FRACTIONAL PERTURBATION TECHNIQUE OF FRACTIONAL DIFFERENTIABLE FUNCTIONS

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In this paper, a fractional perturbation approach is proposed to approximately solve local fractional differential equations. Two examples are used to illustrate the method’s efficiency and convenience.

Key words: fractional differentiable functions; perturbation technique; fractal curves.

1. INTRODUCTION

Based on Cantor-like sets, a local fractional derivative was proposed by Kolwankar and Gangal [1]

\[ D_\alpha^x f(x) = \lim_{\alpha \to 0} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x_0}^{x} (x-\xi)^{n-\alpha} (f(\xi) - f(x_0)) d\xi, \quad (1) \]

where the derivative on the right-hand side is the Riemann-Liouville fractional derivative. With this concept, the local behavior of a fractional Fokker-Planck equation was investigated [1].

Naturally, to find a method for such fractional equations is under taking. The homotopy perturbation method (HPM) [2] provides an effective procedure for numerical solutions of a wide and general class of differential systems representing real physical problems. Many researchers dedicated their effort to applications in fractional differential equations [3–13] and gained much success in fractional calculus. However, on the other hand, since the Caputo derivative and Riemann-Liouville derivative are nonlocal operators, they are not suitable for local fractional differential equations.

Our concern in this work is to directly extend the homotopy perturbation method to local fractional differential equations and obtain non-analytical solution in non-smooth initial boundary problems. This paper aims to derive non-analytical solutions of fractional differential equations via fractional differentiable curves.

2. PROPERTIES OF FRACTIONAL DIFFERENTIABLE FUNCTIONS

Fractional differentiable functions have following properties which are used in this study.

(a) The Taylor series for fractional differentiable functions

If \( f(x) \) is a \( k\alpha \)-differentiable function and \( k \) is an arbitrary positive integer, the Taylor series [14] can be presented as

\[
df = \sum_{i=0}^{\infty} \frac{x^i}{(k\alpha)!} f^{(k\alpha)}(x),
\]

Here \( f(x) \) is \( k\alpha \)-differentiable at the point \( x=0 \). The first two terms on the right side of Eq. (2) are given by Kolwankar and Gangal [1]. The fractional Taylor series is important here since it is used to assume the initial iteration of the following method. Some new results of the fractional differentiable functions can be found in Refs. [15–17].

(b) Fractional Leibniz product law

From the property (2), for the \( \alpha\)-order differential functions \( u \) and \( v \), we can readily find that

\[
D_{x}^{\alpha}(uv) = u^{(\alpha)}v + uv^{(\alpha)}.
\]

Recall that the law here holds for fractional differential functions, which is similar as the one in ordinary calculus.

(c) Fractional Leibniz formulation of fractional differentiable functions

For the fractional differentiable functions, we can have

\[
\alpha I_{x}^{\alpha} D_{x}^{\alpha} f(x) = f(x) - f(0), \quad 0 < \alpha \leq 1,
\]

and

\[
D_{x}^{\alpha} I_{x}^{\alpha} f(x) = f(x), \quad 0 < \alpha \leq 1.
\]

As a result, generalized integration by parts can be used in the fractional calculus operations

\[
\alpha I_{x}^{\alpha} u^{(\alpha)}v = (uv)^{\alpha} - \alpha I_{x}^{\alpha} uv^{(\alpha)}.
\]

Remark: Since Kolwanlar-Gangal’s derivative here is a localized operator, the behaviors of fractal systems act as a non-differential functions whose
differentiability described by fractional differentiable functions. Especially, the fractional order is the Hausdorff dimension of Cantor-like set \([18, 19]\) from which the non-differentiable functions are generated. The fractional differentiable function is illustrated with the evolutions of the space scales \([20]\).

On the other hand, numerical methods in ordinary calculus can be extended to fractional differentiable functions since the localized operator has the similar point to point property of the derivative of integer. Thus, we consider in the next section to extend the HPM method to fractional differentiable functions which can deal with the fractal initial boundary value or non-smooth problems.

3. FRACTIONAL PERTURBATION TECHNIQUE

In order to illustrate the efficiency for fractional partial differential equations, we consider the following three examples.

Example 1. Kolwankar’s local Fokker-Planck equation (an analog of a diffusion equation) \([1]\)

\[
\frac{\partial^\alpha}{\partial t^\alpha} W(x,t) = \chi_c(t) \frac{\partial^2}{\partial x^2} W(x,t), \quad 0 \leq t, 0 \leq x, \quad 0 \leq \alpha < 1,
\]

with the initial value \(W_0 = W(x,0) = e^t\).

As pointed out by Kolwankar and Gangal, even though the variable \(t\) is taking all real positive values the actual evolution takes place only for values of \(t\) in the fractal set \(C\). We take \(\chi_c(t) = 1\) which is a flag function.

Since there is no nonlinear team here, we can construct the following homotopy accord to the homotopy perturbation method \([2]\),

\[
(1 - p)(\frac{\partial^\alpha}{\partial t^\alpha} W(x,t) - \frac{\partial^\alpha}{\partial t^\alpha} W_0) + p(\frac{\partial^\alpha}{\partial t^\alpha} W(x,t) - \chi_c(t) \frac{\partial^2}{\partial x^2} W(x,t)) = 0.
\]

Take \(W = W_0 + W_1 p + W_2 p^2 + \ldots\) into Eq. \((7)\). When \(p = 1\), we can derive the \(n^{th}\) approximate solution of Eq. \((6)\)

\[
W = W_0 + W_1 + W_2 + \ldots + W_n.
\]

Collect the coefficients of \(p\), we can have

\[
p^1 : W_1^{(\alpha)} - W_0^{(\alpha)} = 0, W_1(0) = e^t,
\]

\[
p^2 : W_2^{(\alpha)} - W_1^{(\alpha)} = 0, W_2(0) = e^t,
\]

\[
\ldots
\]

As a result, we can obtain
We can check that \( W(x,t) = \lim_{n \to \infty} \sum_{i=0}^{n} W_i = e^x E_\alpha(t^\alpha) \) is the exact solution of Eq. (6).

Example 2. In order to illustrate the efficiency for higher fractional-order differential equations, we consider the fractional Riccati equation

\( y^{(2\alpha)}(t) = y^2(t) + 1, \quad 0 \leq t, \quad 0 < \alpha \leq 1. \)  \( \tag{8} \)

with the initial value \( y(0) = 0 \) and \( y^{(\alpha)}(0) = 1. \) Here \( y^{(2\alpha)} \) is defined by \( \frac{\partial^{\alpha} y}{\partial t^{\alpha}}. \)

Construct the following homotopy

\( (1-p)(Y^{(2\alpha)} - y_0^{(2\alpha)}) + p[Y^{(2\alpha)} - Y^2 - 1] = 0, \)

or

\( Y^{(2\alpha)} - y_0^{(2\alpha)} + y_0^{(2\alpha)} - p(Y^2 + 1) = 0. \)

Take \( Y = Y_0 + Y_1 p + Y_2 p^2 + ... + Y_n p^n + ... \) and substitute it into Eq. (9). We can collect the coefficients of \( p \)

\( p^0 : y_0^{(2\alpha)} - y_0^{(2\alpha)} = 0, \)

\( p^1 : y_1^{(2\alpha)} + y_0^{(2\alpha)} - Y^2 - 1 = 0, \)

\( p^2 : Y_2^{(2\alpha)} - 2Y_0 Y_1 = 0, \)

\( ... \)

Using the Taylor series [7], we can take the initial iteration or the trial function \( Y_0 = y_0 = \frac{t^\alpha}{\Gamma(1 + \alpha)}. \) Solve the above equations

\( Y_0 = y_0 = \frac{t^\alpha}{\Gamma(1 + \alpha)}, \)

\( Y_1 = \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{2t^{4\alpha}}{\Gamma(1 + 4\alpha)}, \)

\( Y_2 = \frac{2t^{5\alpha}}{\Gamma(1 + 5\alpha)} + \frac{2t^{7\alpha}}{\Gamma(1 + 7\alpha)}, \)

\( ... \)
As a result, if the second order approximate is sufficient, we can derive

\[
Y \approx Y_0 + Y_1 + Y_2 = \frac{t^\alpha}{\Gamma(1 + \alpha)} + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{2t^{4\alpha}}{\Gamma(1 + 4\alpha)} + \frac{6t^{5\alpha}}{\Gamma(1 + 5\alpha)} + \frac{20t^{7\alpha}}{\Gamma(1 + 7\alpha)}.
\]

The same fractional differential equation was also approximately solved by the iteration method [10, 11]. Readers must note that the solution \( Y \) is non-differentiable with the fractal support, i.e., the graph function of \( t^\alpha \) is illustrated which is a non-differentiable function [20]. We only can show the trend of the approximate solution on the large scale. When the parameter \( \alpha \) tends to 1, the function

\[
Y = t + \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^5}{20} + \frac{t^7}{252}
\]

is the approximate solution of the ordinary differentiable equation

\[
y(t) = y^2(t) + 1,
\]

with the initial value problem

\[
y(0) = 0 \quad \text{and} \quad y^0(0) = 1.
\]

Fig. 1 illustrates the approximate solution on the large scale where \( \alpha = 0.9, \alpha = 0.99 \) and \( \alpha = 1 \).

![Fig. 1 – Approximate solution of fractional order \( \alpha = 0.9, \alpha = 0.99 \) and \( \alpha = 1 \).](image)
The discontinuous line (--) is the approximate solution when $\alpha = 0.9$ and the dotted line (…) when $\alpha = 0.99$. The continuous line is the exact solution of the ordinary equation when $\alpha = 1$. The trend of the approximate solutions was also given by Wu [17, 21, 22].

4. CONCLUSIONS

Compared with the fractional decomposition method [21] and the fractional variational iteration method [22], the present method has some obvious merits: (1) the method need not calculate Adomian polynomials; (2) we do not need to determine the generalized multiplier based on fractional variational theory. Besides, our method can solve the fractional initial value problems.

We conclude that, from the viewpoint of the Kolwankar-Gangal's local fractional derivative, the parameter $\alpha$ is the fractal dimension of time. Thus, the approximate solution is generated by some distribute function defined over the fractal sets in some closed interval $[0, 1]$. They are continuous but not differentiable functions with respect to $t$.

REFERENCES