The hydrodynamic fluid description, proposed many years ago by E. Madelung (1927) for quantum mechanics, is used to discuss the class of nonlinear Schrödinger equations. In the case of stationary profile solutions the equation satisfied by the fluid density $\rho = |\Psi|^2$ is integrated and periodic solutions expressed through Jacobi elliptic functions are derived for cubic and cubic + quintic nonlinearities. In the limit case $k^2 = 1$ the solitary wave solution found for the cubic + quintic nonlinearity proves to be much steeper and narrower than the one-soliton solution of the cubic NLS equation.

1. INTRODUCTION

A large class of collective and nonlinear phenomena involving the propagation of quasi-monochromatic waves in weakly nonlinear media are governed by various types of nonlinear Schrödinger equations (NLS eq.). These are relevant for different areas of physics, such as, fluid dynamics [1], plasma physics [2], nonlinear optics [3, 4] and optical fibers [5, 6], electrical transmission lines [7, 8], Bose-Einstein condensates [9, 10], charged-particle beam physics [11], quasi-one-dimensional molecular chains [12, 13], and the list can be continued further on [14, 15].

The general form of the NLS equation we are considering is

$$i \frac{\partial \Psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} + U(|\Psi|^2) \Psi = 0$$

(1)

where $U(|\Psi|^2)$ is a functional of $|\Psi|^2$. If $U = \beta |\Psi|^2$ the equation (1) becomes the well known cubic NLS equation, a completely integrable system, solvable by the inverse scattering transform (IST) method [16–20]. The main interest in the present

*Corresponding author: dgreceu@theory.nipne.ro

paper will be focused on the cubic+quintic nonlinearity

\[ U(|\Psi|^2) = \beta|\Psi|^2 + \gamma|\Psi|^4 \]  

(2)

A hydrodynamic description of quantum mechanics was for the first time formulated by E. Madelung in 1927 [21]. For the one-dimensional Schrödinger equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U(x)\Psi \]  

(3)

Madelung considered a solution of the form

\[ \Psi = \sqrt{\rho} e^{i\theta/h} \]  

(4)

with both \( \rho \) and \( \theta \) depending on \( x \) and \( t \). Introducing (4) in (3) and separating the real and the imaginary part, one obtains

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0 \]  

(5)

and

\[ m \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) v = \frac{\partial}{\partial x} \left[ \frac{\hbar^2}{2m} \left( \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right) - U \right] \]  

(6)

where

\[ v(x,t) = \frac{1}{m} \frac{\partial \theta(x,t)}{\partial x} \]  

(7)

The first equation, (5), is a continuity equation for the fluid density \( \rho = |\Psi|^2 \), while equation (6) is an evolution law for the fluid velocity \( v(x,t) \). In the right hand side of (6), beside the usual force term \(-dU/dx\) there appears another derivative term known in the literature as Bohm’s potential. The interpretation of \( v(x,t) \) as a fluid velocity comes also from the expression of the current density

\[ j = \frac{\hbar}{2im} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) = \rho v \]  

(8)

Many authors have contributed to a better understanding and interpretation of this approach [22–24] and several applications have been considered such as stochastic quantum mechanics [24], quantum cosmology [25, 26], quantum effects in micro-electronic devices [27] only to mention a few examples.

Some year ago, a new approach based on a Madelung fluid description was used by Fedele et al. [28–31] to discuss the NLS equation (1), and a large class of solitary wave solutions was found.
Recently the Madelung’s approach was used to discuss the derivative NLS equations [32,33] and the system of coupled NLS equations (Manakov’s model) [34]. Based on Madelung’s approach a correspondence between generalized NLS equations and generalized Korteweg-de Vries (KdV) equations was discussed in [35, 36].

As mentioned before, in the present paper the Madelung fluid description will be applied mainly to the non-integrable NLS equation with cubic+quintic nonlinearity. In the next section Madelung’s approach to generalized NLS equation will be reviewed. In section three, using this approach periodic and solitary wave (solitons) solutions of the cubic NLS equation will be briefly presented. These results are not new but they are included here for completeness and will serve as a reference for the comparison with those of the fourth section. Section four is dedicated to the study of the NLS equation with cubic+quintic nonlinearity. The last section reviews the original results of the paper and gives a few comments.

2. MADENG FLUID DESCRIPTION OF GENERALIZED NLS EQUATIONS

We seek solutions of equation (1) of the form
\[ \Psi(x,t) = \sqrt{\rho(x,t)} e^{i\theta(x,t)} \] (9)

Using (9) in (1) and separating the imaginary and the real parts one obtains [28, 29]

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \] (10)

and

\[ \frac{\partial \theta}{\partial t} + \frac{1}{2} v^2 = \frac{1}{2} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} + U \] (11)

respectively, where

\[ v = \frac{\partial \theta(x,t)}{\partial x} \]

Derivating (11) with respect to \( x \) we get

\[ \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) v = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right) + \frac{\partial U(\rho)}{\partial x} \] (12)

The equation (10) is a continuity equation for the fluid density \( \rho = |\Psi|^2 \), while (12) is an equation of evolution for the fluid velocity \( v \). The latter, using a set of transformations [28, 29], becomes

\[ -\rho \frac{\partial v}{\partial t} + v \frac{\partial \rho}{\partial t} + 2 \left[ C_0(t) - \int \frac{\partial v}{\partial t} \, dx \right] \frac{\partial \rho}{\partial x} + \frac{1}{4} \frac{\partial^3 \rho}{\partial x^3} + \left( \rho \frac{dU}{d\rho} + 2U \right) \frac{\partial \rho}{\partial x} = 0 \] (13)
Here $C_0(t)$ is an integration quantity, eventually time dependent.

The equation (13) can be solved when two simplifying conditions are met:
- motion with constant velocity $v = v_0 = \text{const.}$
- motion with stationary profile when both $\rho(x,t)$ and $v(x,t)$ are depending only on $\xi = x - u_0 t$.

In the first case, from the continuity equation (10) it follows that $\rho(x,t) = \rho(\xi)$, $\xi = x - v_0 t$. Then (13) becomes

$$\frac{1}{4} \frac{d^3 \rho}{d \xi^3} + E \frac{d \rho}{d \xi} + \left( \rho \frac{dU}{d \rho} + 2U \right) \frac{d \rho}{d \xi} = 0 \tag{14}$$

where $E = 2C_0 - v_0^2$ is an arbitrary constant. Denoting

$$\rho \frac{dU}{d \rho} + 2U = \frac{dG(\rho)}{d \rho} \tag{15}$$

the equation (14) can be integrated once

$$\frac{1}{4} \frac{d^2 \rho}{d \xi^2} + E \rho + G = \frac{1}{2} A \tag{16}$$

Integrating again one obtains

$$\frac{1}{4} \left( \frac{d \rho}{d \xi} \right)^2 = -G(\rho) - E \rho^2 + A \rho + B, \quad G(\rho) = 2 \int G(\rho) d \rho \tag{17}$$

Here $A$ and $B$ are two arbitrary integration constants.

But for constant velocity the equation (12) becomes

$$\frac{1}{2} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} + U(\rho) \sqrt{\rho} = \lambda \sqrt{\rho} \tag{18}$$

and eigenvalue equation for $\sqrt{\rho}$ ($\lambda = \text{const.}$). It can be written in the form

$$\frac{1}{4} \left[ \frac{d^2 \rho}{d \xi^2} - \frac{1}{2 \rho} \left( \frac{d \rho}{d \xi} \right)^2 \right] + \rho U(\rho) = \lambda \rho \tag{19}$$

and using (16) and (17) it becomes

$$\left( \frac{E}{2} + \lambda \right) \rho + \frac{B}{2 \rho} - \left( \frac{G}{2 \rho} + \rho U - G \right) = 0$$
This is a relation which has to be satisfied for any value of $\rho$. Then the following restrictions result

$$B = 0, \quad \lambda = -\frac{E}{2}$$

and

$$\frac{G}{2\rho} + \rho U - G = 0$$

The last condition is satisfied for any expression of $U(\rho)$ of the form $U(\rho) = \beta \rho^p$ with $p$ a number, not necessarily an integer, except the value $p = -1$ (when $G = \beta \ln \rho$). For $p = -2$ (the anticubic case) as $dG/d\rho = 0$ the equation (14) becomes linear.

The phase $\theta(x, t)$ is easily calculated using (11), (18), (19) and the definition of $v$. One obtains

$$\theta(x, t) = v_0 \xi - \frac{1}{2} \left( v_0^2 + E \right) t$$

In the second case the integration of (10) gives

$$v = u_0 + \frac{A_0}{\rho}$$

where $A_0$ is an integration constant (it has to be taken equal to zero for solitary wave solutions which vanish at infinity), and the equation (10) transforms into the same equation (14) with a different expression for $E$, $E = 2C_0 + u_0^2$. Contrary to the previous case no supplementary conditions exist now and this will lead to a larger class of solutions. The phase $\theta(x, t) = \theta(\xi)$ will also have a more complex expression which can be derived by performing the integral in the following expression

$$\theta(\xi) = u_0 \xi + A_0 \int_0^\xi \frac{d\xi'}{\rho(\xi')}$$

3. CUBIC NLS EQUATION

The first case we shall discuss is that of cubic nonlinearity, $U = \beta \rho$, where $\beta = \pm 1$ (the magnitude of $\beta$ is included in the definition of $|\Psi|$). $\beta = +1$ corresponds to the focusing NLS equation while $\beta = -1$ to the defocussing case. Then the equation (14) becomes

$$\frac{1}{4} \frac{d^3 \rho}{d\xi^3} + E \frac{d\rho}{d\xi} + 3\beta \rho \frac{d\rho}{d\xi} = 0$$

which integrated twice gives

$$\frac{1}{4} \left( \frac{d\rho}{d\xi} \right)^2 = P_3(\rho), \quad P_3(\rho) = -\beta \rho^3 - E \rho^2 + A \rho + B$$
Integrating (25) one has to keep in mind that the physically acceptable solutions correspond to real, positive and finite $\rho$ and to the domains where $P_3(\rho) > 0$. Let us denote by $\rho_1 > \rho_2 > \rho_3$ the three real roots of $P_3(\rho)$. When $\beta = +1$ the previous requirements are realized if at least two of the roots are positive, $\rho_1 > \rho_2 > 0$, and $\rho \in [\rho_2, \rho_1]$. Then the solution of (24) is written in terms of Jacobi elliptic functions ([37], p. 79)

$$
\int_{\rho}^{\rho_1} \frac{dt}{\sqrt{(\rho_1 - t)(t - \rho_2)(t - \rho_3)}} = g u = 2 \xi
$$

$$
\text{sn}^2 u = \frac{\rho_1 - \rho}{\rho_1 - \rho_2}, \quad \rho = \rho_1 - (\rho_1 - \rho_2) \text{sn}^2 u, \quad u = \frac{2}{g} \xi
$$

(26)

$$
k^2 = \frac{\rho_1 - \rho_2}{\rho_1 - \rho_3}, \quad g = \frac{1}{\sqrt{\rho_1 - \rho_3}}
$$

In the limit case $\rho_3 = \rho_2$, $k^2 = 1$ the solution becomes

$$
\rho = \rho_1 - (\rho_1 - \rho_2) \tanh^2 u
$$

(27)

describing a shifted bright soliton (a bright type soliton with a nonvanishing value at infinity). In the case of constant velocity the supplementary condition $B = 0$ has to be imposed. This can be respected if any $\rho_2$ or $\rho_3$ is zero. In the degenerate case $\rho_2 = \rho_3$ the solution (27) transforms into the bright soliton solution

$$
\rho = \rho_1 \frac{1}{\cosh^2 u}, \quad u = \sqrt{\rho_1} \xi
$$

(28)

It is interesting to note that the equation (25) can be solved also in an apparently different way ([37], p. 77)

$$
\int_{\rho_2}^{\rho_1} \frac{dt}{\sqrt{(\rho_1 - t)(t - \rho_2)(t - \rho_3)}} = g u = 2 \xi
$$

(29)

$$
k^2 \text{sn}^2 u = \frac{\rho - \rho_2}{\rho - \rho_3}, \quad \rho = \frac{\rho_2 - \rho_3 k^2 \text{sn}^2 u}{1 - k^2 \text{sn}^2 u}
$$

with the same definitions for $k^2$ and $g$. Actually the two solutions are not independent. Indeed adding the two integrals (26) and (29) we get ($k^2 \neq 1$)

$$
u_1 + u_2 = K(k)
$$

where $K(k)$ is the complete elliptic integral of first kind, and by $u_1$ and $u_2$ we denoted the values of the integral (26) and (29) respectively. Using the addition formula

$$
\text{sn}(u - v) = \frac{\text{sn} u \text{cn} v \text{dn} v - \text{sn} v \text{cn} u \text{dn} u}{1 - k^2 \text{sn}^2 u \text{sn}^2 v}
$$
with \( u = u_1 \) and \( v = k \) we get
\[
\text{sn}^2 u_2 = \frac{1 - \text{sn}^2 u_1}{1 - k^2 \text{sn}^2 u_1}
\]
which using the expressions (26) for \( \text{sn}^2 u_1 \) and (29) for \( \text{sn}^2 u_2 \), becomes an identity. Although of different forms, the two solutions (26) and (29) represent the same one, the second being the first translated in \( u \) space by \( K(k) \).

If \( \beta = -1 \) the two requirements mentioned previously are satisfied only when all the three roots of \( P_3(\rho) \) are positive and \( \rho \in [\rho_3, \rho_2] \). Then the solution is ([37], p. 72)

\[
\int_{\rho_2}^{\rho} \frac{dt}{\sqrt{(\rho_1 - t)(\rho_2 - t)(\rho_3 - t)}} = g u = 2 \xi
\]

In the limit case \( \rho_1 = \rho_2, \ k^2 = 1 \) we have
\[
\rho = \rho_3 + (\rho_1 - \rho_3) \tanh^2 u
\]
representing a gray soliton \( (\rho(0) = \rho_3, \ \rho(\infty) = \rho_2) \). For constant velocity, the condition \( B = 0 \) implies \( \rho_3 = 0 \) and (31) becomes the dark soliton solution
\[
\rho = \rho_1 \tanh^2 u, \quad u = \sqrt{\rho_1} \xi
\]
The phase \( \theta(x, t) = \theta(\xi) \), using (23), is given by
\[
\theta(\xi) = u_0 \xi + \tilde{A} \int_0^u \frac{dt}{1 - \alpha^2 \text{sn}^2 t}
\]
where \( \tilde{A} = A_0 g/(2 \rho_1) \), \( \alpha^2 = (\rho_1 - \rho_2)/\rho_1 \) for \( \beta = 1 \) and \( \tilde{A} = A_0 g/(2 \rho_3) \), \( \alpha^2 = -(\rho_2 - \rho_3)/\rho_3 \) for \( \beta = -1 \). The integral from the right-hand side of (33) is nothing else that the incomplete elliptic integral of third kind ([37], p. 223)

\[
\Pi(\varphi, \alpha^2, k) = \int_0^u \frac{dt}{1 - \alpha^2 \text{sn}^2 t}
\]
where \( \sin \varphi = \text{sn} u \) and \( u = 2 \xi/g \) and the modulus \( k \) of the elliptic integral is given in (26) for \( \beta = +1 \) and in (30) for \( \beta = -1 \). For \( \beta = 1 \) as \( 0 < \alpha^2 < k^2 \) we are in the so called hyperbolic case, while for \( \beta = -1 \), \( 0 < -\alpha^2 < \infty \) we are in the circular case [37].
3.1. DIRECT METHOD

Similar results are obtained by seeking solutions of the equation (24) of the form
\[ \rho = A + By, \quad y = \text{sn}^2 u, \quad u = \mu \xi \] (35)
The positive character of \( \rho \) implies \( A > 0 \) if \( B > 0 \) and \( A > |B| \) if \( B < 0 \). Using well known relations valid for Jacobi elliptic functions [37], the derivatives of \( \rho \) with respect to \( \xi \) are easily calculated, namely
\[
\begin{align*}
\frac{dy}{d\xi} &= 2\mu \text{sn} \text{cn} \text{dn} u \\
\frac{d^2y}{d\xi^2} &= 2\mu^2 \left[ 3k^2 y^2 - 2(1 + k^2) y + 1 \right] \\
\frac{d^3y}{d\xi^3} &= 4\mu^2 \left[ 3k^2 y - (1 + k^2) \right] \frac{dy}{d\xi}
\end{align*}
\] (36)
When these are introduced into (24) one obtains
\[
\mu^2 \left[ 3k^2 y - (1 + k^2) \right] + E + 3\beta(A + By) = 0
\]
which has to be satisfied for any value of \( y \). Therefore the following two conditions result
\[
\begin{align*}
3\beta A &= \mu^2(1 + k^2) - E = 3e, \quad A = \beta e \\
\beta B &= -\mu^2k^2, \quad B = -\beta \mu^2k^2
\end{align*}
\] (37)
The positivity of \( A \) implies that \( e > 0 \) when \( \beta = +1 \) and \( e < 0 \) if \( \beta = -1 \). As for \( \beta = +1 \), \( B < 0 \), \( \rho \) remains positive if \( e \geq \mu^2k^2 \). Then the solutions are
\[
\begin{align*}
\rho &= e - \mu^2k^2 \text{sn}^2 u, \quad e \geq \mu^2k^2, \quad \text{for } \beta = 1 \\
\rho &= |e| + \mu^2k^2 \text{sn}^2 u, \quad \text{for } \beta = -1
\end{align*}
\] (38)
These are identical with the solutions found previously if we identify \( e = \rho_1, \rho_1 - \rho_2 = \mu^2k^2, \rho_2 = \rho_1 - \rho_3 \) for \( \beta = +1 \) and \( |e| = \rho_3, \rho_2 - \rho_3 = \mu^2k^2, \mu^2 = \rho_1 - \rho_3 \) for \( \beta = -1 \).

4. CUBIC + QUINTIC NONLINEARITY

In this section a detailed discussion of the cubic+quintic nonlinearity in the NLS equation will be presented. With
\[
U(\rho) = \beta \rho + \frac{3}{2} \gamma \rho^2
\] (39)
the equation (14) writes
\[
\frac{1}{4} \frac{d^3 \rho}{d \xi^3} + E \frac{d \rho}{d \xi} + \frac{d}{d \xi} \left( \frac{3}{2} \beta \rho^2 + 2 \gamma \rho^3 \right) = 0
\] (40)
which integrated twice becomes
\[
\frac{1}{4} \left( \frac{d \rho}{d \xi} \right)^2 = P_4(\rho), \quad P_4(\rho) = -\gamma \rho^4 - \beta \rho^3 - E \rho^2 + A \rho + B
\] (41)
Here \( E, A, B \) are arbitrary constants. As in the previous section we consider the cases \( \beta = \pm 1 \). In integrating (41) one has to choose positive solutions \( \rho \) and those domains of \( \rho \) where \( P_4(\rho) > 0 \). Further on we shall discuss separately the influence of the sign of \( \gamma \).

4.1. CASE \( \gamma > 0 \)

Let us consider that the polynomial \( P_4(\rho) \) has four distinct, real roots \( \rho_1 > \rho_2 > \rho_3 > \rho_4 \) and at least two of them are positive \( \rho_1 > \rho_2 > 0 \). The conditions mentioned above are realized if \( \rho \in [\rho_2, \rho_1] \). The solution of (41) is given by \( [37], p. 124 \)
\[
\int_{\rho}^{\rho_1} \frac{dt}{\sqrt{(\rho_1 - t)(t - \rho_2)(t - \rho_3)(t - \rho_4)}} = g u = 2 \sqrt{\gamma} \xi
\]
\[
\alpha^2 \text{sn}^2 u = \frac{\rho_1 - \rho}{\rho - \rho_4}, \quad \rho = \frac{\rho_1 + \rho_4 \alpha^2 \text{sn}^2 u}{1 + \alpha^2 \text{sn}^2 u}, \quad \rho = \frac{2 \sqrt{\gamma} \xi}{g}
\]
\[
k^2 = \frac{\alpha^2 \rho_3 - \rho_4}{\rho_1 - \rho_3}, \quad \alpha^2 = \frac{\rho_1 - \rho_2}{\rho_2 - \rho_4}, \quad g = \frac{2}{\sqrt{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}}
\] (42)
In the limit case \( \rho_2 = \rho_3, \ k^2 = 1 \), the solution becomes
\[
\rho = \frac{\rho_1 + \rho_4 \alpha^2 \tanh^2 u}{1 + \alpha^2 \tanh^2 u}
\]
(43)
describing a shifted bright solitary wave \( \rho(0) = \rho_1, \ \rho(\infty) = \rho_2 \).

For constant velocity, the additional condition \( B = 0 \) has to be imposed, and this can be satisfied if one of the roots \( \rho_2, \rho_3, \rho_4 \) is equal to zero. Let us consider only the limit case \( \rho_2 = \rho_3 \) and take \( \rho_2 = 0 \) \( \rho_4 < 0 = \rho_3 = \rho_2 < \rho_1 \). Then \( |\rho_4| \alpha^2 = \rho_1 \), \( g = 2/\sqrt{\rho_1 |\rho_4|} \) and the solution (43) is the bright solitary wave
\[
\rho = \frac{1}{1 + \alpha^2 \tanh^2 u} \rho_1 \cosh^2 u, \quad u = \sqrt{\gamma \rho_1 |\rho_4|} \xi
\]
(44)
This can be compared with the bright soliton solution (28) of the cubic NLS equation. Using the relation between roots and coefficients of a polynomial equation, in the present case one finds $|\rho_4| = \rho_1 + 1/\gamma > \rho_1$. Then the domain of the variable $u_c$ in (28) is boosted for $u_{c+q}$ in (44), $u_{c+q} = \sqrt{\gamma|\rho_4|}u_c$ and consequently the bright solitary solution (44) is much steeper than the bright soliton (28). In the figure 1 the two solutions are represented for $\rho_1 = 1$, $\gamma = 1/2$ ($|\rho_4| = 3$, $\alpha^2 = 1/3$). As the phase is concerned, it is given in both cases by the expression (21), but with different values of the constant $E$, namely $E_c = -\rho_1$ and $E_{c+q} = -\gamma\rho_1|\rho_4|$ for the cubic NLS and the cubic + quintic case respectively. As a general rule the phase for the solution of the cubic + quintic NLS equation has a more rapid variation than for the cubic case.

If all the roots are positive (when $\gamma > 0$ this is possible only if $\beta = -1$), a second acceptable situation exists if $\rho \in [\rho_4, \rho_3]$ and the solution is given by ([37], p. 103)

$$\int_{\rho_4}^{\rho} \frac{dt}{\sqrt{(\rho_1-t)(\rho_2-t)(\rho_3-t)(t-\rho_4)}} = gu = 2\sqrt{\gamma}\xi$$

$$\alpha^2 \sin^2 u = \frac{\rho - \rho_4}{\rho_1 - \rho}, \quad \rho = \frac{\rho_4 + \rho_1\alpha^2 \sin^2 u}{1 + \alpha^2 \sin^2 u}, \quad u = \frac{2\sqrt{\gamma}}{g}\xi$$

$$k^2 = \alpha^2 \frac{\rho_1 - \rho_2}{\rho_2 - \rho_4}, \quad \alpha^2 = \frac{\rho_3 - \rho_4}{\rho_1 - \rho_3}, \quad g = \frac{2}{\sqrt{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}}$$

The limit case $k^2 = 1$ is obtained when $\rho_2 = \rho_3$, and the solution transforms to

$$\rho = \frac{\rho_4 + \rho_1\alpha^2 \tanh^2 u}{1 + \alpha^2 \tanh^2 u}$$

(describing a gray solitary wave ($\rho(0) = \rho_4$, $\rho(\infty) = \rho_3$). In the case of constant velocity, the condition $B = 0$ can be realized only if $\rho_4 = 0$. Then the solution (46)
becomes
\[ \rho = \frac{\rho_1 \alpha^2 \tanh^2 u}{1 + \alpha^2 \tanh^2 u}, \quad \alpha^2 = \frac{\rho_2}{\rho_1 - \rho_2}, \quad u = \sqrt{\gamma} \rho_2 (\rho_1 - \rho_2) \xi \]  
(47)
describing a dark solitary wave.

The case when \( P_1(\rho) \) has two positive roots \( \rho_1 > \rho_2 > 0 \) and the other two complex conjugated (\( \rho_3 = b + ia, \rho_4 = \rho_3^* = b - ia \)) represents another acceptable situation (with \( b \leq \rho_2 \)). Then the solution of (41) is (\[37\], p. 133)

\[
\int_{\rho_2}^{\rho} \frac{dt}{\sqrt{(\rho_1 - t)(\rho_2 - t)(t - \rho_3)(t - \rho_4)}} = gu = 2\sqrt{\gamma} \xi
\]
\[\text{cn} u = \frac{(\rho_1 - \rho)B - (\rho - \rho_2)A}{(\rho_1 - \rho)B + (\rho - \rho_2)A}, \quad \rho = \frac{(\rho_2A + \rho_1B) + (\rho_2A - \rho_1B)\text{cn} u}{(A + B) + (A - B)\text{cn} u} \]  
(48)
\[k^2 = \frac{(\rho_1 - \rho_2)^2 - (A - B)^2}{4AB}, \quad g = \frac{1}{\sqrt{AB}}, \quad u = \frac{2\sqrt{\gamma}}{g} \xi
\]
\[A^2 = (\rho_1 - b)^2 + a^2, \quad B^2 = (\rho_2 - b)^2 + a^2 \]

The limit case \( k^2 = 1 \) is attained when \( \rho_1 - \rho_2 = A + B \) and \( \rho_1 + \rho_2 - 2b = A - B \). Then \( A = \rho_1 - b, B = -(\rho_2 - b), \rho_2A + \rho_1B = b(\rho_1 - \rho_2) \) and \( \rho_2A - \rho_1B = 2\rho_1\rho_2 - b(\rho_1 + \rho_2) \). As \( \lim_{k^2 \to 1} \text{cn} u = \text{sech} u \), the solution (48) writes

\[\rho = \frac{b(b(\rho_1 - \rho_2) + [2\rho_1\rho_2 - b(\rho_1 + \rho_2)]\text{sech} u)}{(\rho_1 - \rho_2) + (\rho_1 + \rho_2 - 2b)\text{sech} u} \]  
(49)

For \( u = 0, \rho(0) = \rho_2 \) while when \( u \to \infty, \rho(\infty)b < \rho_2 \), and (47) describes a shifted bright solitary wave. For \( b = 0 \) it becomes a bright solitary wave

\[\rho = \frac{2\rho_1\rho_2}{(\rho_1 + \rho_2) + (\rho_1 - \rho_2)\cosh u} \]  
(50)

4.2. CASE \( \gamma < 0 \)

The two conditions mentioned before (\( \rho > 0, P_1(\rho) > 0 \)) can be satisfied if \( P_4(\rho) \) has at least three positive, distinct roots \( \rho_1 > \rho_2 > \rho_3 > 0 \) and \( \rho \in [\rho_3, \rho_2] \). Then the solution is (\[37\], p. 116)

\[
\int_{\rho}^{\rho_3} \frac{dt}{\sqrt{(\rho_1 - t)(\rho_2 - t)(t - \rho_3)(t - \rho_4)}} = gu = 2\sqrt{|\gamma|} \xi
\]
\[\alpha^2 \text{sn}^2 u = \frac{\rho_2 - \rho}{\rho_1 - \rho}, \quad \rho = \frac{\rho_2 - \rho_1 \alpha^2 \text{sn}^2 u}{1 - \alpha^2 \text{sn}^2 u}, \quad u = \frac{2\sqrt{|\gamma|}}{g} \xi \]  
(51)
\[k^2 = \frac{\rho_1 - \rho_4}{\rho_2 - \rho_4}, \quad \alpha^2 = \frac{\rho_2 - \rho_3}{\rho_1 - \rho_3}, \quad g = \frac{2}{\sqrt{(\rho_1 - \rho_3)(\rho_2 - \rho_4)}} \]
The degenerate case \( k^2 = 1 \) is obtained when \( \rho_3 = \rho_4 \). Then the solution (51) becomes
\[
\rho = \frac{\rho_2 - \rho_1 \alpha^2 \tanh^2 u}{1 - \alpha^2 \tanh^2 u}
\]
describing a shifted bright solitary wave \((\rho(0) = \rho_2, \rho(\infty) = \rho_3)\). In the case of constant velocity, the condition \( B = 0 \) implies that one of the two roots \( \rho_3 \) and \( \rho_4 \) is zero. In the limit case \( k^2 = 1 \), \( \rho_3 = \rho_4 = 0 \) and (52) transforms into the bright soliton solution
\[
\rho = \frac{1}{1 - \alpha^2 \tanh^2 u} \quad \alpha^2 = \frac{\rho_2}{\rho_1}, \quad u = \sqrt{|\gamma|} \rho_2 (\rho_1 - \rho_2) \xi
\]
(53)

The phase \( \theta(x,t) = \theta(\xi) \) is calculated using (23) and the result is expressed again in terms of an incomplete elliptic integral of third kind. Here we give the result only for the case \( \gamma > 0 \), using the expression (42) for \( \rho(\xi) \). It writes
\[
\theta(\xi) = \left( u_0 + \frac{A_0}{\rho_4} \right) \xi - A_0 \frac{g}{2\sqrt{\gamma}} \frac{\rho_1 - \rho_4}{\rho_1 \rho_4} \Pi \left( \varphi, -\frac{\rho_1}{\rho_1 \alpha^2}, k \right)
\]
(54)
with \( \sin \varphi = sn \ u \) and \( u = \frac{2\sqrt{\gamma}}{g} \xi \). The incomplete elliptic integral of the third kind appearing in (54) is in the circular case. In the limit case \( k^2 = 1 \) the expression of the phase \( \theta(\xi) \) becomes
\[
\theta(\xi) = \left( u_0 - \frac{A_0}{|\rho_4|} \right) \xi + A_0 \frac{g}{2\sqrt{\gamma}} \frac{\rho_1 + |\rho_4|}{\rho_1 |\rho_4|} \int_0^u \frac{dt}{1 - \eta^2 \tanh^2 t}
\]
(55)
where we denoted \( \eta^2 = \frac{|\rho_4|}{\rho_1} \alpha^2 \leq 1 \) (\( \eta^2 = 1 \) for the constant velocity case). The integral is easily calculated with the change of variable \( x = \tanh t \)
\[
\int_0^u \frac{dt}{1 - \eta^2 \tanh^2 t} = \int_0^X \frac{dx}{1 - x^2} \frac{1}{1 - \eta^2 x^2}, \quad X = \tanh u \leq 1
\]
(56)
\[
= \frac{1}{1 - \eta^2} \left\{ \frac{1}{2} \ln \frac{1 + X}{1 - X} - \eta^2 \ln \frac{1 + \eta X}{1 - \eta X} \right\} = \frac{1}{1 - \eta^2} [u - \eta \arctanh(\eta \tanh u)].
\]
For \( u \to \pm \infty \), the second term is finite, namely \( \pm \arctanh \eta \).

5. SUMMARY

In this paper the Madelung fluid description was used to find periodic and solitary wave solutions of the generalized NLS equations. The main interest was the study of a cubic + quintic nonlinearity. It was found that the pulse solution (bright
type solitary wave in the case of cubic + quintic nonlinearity (for $\gamma > 0$, $\beta > 0$) is much steeper and narrower than in the simple case of cubic nonlinearity (bright soliton), representing a better model to describe the propagation of short laser pulses in nonlinear dielectric media. At the same time a larger class of solutions can exist in this case compared to the usual cubic NLS equation. The present results are in agreement with those already found in the literature which were derived by different methods (see [38] and references therein). Also they fit well with the discussion of [35, 36] concerning the correspondence between generalized NLS equations and generalized KdV ones. As in the case of cubic NLS equation, a direct method can be used to find solutions of (40). It is easily seen that a solution of the form (35) is not possible and we are obliged to consider

$$\rho(\xi) = \frac{A + By}{1 + Cy}, \quad y = \text{sn}^2 \mu \xi$$

Indeed all the solutions we found previously (see (43), (46), (52)) are of this form.

A qualitative discussion for traveling wave solutions of an arbitrary polynomial nonlinearity (of order $n$) is possible. Indeed, integrating twice the equation (14) one gets

$$(\rho \xi)^2 - P_{n+2}(\rho) = 0 \quad (57)$$

This has a very simple interpretation if we consider $\xi$ as a time variable, and $\rho$ as a position variable. Then $(\rho \xi)^2$ is a kinetic energy and $-P_{n+2}(\rho)$ a potential energy and $\rho$ evolves on a zero-energy surface in the phase space belonging to $P_{n+2}(\rho)$. This is known as the potential representation of the nonlinear dispersive evolution equation. The concept was introduced by Rosenau [39, 40], and was applied for a large class of nonlinear evolution equations in [41]. Closed trajectories in bounded regions of the phase space correspond to periodic solutions, whereas solitary wave solutions are represented in the phase space by separatrix trajectories [42]. Unfortunately explicit expressions for periodic solutions when the nonlinearity degree $n > 2$ are not known as the integration of the equation (57) leads to hyperelliptic functions of complex nature. Probably it could be a useful starting point for numerical calculations. In spite of this, it is proved in the present paper that the Madelung fluid description is a fruitful method to find periodic and solitary wave solutions for generalized NLS equations of practical interest.

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