

QUANTUM COHERENCE OF TWO-MODE SYSTEMS IN A THERMAL ENVIRONMENT

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In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we address the quantification of coherence in Gaussian states of continuous variable systems. We give a description of quantum coherence by using the relative entropy as a measure for quantum correlations between two non-interacting non-resonant bosonic modes embedded in a thermal environment. We evaluate the relative entropy of coherence by simulating the effects of a thermal bath in time on the system. This provides a picture of the decoherence of simple open systems.

Key words: Quantum coherence, Gaussian states, Open quantum systems.

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1. INTRODUCTION

Quantum correlations represent an important physical resource for many quantum information processing and communication protocols and tasks [1, 2]. During the last decades a lot of attention was paid to the study of quantum correlations, including quantum entanglement and quantum discord, of bipartite or multipartite states. For example, the time evolution of quantum correlations in systems consisting of two-bosonic modes interacting with thermal environments was studied in the recent years in Refs. [3–5].

A huge boost in the development of quantum information theory came from a resource theory approach to quantum entanglement. This lucrative formulation treats what used to be considered constraints of the theory as useful resources. An active field of research is the expansion of this framework to the theory of discord and general quantum correlations.

Coherence is a necessary condition for all types of quantum correlations, yet for the most part it has been ignored in the study of quantum information theory. Recently, a framework for the quantification of coherence has been established [6], in which quantum coherence is treated as a resource in a manner similar to quantum entanglement. The main results so far, apply mostly to the finite dimensional setting,

which do not describe many physical relevant situations such as quantum optics. In this paper we address the quantification of coherence in the framework of completely positive dynamical semigroups, which describe the time evolution of quantum open systems of continuous variables.

2. GKLS MASTER EQUATION

The most general description of the irreversible time evolution of an open system is given by the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) quantum master equation. This is done in the Markovian approximation, meaning that there are no memory effects. The GKLS quantum Markovian master equation for the density operator $\rho(t)$ in the Schrödinger representation is [7–9]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j (2V_j \rho(t) V_j^\dagger - \{\rho(t), V_j^\dagger V_j\}_+). \quad (1)$$

Here, H denotes the Hamiltonian of the open system and the operators V_j, V_j^\dagger , defined on the Hilbert space of H , represent the interaction of the open system with the environment and \dagger denotes Hermitian conjugation.

GKLS is based on the axiomatic formalism with completely positive quantum dynamical semigroups, which preserve the Gaussian nature of the state during time evolution of the system [10]. In this case the model represents a Gaussian lossy channel.

The Hamiltonian of two uncoupled non-resonant, harmonic oscillators of identical mass m and frequencies ω_1 and ω_2 is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m}{2}(\omega_1^2 x^2 + \omega_2^2 y^2), \quad (2)$$

where x, y are the coordinates and p_x, p_y are the momenta of the two bosonic modes, and V_j, V_j^\dagger are taken polynomials of first degree in these canonical observables ($j = 1, 2, 3, 4$) [10]:

$$V_j = a_{xj} p_x + a_{yj} p_y + b_{xj} x + b_{yj} y, \quad (3)$$

$$V_j^\dagger = a_{xj}^* p_x + a_{yj}^* p_y + b_{xj}^* x + b_{yj}^* y, \quad (4)$$

where $a_{xj}, a_{yj}, b_{xj}, b_{yj} \in \mathbf{C}$ and $*$ denotes complex conjugation.

3. COVARIANCE MATRIX FORMALISM

In this paper we consider bipartite two-mode Gaussian states ρ_{AB} , i.e. states with Gaussian Wigner distribution on the quantum phase space of two quantized

modes. Without loss of generality, we can assume that all the first moments (expected values) of the canonical phase space operators are set to zero by means of local unitary operations, because we are aiming at computing correlations between the modes, which are not affected by local displacements. The states under consideration are then entirely specified by the second order moments (covariances), grouped together in the following 4×4 bimodal covariance matrix:

$$\sigma(t) = \begin{pmatrix} \sigma_{xx}(t) & \sigma_{xp_x}(t) & \sigma_{xy}(t) & \sigma_{xp_y}(t) \\ \sigma_{xp_x}(t) & \sigma_{p_x p_x}(t) & \sigma_{yp_x}(t) & \sigma_{p_x p_y}(t) \\ \sigma_{xy}(t) & \sigma_{yp_x}(t) & \sigma_{yy}(t) & \sigma_{yp_y}(t) \\ \sigma_{xp_y}(t) & \sigma_{p_x p_y}(t) & \sigma_{yp_y}(t) & \sigma_{p_y p_y}(t) \end{pmatrix} \equiv \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad (5)$$

which can be written in standard form as:

$$\sigma(t) = \begin{pmatrix} a(t) & 0 & c(t) & 0 \\ 0 & a(t) & 0 & d(t) \\ c(t) & 0 & b(t) & 0 \\ 0 & d(t) & 0 & b(t) \end{pmatrix}, \quad (6)$$

where $\det A$, $\det B$, $\det C$ and $\det \sigma(t)$ are invariant quantities.

The covariance matrix is a real, symmetric and positive matrix. Its elements, which up to local displacements fully characterize any Gaussian state of a bipartite system, are given by

$$\sigma_{ij} = \frac{1}{2} \text{Tr}(\rho_{AB} \{R_i, R_j\}_+), \quad (7)$$

where $i, j = 1, \dots, 4$ and $\mathbf{R} = (x, p_x, y, p_y)$ is the vector of phase space operators satisfying the canonical commutation relations $[R_i, R_j] = i\Omega_{ij}$. Here $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus 2}$ is the symplectic matrix [11].

A , B and C are 2×2 Hermitian matrices: A and B denote the symmetric covariance matrices for the individual reduced one-mode states, while the matrix C contains correlations between the modes (T denotes the transposed matrix).

4. TIME EVOLUTION OF THE SYSTEM

The equations of motion written in terms of the covariance matrix are as follows [10]:

$$\frac{d\sigma(t)}{dt} = X\sigma(t) + \sigma(t)X^T + 2D, \quad (8)$$

with the solution given by

$$\sigma(t) = \exp(Xt)[\sigma(0) - \sigma(\infty)]\exp(Xt) + \sigma(\infty). \quad (9)$$

The values at the infinity of time are extracted from the following equation:

$$X\sigma(\infty) + \sigma(\infty)X^T = -2D, \quad (10)$$

with X in terms of mass m , frequencies ω_1 and ω_2 , and dissipation constant λ :

$$X = \begin{pmatrix} -\lambda & \frac{1}{m} & 0 & 0 \\ -m\omega_1^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & \frac{1}{m} \\ 0 & 0 & -m\omega_2^2 & -\lambda \end{pmatrix} \quad (11)$$

and D the diffusion matrix

$$D = \begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_x} & D_{p_x p_x} & D_{yp_x} & D_{p_x p_y} \\ D_{xy} & D_{yp_x} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{p_x p_y} & D_{yp_y} & D_{p_y p_y} \end{pmatrix}. \quad (12)$$

If we assume that the asymptotic state is a Gibbs state corresponding to the two modes in thermal equilibrium at temperature T , then the diffusion matrix becomes [9]:

$$D = \begin{pmatrix} \frac{\lambda}{2m\omega_1} \coth \frac{\omega_1}{2KT} & 0 & 0 & 0 \\ 0 & \frac{\lambda m \omega_1}{2} \coth \frac{\omega_1}{2KT} & 0 & 0 \\ 0 & 0 & \frac{\lambda}{2m\omega_2} \coth \frac{\omega_2}{2KT} & 0 \\ 0 & 0 & 0 & \frac{\lambda m \omega_2}{2} \coth \frac{\omega_2}{2KT} \end{pmatrix}. \quad (13)$$

With this assumption the system is fully analytically solvable.

5. RELATIVE ENTROPY OF COHERENCE

By definition, relative entropy has the following form:

$$S(\rho_1|\rho_2) = -S(\rho_1) - \text{Tr}(\rho_1 \log \rho_2), \quad (14)$$

where $S(\rho)$ is the von Neumann entropy $S(\rho) = -\text{Tr}(\rho \log \rho)$, the quantum analogue of Shannon entropy from classical information theory. Relative entropy is a pseudo distance, in the sense that it is semi positive and contractive, but not symmetric:

$$S(\rho_1|\rho_2) \geq 0, \quad (15)$$

$$S(\rho_1|\rho_2) \geq S(\Gamma \rho_1|\Gamma \rho_2), \quad (16)$$

$$S(\rho_1|\rho_2) \neq S(\rho_2|\rho_1), \quad (17)$$

where Γ is a completely positive trace preserving (CPT) map. Pseudo distances are often used as measures of resources. As such, relative entropy can be used to define a measure for quantum coherence, namely the relative entropy of coherence [12]. This is done by taking the relative entropy between a given Gaussian state and the closest Gaussian thermal state. However, this is only an upper bound because the Gaussian

set is not convex. Ideally one should search for the closest incoherent quantum state, which in general is not of Gaussian form. With this in mind, the explicit formula for the relative entropy of coherence can be written as:

$$C(\rho) = \inf_{\delta} S(\rho|\delta), \quad (18)$$

where δ is a Gaussian thermal state. Calculating $\sup_{\delta} \text{Tr}(\rho \log \delta)$ gives $\delta = \rho_{diag}$, therefore the formula becomes:

$$C(\rho) = S(\rho_{diag}) - S(\rho). \quad (19)$$

The corresponding covariance matrix for the closest thermal state in the two-mode case, is described by the average numbers of thermal photons n_1 and n_2 :

$$n_1 = \frac{1}{4}(A_{11} + A_{22} - 2),$$

$$n_2 = \frac{1}{4}(B_{11} + B_{22} - 2).$$

This leads to the final expression for the relative entropy of coherence of a two-mode Gaussian state [12]:

$$C(\rho) = S(\rho_{diag}) - S(\rho) = \sum_{i=1}^2 [g(2n_i + 1) - g(k_i)], \quad (20)$$

where

$$g(x) = \frac{x+1}{2} \log \frac{x+1}{2} - \frac{x-1}{2} \log \frac{x-1}{2}. \quad (21)$$

Here k_1 and k_2 are the symplectic eigenvalues of the covariance matrix

$$\sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix} \quad (22)$$

and are of the form

$$2k_1^2 = \Delta - \sqrt{\Delta^2 - 4 \det \sigma}, \quad (23)$$

$$2k_2^2 = \Delta + \sqrt{\Delta^2 - 4 \det \sigma}, \quad (24)$$

where Δ is symplectic invariant (seralian), given by

$$\Delta = \det A + \det B + 2 \det C. \quad (25)$$

We take as the initial Gaussian state of our system a two-mode squeezed thermal state, which has the corresponding covariance matrix [13]

$$\sigma(0) = \begin{pmatrix} a(0) & 0 & c(0) & 0 \\ 0 & a(0) & 0 & -c(0) \\ c(0) & 0 & b(0) & 0 \\ 0 & -c(0) & 0 & b(0) \end{pmatrix}, \quad (26)$$

with

$$a(0) = n_1 \cosh^2 r + n_2 \sinh^2 r + \frac{1}{2} \cosh 2r, \quad (27)$$

$$b(0) = n_1 \sinh^2 r + n_2 \cosh^2 r + \frac{1}{2} \cosh 2r, \quad (28)$$

$$c(0) = \frac{1}{2}(n_1 + n_2 + 1) \sinh 2r, \quad (29)$$

where n_1 and n_2 are the average numbers of thermal photons associated with the two modes and r denotes the squeezing parameter. In the particular case where $n_1 = 0$ and $n_2 = 0$, (26) becomes the covariance matrix of the two-mode squeezed vacuum state.

We compute the relative entropy of coherence C as a function of time and temperature. The evolution of C is illustrated in figures 1-4. As shown, it has finite values and it decays asymptotically to zero in time under the effect of the thermal bath. It is not strictly monotonous because the measure provides only an upper bound to coherence by only searching for the closest Gaussian thermal state and not a general quantum incoherent one. If the squeezing parameter r is zero, then C is also zero as expected.

6. SUMMARY

In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we described the time evolution of the relative entropy of coherence for a system composed of two non-interacting modes embedded in a thermal bath. It decreases asymptotically to zero in time under the influence of the thermal bath. For a strictly monotonous time evolution, non-Gaussian states must be considered.

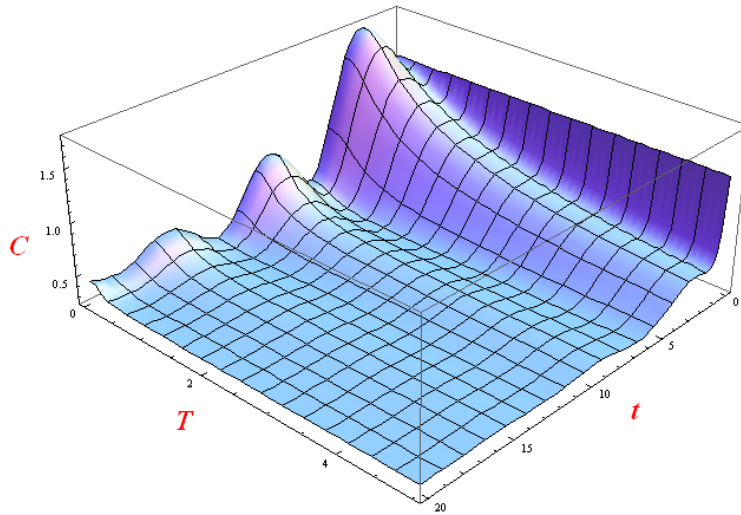


Fig. 1 – Relative entropy of coherence $C(t, T)$ versus time t and temperature T for two bosonic modes with frequencies $\omega_1 = 1$ and $\omega_2 = 0.5$, average photon numbers $n_1 = 0.5$ and $n_2 = 1$, dissipation coefficient $\lambda = 0.1$ and squeezing parameter $r = 0.5$.

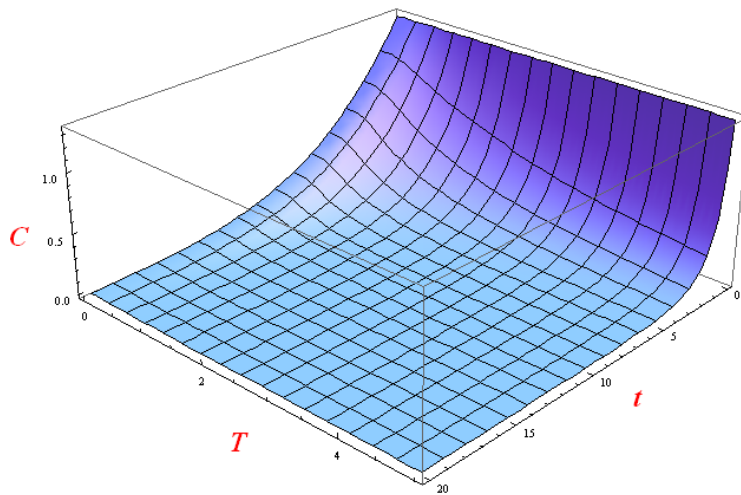


Fig. 2 – Relative entropy of coherence $C(t, T)$ versus time t and temperature T for two bosonic modes with frequencies $\omega_1 = 1$ and $\omega_2 = 1$, average photon numbers $n_1 = 1$ and $n_2 = 2$, dissipation coefficient $\lambda = 0.1$ and squeezing parameter $r = 0.5$.

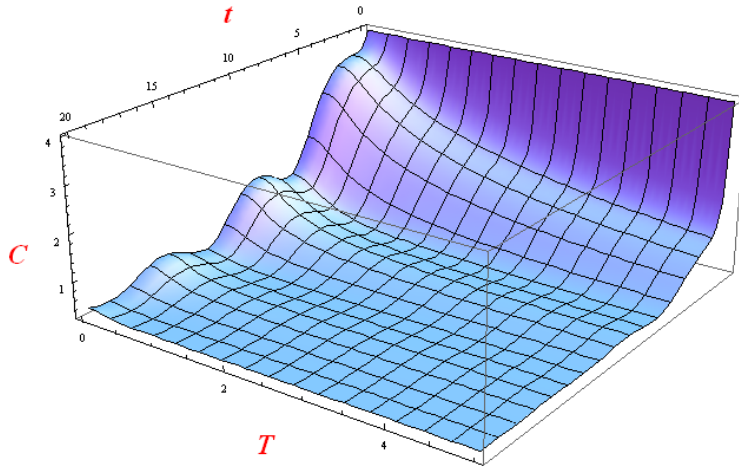


Fig. 3 – Relative entropy of coherence $C(t, T)$ versus time t and temperature T for two bosonic modes with frequencies $\omega_1 = 1$ and $\omega_2 = 0.5$, average photon numbers $n_1 = 1$ and $n_2 = 2$, dissipation coefficient $\lambda = 0.1$ and squeezing parameter $r = 1$.

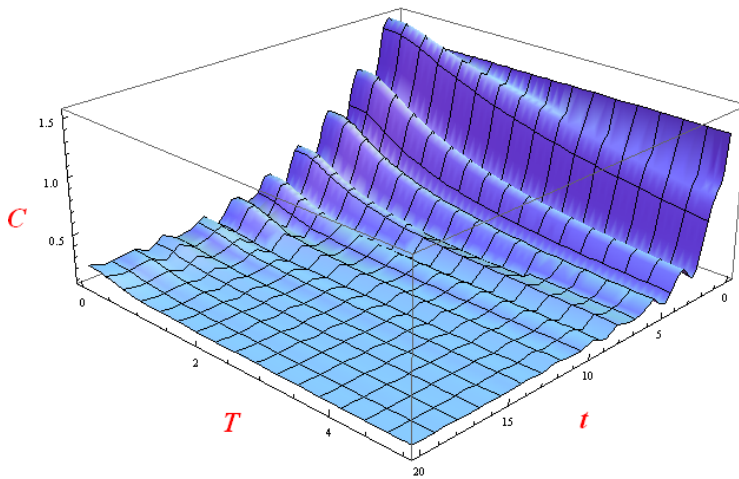


Fig. 4 – Relative entropy of coherence $C(t, T)$ versus time t and temperature T for two bosonic modes with frequencies $\omega_1 = 1$ and $\omega_2 = 1.5$, average photon numbers $n_1 = 1$ and $n_2 = 2$, dissipation coefficient $\lambda = 0.1$ and squeezing parameter $r = 0.5$.

REFERENCES

1. M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
2. R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
3. T. Mihaescu, A. Isar, *Rom. J. Phys.* **60**, 853 (2015).
4. S. Suciu, A. Isar, *Rom. J. Phys.* **60**, 859 (2015).
5. Hoda Alijanzadeh Boura, A. Isar, *Rom. J. Phys.* **60**, 1278 (2015).
6. T. Baumgratz, M. Cramer, M.B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
7. V. Gorini, A. Kossakowski, E. C. G. Sudarshan, *J. Math. Phys.* **17**, 821 (1976).
8. G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976).
9. A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu, W. Scheid, *Int. J. Mod. Phys. E* **3**, 635 (1994).
10. A. Sandulescu, H. Scutaru, W. Scheid, *J. Phys. A: Math. Gen.* **20**, 2121 (1987).
11. G. Adesso, F. Illuminati, *J. Phys. A* **40** 7821, (2007).
12. J. Xu, *Phys. Rev. A* **93**, 032111 (2016)
13. P. Marian, T. A. Marian, H. Scutaru, *Phys. Rev. A* **68**, 062309 (2003).