannot depend on the coordinates, \( \theta(p) \). The Darboux transformation is

\[ Q_i = q_i - A_{i+2}(p), \quad P_i = p_i, \]

\[ \partial_{p_i} A_{i+2} - \partial_{p_j} A_{j+2} = \theta_{ij}(p). \]

Upon replacement in \( 2H = p^2 + q^2 \) a ‘P-space’ magnetic field results, which upon the subsequent transformation (canonical this time) \( P' = Q, \quad Q' = -P \) is mapped into a real one. If \( V(q) \) contains higher than quadratic terms a similar picture applies, but with nonquadratic kinetic term in the end.

3. GENERAL FRAMEWORK

Let us introduce the formalism for the case \( F \neq 0 \) in its simplest form first, namely for \( F \) and \( \theta \) constant. Denote by \( x_a, a = 1, 2, 3, 4 \) the phase space coordinates, \( x_1, x_2, x_3, x_4 = q_1, q_2, p_1, p_2 \). Since no risk of confusion exists, all indices are put down. Eqs. (1) can then be rewritten as \( \{x_a, x_b\} = \Theta_{ab} \), where

\[
\Theta = \begin{pmatrix}
0 & 0 & 1 & 0 \\
-\theta & 0 & 0 & 1 \\
-1 & 0 & 0 & F \\
0 & -1 & -F & 0 \\
\end{pmatrix}
\]

i.e. \( \omega = \frac{1}{1-\theta F} \begin{pmatrix}
0 & -F & 1 & 0 \\
F & 0 & 0 & 1 \\
-1 & 0 & 0 & -\theta \\
0 & -1 & \theta & 0 \\
\end{pmatrix} \). \( \Theta_{ab} = (\omega^{-1})_{ab} \), and \( \omega \) is the symplectic form, which enters the action

\[
S = \int dt \left( \frac{1}{2} \omega_{ab} x_a \dot{x}_b - H(x) \right),
\]

out of which one variationally derives [2] the equations of motion

\[
\dot{x}_a = \{x_a, H\} = \Theta_{ab} \frac{\partial H}{\partial x_b}.
\]

If \( F \neq 0 \) two different possibilities appear: We can put the \( \det \Theta^{-1} = 1/(1-\theta F) \) factor either in the symplectic form – like in Eq. (8), or into the commutation relations – by transferring it from \( \omega \) to \( \Theta \) in (8).
Both cases – \((1-\Theta F)\) dividing either \(\Theta\) or \(\omega\), the other being in turn simpler – present interest, and will be discussed in the next two sections for generic \(\theta(x)\) and \(F(x)\).

The extension to nonconstant \(\theta(x)\) can in turn either start from the general Eqs. (1), or from the generalization of Eq. (9),

\[
\tilde{S} = \int dt \left[ \mathcal{A}_a(x) \dot{x}_a - H(x) \right],
\]

with
\[
\partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a = \omega_{ab}(x)
\]
and \(\omega_{ab}(x)\) having the form in (8) but without the \((1-\Theta F)^{-1}\) factor in front – which now multiplies \(\Theta \equiv \omega^{-1}\). In brief, in Section 3 we will have \(\det \Theta = (1-\Theta F)\) and \(\det \omega = 1\), whereas in Section 4 the reverse will apply, namely \(\det \Theta = 1\) and \(\det \omega = 1-\Theta F\).

### 4. Starting from the commutation relations

Consider first a generalized electromagnetic background \(F(q, p)\), living on a space with noncommutativity field \(\theta(q, p)\), and flat metric \(g_{ij}\):

\[
\{q^i, q^j\} = 0, \quad \{q^i, p^j\} = \delta^{ij}, \quad \{p^i, p^j\} = F_{ij}(q, p).
\]

The Jacobi identities read

\[
\{q^k, F_{ij}\} - \frac{\partial F_{ij}}{\partial p_k} - \frac{\partial F_{ij}}{\partial q^m} \theta^{mk} = 0
\]

\[
\{\theta^{ij}, p_k\} = \frac{\partial \theta^{ij}}{\partial q^k} + \frac{\partial \theta^{ij}}{\partial p^m} F_{mk} = 0
\]

\[
\{F_{ij}, p_k\} + \text{cyclic} = \left( \frac{\partial F_{ij}}{\partial q^k} + \frac{\partial F_{ij}}{\partial p^m} F_{mk} \right) + \text{cyclic} = 0
\]

\[
\{q^k, \theta^{ij}\} + \text{cyclic} = \left( \frac{\partial \theta^{ij}}{\partial p_k} - \frac{\partial \theta^{ij}}{\partial q^m} \theta^{mk} \right) + \text{cyclic} = 0.
\]

The Jacobi identities ensure the invariance of the commutation relations under time evolution, for a generic Hamiltonian \(H(p, q)\). Explicitely:

\[
\{\{q^m, p_n\} - \delta_n^m, H\} = -\partial_p H\{F_{mn}, q^m\} + \partial_q H\{\theta^{mn}, p_n\},
\]

\[
\{\{q^m, q^n\} - \theta^{mn}, H\} = -\partial_p H\{\theta^{mn}, p_n\} - \partial_q H\{\{\theta^{mn}, q^m\} + \text{cyclic}\},
\]
\{ \{ p_m, p_n \} - F_{mn}, H \} = -\partial_s H \{ F_{mn}, q^s \} + \partial_{p_s} H \{ F_{mn}, p_s \} + \text{cyclic}, \quad (20)

and Eqs. (14–17) ensure the right hand side in (18–19) to be zero. The Jacobi identities imply that \( F \)'s and \( \theta \)'s should depend on both \( q \)'s and \( p \)'s – unless they are constant. They consequently mix \( q \)- and \( p \)-dependence. In the two-dimensional case, \( q_1, q_2, p_1, p_2 \) depending on \( t \), \( F_{12} = F \), \( \theta_{12} = \theta \), Eqs. (16, 17) are identically satisfied, whereas (14, 15) read:

\[
F \partial_{p_1} \theta = \partial_1 \theta, \quad F \partial_{p_2} \theta = -\partial_2 \theta \quad (21)
\]

\[
\partial_{p_1} F = \theta \partial_1 F, \quad \partial_{p_2} F = -\theta \partial_2 F. \quad (22)
\]

\( q_1 \) and \( p_2 \), respectively \( q_2 \) and \( p_1 \), appear in pairs in \( \theta \) and \( F \).

Let us search for canonical coordinates \( Q_i, P_j \). The simplest possible Ansatz is

\[
Q_1 = q_1 + f(q, p), \quad Q_2 = q_2, \quad P_1 = p_1, \quad P_2 = p_2 + g(q, p). \quad (23)
\]

Imposing \( \{ Q_1, Q_2 \} = 0 \) and \( \{ Q_1, P_1 \} = 1 \) leads to

\[
\theta \frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial p_2} = 0, \quad \frac{\partial f}{\partial q_1} - F \frac{\partial f}{\partial p_2} = 0, \quad (24)
\]

leading to

\[
\frac{\partial f}{\partial q_1} = \frac{F \theta}{1 - \theta F}, \quad \frac{\partial f}{\partial p_2} = \frac{\theta}{1 - \theta F}. \quad (25)
\]

The integrability condition \( \partial_{p_2} \frac{F \theta}{1 - \theta F} = \partial_{q_1} \frac{\theta}{1 - \theta F} \) is satisfied thanks to the Jacobi identities, and one obtains

\[
f = \int dp_2 \frac{\theta}{1 - \theta F} = \int dq_1 \frac{F \theta}{1 - \theta F}. \quad (26)
\]

Similarly the integrability condition for

\[
\frac{\partial g}{\partial q_1} = \frac{F}{1 - \theta F}, \quad \frac{\partial g}{\partial p_2} = \frac{F \theta}{1 - \theta F}. \quad (27)
\]

is automatically satisfied due to the Jacobi identities and one has

\[
g = \int dp_2 \frac{F}{1 - \theta F} = \int dq_1 \frac{F}{1 - \theta F}. \quad (28)
\]

Since \( \{ Q_2, P_1 \} = 0 \) is trivially satisfied, the last condition to impose is \( \{ Q_1, P_2 \} = 0 \). The resulting nonlinear partial differential equation linearizes when the partial derivatives of \( f, g \) with respect to \( q_1, p_2 \) obtained above are put in. It simplifies then to
which is automatically obeyed by the solutions (26, 28) above, again due to the
Jacobi identities! The Darboux coordinates are thus found in the general case,
being given by Eqs. (23, 26, 28).

It is however important to observe that the case \( \theta, F \) constant does not
follow smoothly from the above solutions. The reason is that when \( \theta \) and \( F \) are
constant, Eqs. (25, 27) are not solved anymore by (26, 28), but simply by

\[
f = q_1 \frac{\theta F}{1 - F\theta} + p_2 \frac{\theta}{1 - F\theta}, \quad g = q_1 \frac{F}{1 - F\theta} + p_2 \frac{F\theta}{1 - F\theta}.
\]

Inverting (30) one recovers (2).

Finally, we show how to invert the Darboux transformations (23, 26, 28) in the
case in which \( F \) and \( \theta \) are variable but small, in the sense that in the equations

\[
Q_1 - q_1 = \int_{q_1}^{q_1} \frac{F\theta}{1 - F\theta} = \int_{p_2}^{p_2} \frac{\theta}{1 - F\theta} = I_1(q, p),
\]

\[
P_2 - p_2 = \int_{p_1}^{p_1} \frac{F}{1 - F\theta} = \int_{p_2}^{p_2} \frac{F\theta}{1 - F\theta} = I_2(q, p),
\]

and for \( \Delta Q \) and \( \Delta P \) being the relevant length and momentum scales of the
problem at hand, one assumes the regime

\[|I_1(q, p)| \ll \Delta Q, \quad |I_2(q, p)| \ll \Delta P.\]

Then one can expand \( I_1(q, p) \) and \( I_2(q, p) \) in partial derivatives with respect to
\( Q_1 \) and \( P_2 \), around \( I_1(Q, P) \) and \( I_2(Q, P) \), to obtain

\[
Q_1 - q_1 = I_1(Q, P) + \frac{\theta F}{1 - F\theta} (Q_1 - q_1) + \frac{\theta}{1 - F\theta} (P_2 - p_2)
\]

\[
P_2 - p_2 = I_2(Q, P) + \frac{F}{1 - F\theta} (Q_1 - q_1) + \frac{F\theta}{1 - F\theta} (P_2 - p_2).
\]

The expressions for \( p_2 \) and \( q_1 \) follow immediately, together with the expression
for the Hamiltonian \( H(Q, P) \), written in terms of canonical phase-space
coordinates.

\section*{5. STARTING FROM THE SYMPLECTIC STRUCTURE}

The Poisson brackets are now

\[
\{q_1, q_2\} = \frac{\theta}{1 - \theta F}, \quad \{p_1, p_2\} = \frac{F}{1 - \theta F}, \quad \{q_i, p_j\} = \frac{\delta_{ij}}{1 - \theta F}.
\]

The important point is that the Jacobi identities now imply that $F(q)$ and $\theta(p)$ (with then usual Jacobi), and thus decouple the $q$- and $p$-dependence.

To find a Darboux transformation for arbitrary $F_{12}(q)$ and $\theta_{12}(p)$, define

$$P_1 = p_1 + A_1(q), \quad P_2 = p_2 + A_2(q), \quad Q_1 = q_1 - A_3(p), \quad Q_2 = q_2 - A_4(p),$$

where we are guided by the Ansatz

$$\frac{\partial A_2(q)}{\partial q_1} - \frac{\partial A_2(q)}{\partial q_1} + \{A_1, A_2\} = F(q),$$
$$\frac{\partial A_4(p)}{\partial p_2} - \frac{\partial A_3(p)}{\partial p_2} + \{A_3, A_4\} = \theta(p),$$

reminiscent of the Non-Abelian definition of field strenghts. It is simplest to achieve canonical Poisson brackets between the $Q$’s and the $P$’s through, for instance,

$$A_1 = 0, \quad A_4 = 0, \quad \frac{\partial A_2(q)}{\partial q_1} = F(q), \quad \frac{\partial A_3(p)}{\partial p_2} = -\theta(p).$$

It is easy to check that the new phase-space coordinates $Q_{1,2}, P_{1,2}$ are canonical. In consequence one possible Darboux transformation is

$$Q_1 = q_1 + \int p_1 \theta(p), \quad Q_2 = q_2, \quad P_1 = p_2, \quad P_2 = p_2 + \int q_1 F(q).$$

We proceed to invert it in some cases.

For $F, \theta$ constant (41) becomes $Q_1 = q_1 + p_2 \theta, \quad P_2 = p_2 + q_1 F,$ and the inversion formulas are immediate $q_1 = \frac{Q_1 - \theta P_2}{1 - F \theta}, \quad p_2 = \frac{P_2 - F Q_1}{1 - F \theta}$. They are to be used in the initial Hamiltonian $H(p, q)$ to provide the canonical one $H(P, Q)$.

If $\theta$ (or $F$) is either constant or small, one cannot invert explicitly for generic functions. If however $\theta$ is small and constant, (41) leads to $Q_1 - q_1 = \theta p_2 \ll |Q_1|$ and thus

$$P_2 - p_2 = \int q_1 F(q) = I(q_1, Q_2) = I(Q_1, Q_2) + \frac{\partial I(Q_1, Q_2)}{\partial Q_1}(q_1 - Q_1),$$

and one uses $\frac{\partial I(Q_1, Q_2)}{\partial Q_1} = \int q_1 F(Q) + F(Q)(q_1 - Q_1)$. Solving the ensuing two linear equations in the unknowns $(P_2 - p_2)$ and $(Q_1 - q_1)$ one immediately reaches
consistent with the constant $F$, $\theta$ case.

A third case in which (41) is explicitly invertible is the one in which both $F$ and $\theta$ are variable but small with respect to all corresponding scales of the problem, in a way similar to Eq. (33). Then (41) becomes

\begin{equation}
\begin{align*}
q_1 &= Q_1 - \frac{\theta P_2 - \theta \int_{\hat{Q}_1} F(Q) \, dQ}{1 - F \theta}, \\
p_2 &= P_2 - \frac{\int_{\hat{Q}_1} F(Q) - F \theta P_2}{1 - F \theta},
\end{align*}
\end{equation}

leading to

\begin{equation}
\begin{align*}
q_1 &= Q_1 - \frac{\int_{\hat{P}_2} \theta(P) - \int_{\hat{Q}_1} F(Q)}{1 - F \theta}, \\
p_2 &= P_2 - \frac{\int_{\hat{Q}_1} F(Q) - F \int_{\hat{P}_2} \theta(P)}{1 - F \theta}.
\end{align*}
\end{equation}

Together with $q_2 = Q_2$ and $p_1 = P_1$, they determine the canonical Hamiltonian $H(Q, P)$, once $H_0(q, p)$ is given.

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