

# THE HIGGS SECTOR IN THE CAUSAL APPROACH

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We consider the electro-weak sector of the standard model up to the second order of the perturbation theory (in the causal approach) and derive the most general form of the interaction Lagrangian for an arbitrary number of Higgs fields.

Some restrictions can be imposed using the hypothesis of super-renormalizability.

*Key words:* Perturbation theory, Standard Model.

## 1. INTRODUCTION

The general framework of perturbation theory consists in the construction of the chronological products such that Bogoliubov axioms are verified [1–3]; for every set of Wick monomials  $W_1(x_1), \dots, W_n(x_n)$  acting in the Fock space  $\mathcal{H}$  one associates the distribution-valued operators

$$T^{W_1, \dots, W_n}(x_1, \dots, x_n) \equiv T(W_1(x_1), \dots, W_n(x_n))$$

called chronological products. The construction of the chronological products can be done recursively according to Epstein-Glaser prescription [2, 4] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [10] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined but there are some natural limitation on the arbitrariness. If the arbitrariness does not grow with  $n$  we have a renormalizable theory. An equivalent point of view uses retarded products [11].

Gauge theories describe particles of higher spin. Usually such theories are not renormalizable. However, one can save renormalizability using ghost fields. Such theories are defined in a Fock space  $\mathcal{H}$  with indefinite metric, generated by physical and un-physical fields (called *ghost fields*). One selects the physical states assuming the existence of an operator  $Q$  called *gauge charge* which verifies  $Q^2 = 0$  and such that the *physical Hilbert space* is by definition  $\mathcal{H}_{\text{phys}} \equiv \text{Ker}(Q)/\text{Im}(Q)$ . The space  $\mathcal{H}$  is endowed with a grading (usually called *ghost number*) and by construction the gauge charge is raising the ghost number of a state. Moreover, the space of Wick

monomials in  $\mathcal{H}$  is also endowed with a grading which follows by assigning a ghost number to every one of the free fields generating  $\mathcal{H}$ . The graded commutator  $d_Q$  of the gauge charge with any operator  $A$  of fixed ghost number

$$d_Q A = [Q, A] \quad (1)$$

is raising the ghost number by a unit. It means that  $d_Q$  is a co-chain operator in the space of Wick polynomials. From now on  $[\cdot, \cdot]$  denotes the graded commutator.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number  $T(x)$  called *the interaction Lagrangian* such that

$$[Q, T] = i\partial_\mu T^\mu \quad (2)$$

for some other Wick polynomials  $T^\mu$ . This relation means that the expression  $T$  leaves invariant the physical states, at least in the adiabatic limit. Indeed, if this is true we have:

$$T(f) \mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{phys}} \quad (3)$$

up to terms which can be made as small as desired (making the test function  $f$  flatter and flatter). In all known models one finds out that there exist a chain of Wick polynomials  $T^\mu, T^{\mu\nu}, T^{\mu\nu\rho}, \dots$  such that:

$$[Q, T] = i\partial_\mu T^\mu, \quad [Q, T^\mu] = i\partial_\nu T^{\mu\nu}, \quad [Q, T^{\mu\nu}] = i\partial_\rho T^{\mu\nu\rho}, \dots \quad (4)$$

It so happens that for all these models the expressions  $T^{\mu\nu}, T^{\mu\nu\rho}, \dots$  are completely antisymmetric in all indexes; it follows that the chain of relation stops at the step 4 (if we work in four dimensions). We can also use a compact notation  $T^I$  where  $I$  is a collection of indexes  $I = [\nu_1, \dots, \nu_p]$  ( $p = 0, 1, \dots$ ) and the brackets emphasize the complete antisymmetry in these indexes. All these polynomials have the same canonical dimension

$$\omega(T^I) = \omega_0, \quad \forall I \quad (5)$$

and because the ghost number of  $T \equiv T^\emptyset$  is supposed null, then we also have:

$$gh(T^I) = |I|. \quad (6)$$

One can write compactly the relations (4) as follows:

$$d_Q T^I = i \partial_\mu T^{I\mu}. \quad (7)$$

For concrete models the equations (4) can stop earlier: for instance in the Yang-Mills case we have  $T^{\mu\nu\rho} = 0$  and in the case of gravity  $T^{\mu\nu\rho\sigma} = 0$ .

Now we can construct the chronological products

$$T^{I_1, \dots, I_n}(x_1, \dots, x_n) \equiv T(T^{I_1}(x_1), \dots, T^{I_n}(x_n))$$

according to the recursive procedure. We say that the theory is gauge invariant in all

orders of the perturbation theory if the following set of identities generalizing (7):

$$d_Q T^{I_1, \dots, I_n} = i \sum_{l=1}^n (-1)^{s_l} \frac{\partial}{\partial x_l^\mu} T^{I_1, \dots, I_l \mu, \dots, I_n} \quad (8)$$

are true for all  $n \in \mathbb{N}$  and all  $I_1, \dots, I_n$ . Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1} |I_j|. \quad (9)$$

In particular, the case  $I_1 = \dots = I_n = \emptyset$  it is sufficient for the gauge invariance of the scattering matrix, at least in the adiabatic limit: we have the same argument as for relation (3).

Such identities can be usually broken by *anomalies i.e.* expressions of the type  $A^{I_1, \dots, I_n}$  which are quasi-local and might appear in the right-hand side of the relation (8). If one eliminates the anomalies, some restrictions must be imposed on the interaction Lagrangean, besides those following from (2).

In this paper we consider all these restrictions up to the second order of the perturbation theory and determine the most general form for  $T$ . This problem was previously analyzed in great detail in [9], but no general solution was found. We find three type of solutions corresponding to the parameter  $\gamma \equiv \frac{m_3 \cos \theta}{m_1}$  taking the values 1,  $> 1$  and  $< 1$  respectively. The first case is relevant for the usual standard model; more precisely we are investigating here only the electro-weak sector. To discriminate between these possibilities we can use the super-renormalizability assumption introduced in [6]. In this way, only the case  $\gamma = 1$  remains, but we have to add some extra Higgs scalars.

In the next Section we provide the basis of the causal formalism and its use for the standard model. In Section 3 we describe in detail the conditions obtained from gauge invariance in the first two orders of perturbation theory. In Section 4 we obtain three types of basic solutions. In Section 5 we derive the real, antisymmetric, irreducible representations of the group  $so(3)$  which are needed for the general solution.

## 2. THE COHOMOLOGY OF THE GAUGE CHARGE OPERATOR

To fix the notations, we give in the first five subsections a short account of the description of Yang-Mills fields in the causal formalism following [5].

### 2.1. MASSLESS PARTICLES OF SPIN 1 (PHOTONS)

We consider a vector space  $\mathcal{H}$  of Fock type generated (in the sense of Borchers theorem) by the vector field  $v_\mu$  (with Bose statistics) and the scalar fields  $u, \tilde{u}$  (with

Fermi statistics). The Fermi fields are usually called *ghost fields*. We suppose that all these (quantum) fields are of null mass. Let  $\Omega$  be the vacuum state in  $\mathcal{H}$ . In this vector space we can define a sesquilinear form  $\langle \cdot, \cdot \rangle$  in the following way: the (non-zero) 2-point functions are by definition:

$$\begin{aligned} \langle \Omega, v_\mu(x_1)v_\mu(x_2)\Omega \rangle &= i \eta_{\mu\nu} D_0^{(+)}(x_1 - x_2), \\ \langle \Omega, u(x_1)\tilde{u}(x_2)\Omega \rangle &= -i D_0^{(+)}(x_1 - x_2), \\ \langle \Omega, \tilde{u}(x_1)u(x_2)\Omega \rangle &= i D_0^{(+)}(x_1 - x_2) \end{aligned} \quad (10)$$

and the  $n$ -point functions are generated according to Wick theorem. Here  $\eta_{\mu\nu}$  is the Minkowski metrics (with diagonal  $1, -1, -1, -1$ ) and  $D_0^{(+)}$  is the positive frequency part of the Pauli-Jordan distribution  $D_0$  of null mass. To extend the sesquilinear form to  $\mathcal{H}$  we define the conjugation by

$$v_\mu^\dagger = v_\mu, \quad u^\dagger = u, \quad \tilde{u}^\dagger = -\tilde{u}. \quad (11)$$

Now we can define in  $\mathcal{H}$  the operator  $Q$  according to the following formulas:

$$\begin{aligned} &= i \partial_\mu u, \quad [Q, u] = 0, \\ [Q, \tilde{u}] &= -i \partial_\mu v^\mu, \quad Q\Omega = 0 \end{aligned} \quad (12)$$

where by  $[\cdot, \cdot]$  we mean the graded commutator. One can prove that  $Q$  is well defined. Indeed, we have the causal commutation relations

$$[v_\mu(x_1), v_\mu(x_2)] = i \eta_{\mu\nu} D_0(x_1 - x_2) \cdot I, \quad [u(x_1), \tilde{u}(x_2)] = -i D_0(x_1 - x_2) \cdot I \quad (13)$$

and the other commutators are null. The operator  $Q$  should leave invariant these relations, in particular

$$[Q, [v_\mu(x_1), \tilde{u}(x_2)]] + \text{cyclic permutations} = 0 \quad (14)$$

which is true according to (12). It is useful to introduce a grading in  $\mathcal{H}$  as follows: every state which is generated by an even (odd) number of ghost fields and an arbitrary number of vector fields is even (resp. odd). We denote by  $|f|$  the ghost number of the state  $f$ . We notice that the operator  $Q$  raises the ghost number of a state (of fixed ghost number) by an unit. The usefulness of this construction follows from:

**Theorem 2.1** *The operator  $Q$  verifies  $Q^2 = 0$ . The factor space  $Ker(Q)/Ran(Q)$  is isomorphic to the Fock space of particles of zero mass and helicity 1 (photons).*

## 2.2. MASSIVE PARTICLES OF SPIN 1 (HEAVY BOSONS)

We repeat the whole argument for the case of massive photons *i.e.* particles of spin 1 and positive mass.

We consider a vector space  $\mathcal{H}$  of Fock type generated (in the sense of Borchers theorem) by the vector field  $v_\mu$ , the scalar field  $\Phi$  (with Bose statistics) and the scalar

fields  $u, \tilde{u}$  (with Fermi statistics). We suppose that all these (quantum) fields are of mass  $m > 0$ . In this vector space we can define a sesquilinear form  $\langle \cdot, \cdot \rangle$  in the following way: the (non-zero) 2-point functions are by definition:

$$\begin{aligned} \langle \Omega, v_\mu(x_1)v_\mu(x_2)\Omega \rangle &= i \eta_{\mu\nu} D_m^{(+)}(x_1 - x_2), \\ \langle \Omega, \Phi(x_1)\Phi(x_2)\Omega \rangle &= -i D_m^{(+)}(x_1 - x_2) \\ \langle \Omega, u(x_1)\tilde{u}(x_2)\Omega \rangle &= -i D_m^{(+)}(x_1 - x_2), \\ \langle \Omega, \tilde{u}(x_1)u(x_2)\Omega \rangle &= i D_m^{(+)}(x_1 - x_2) \end{aligned} \quad (15)$$

and the  $n$ -point functions are generated according to Wick theorem. Here  $D_m^{(+)}$  is the positive frequency part of the Pauli-Jordan distribution  $D_m$  of mass  $m$ . To extend the sesquilinear form to  $\mathcal{H}$  we define the conjugation by

$$v_\mu^\dagger = v_\mu, \quad u^\dagger = u, \quad \tilde{u}^\dagger = -\tilde{u}, \quad \Phi^\dagger = \Phi. \quad (16)$$

Now we can define in  $\mathcal{H}$  the operator  $Q$  according to the following formulas:

$$\begin{aligned} [Q, v_\mu] &= i \partial_\mu u, & [Q, u] &= 0, & [Q, \tilde{u}] &= -i (\partial_\mu v^\mu + m \Phi) \\ [Q, \Phi] &= i m u, & Q\Omega &= 0. \end{aligned} \quad (17)$$

One can prove that  $Q$  is well defined. We have a result similar to the first theorem of this Section:

**Theorem 2.2** *The operator  $Q$  verifies  $Q^2 = 0$ . The factor space  $\text{Ker}(Q)/\text{Ran}(Q)$  is isomorphic to the Fock space of particles of mass  $m$  and spin 1 (massive photons).*

### 2.3. THE GENERIC YANG-MILLS CASE

The situations described above (of massless and massive photons) are susceptible of the following generalizations. We can consider a system of  $r_1$  species of particles of null mass and helicity 1 if we use in the first part of this Section  $r_1$  triplets  $(v_a^\mu, u_a, \tilde{u}_a), a \in I_1$  of massless fields; here  $I_1$  is a set of indexes of cardinal  $r_1$ . All the relations have to be modified by appending an index  $a$  to all these fields.

In the massive case we have to consider  $r_2$  quadruples  $(v_a^\mu, u_a, \tilde{u}_a, \Phi_a), a \in I_2$  of fields of mass  $m_a$ ; here  $I_2$  is a set of indexes of cardinal  $r_2$ .

We can consider now the most general case involving fields of spin not greater than 1. We take  $I = I_1 \cup I_2 \cup I_3$  a set of indexes and for any index we take a quadruple  $(v_a^\mu, u_a, \tilde{u}_a, \Phi_a), a \in I$  of fields with the following conventions: (a) For  $a \in I_1$  we impose  $\Phi_a = 0$  and we take the masses to be null  $m_a = 0$ ; (b) For  $a \in I_2$  we take the all the masses strictly positive:  $m_a > 0$ ; (c) For  $a \in I_3$  we take  $v_a^\mu, u_a, \tilde{u}_a$  to be null and the fields  $\Phi_a \equiv \phi_a^H$  of mass  $m_a \equiv m_a^H \geq 0$ . The fields  $\phi_a^H$  are called *Higgs fields*.

If we define  $m_a = 0, \forall a \in I_3$  then we can define in  $\mathcal{H}$  the operator  $Q$  according

to the following formulas for all indexes  $a \in I$  :

$$\begin{aligned} [Q, v_a^\mu] &= i \partial^\mu u_a, & [Q, u_a] &= 0, \\ [Q, \tilde{u}_a] &= -i (\partial_\mu v_a^\mu + m_a \Phi_a) & [Q, \Phi_a] &= i m_a u_a, \\ & & Q\Omega &= 0. \end{aligned} \quad (18)$$

If we consider matter fields also *i.e.* some set of Dirac fields with Fermi statistics:  $\Psi_A, A \in I_4$  then we impose

$$d_Q \Psi_A = 0 \quad (19)$$

and the space  $\mathcal{P}_0$  is generated by  $\Psi_A$  and  $\bar{\Psi}_A$  also.

#### 2.4. THE YANG-MILLS LAGRANGIAN

Now we consider the framework and notations from the end of the preceding Section. Then we have the following result which describes the most general form of the Yang-Mills interaction. Summation over the dummy indexes is used everywhere.

Let  $T$  be a relative cocycle for  $d_Q$  which is at least tri-linear in the fields and is of canonical dimension  $\omega(T) \leq 4$  and ghost number  $gh(T) = 0$ . Then: (i)  $T$  is (relatively) cohomologous to a non-trivial co-cycle of the form:

$$\begin{aligned} T &= f_{abc} \left( \frac{1}{2} v_{a\mu} v_{b\nu} F_c^{\nu\mu} + u_a v_b^\mu d_\mu \tilde{u}_c \right) \\ &\quad + f'_{abc} (\Phi_a \phi_b^\mu v_{c\mu} + m_b \Phi_a \tilde{u}_b u_c) \\ &+ \frac{1}{3!} f''_{abc} \Phi_a \Phi_b \Phi_c + \frac{1}{4!} \sum_{a,b,c,d \in I_3} g_{abcd} \Phi_a \Phi_b \Phi_c \Phi_d + j_a^\mu v_{a\mu} + j_a \Phi_a; \end{aligned} \quad (20)$$

(ii) The relation  $d_Q T = i \partial_\mu T^\mu$  is verified by:

$$T^\mu = f_{abc} \left( u_a v_{b\nu} F_c^{\nu\mu} - \frac{1}{2} u_a u_b d^\mu \tilde{u}_c \right) + f'_{abc} \Phi_a \phi_b^\mu u_c + j_a^\mu u_a \quad (21)$$

(iii) The relation  $d_Q T^\mu = i \partial_\nu T^{\mu\nu}$  is verified by:

$$T^{\mu\nu} \equiv \frac{1}{2} f_{abc} u_a u_b F_c^{\mu\nu}. \quad (22)$$

Here

$$\phi_a^\mu \equiv \partial^\mu \phi_a - m v_a^\mu. \quad (23)$$

There are various restrictions on the constants appearing in the preceding expressions. We are interested in the structure of the coefficients  $f_{abc}$  and  $f'_{abc}$  determining the electro-weak sector. We can impose the following restrictions:

$$f'_{abc} = -(a \leftrightarrow b) \quad (24)$$

$$f'_{abc} = 0, \quad \forall c \in I_3 \quad (25)$$

$$f'_{abc} = 0, \quad \forall a \in I_1 \quad (26)$$

The preceding expressions  $T^I$  are self-adjoint if the constants  $f_{abc}$ ,  $f'_{abc}$  are real.

## 2.5. FIRST AND SECOND ORDER GAUGE INVARIANCE

It can be proved that first order gauge invariance (in the electro-weak sector) leads to:

$$f'_{cab} m_a - f'_{cba} m_b = f_{abc} m_c, \quad \forall a, b \in I_1 \cup I_2, c \in I_2 \cup I_3 \quad (27)$$

From here:

$$f'_{abc} = \frac{m_a^2 + m_b^2 - m_c^2}{2m_a m_b} f_{abc}, \quad a, b, c \in I_2 \quad (28)$$

$$f_{abc} (m_a - m_b) = 0, \quad c \in I_1, a, b \in I_2 \quad (29)$$

$$f'_{abc} = f_{abc}, \quad a, b \in I_2, c \in I_1 \quad (30)$$

$$f'_{abc} = m_c g_{abc}, \quad g_{abc} = g_{acb}, \quad a \in I_3, b, c \in I_2 \quad (31)$$

$$f'_{abc} = 0, \quad a \in I_3, b \in I_2, c \in I_1. \quad (32)$$

It also can be proved that second order gauge invariance (in the electro-weak sector) leads to:

$$\sum_c (f_{abc} f_{dec} + f_{bdc} f_{aec} + f_{dac} f_{bec}) = 0 \quad (33)$$

(which is the Jacobi identity) and

$$\sum_c [f'_{dca} f'_{ceb} - (a \leftrightarrow b)] = - \sum_c f_{abc} f'_{dec}, \quad a, b \in I_1 \cup I_2, d, e \in I_2 \cup I_3. \quad (34)$$

## 2.6. EXPLICIT FORM OF (34)

To simplify the analysis of the restrictions listed above we use the vectorial notations and we denote the indices from  $I_3$  by  $j, k, \dots$ . We define the vectors  $g_{ab} \in \mathbb{R}^{|I_3|}$  by:

$$(g_{ab})_j \equiv g_{jab}, \quad \forall j \in I_3, \forall a, b = 1, 2, 3 \quad (35)$$

and according to (31) we have symmetry in  $a \leftrightarrow b$ . We also define the matrices  $T_a \in M_{\mathbb{R}}(|I_3|, |I_3|)$  according to:

$$(T_a)_{jk} \equiv f'_{jka} \quad (36)$$

and observe that these matrices are antisymmetric. Finally we define  $G \in M_{\mathbb{R}}(|I_2|, |I_2|)$  by

$$G_{ab} \equiv m_a m_b \sum_{c \in I_2} (g_{ac} g_{bc}^T - g_{bc} g_{ac}^T), \quad \forall a, b \in I_2 \quad (37)$$

and note that we have symmetry in  $a \leftrightarrow b$ . From (34) we obtain in detail the following cases:

1)  $a, b, d, e \in I_2$

$$m_a m_b (\langle g_{ad}, g_{be} \rangle - \langle g_{bd}, g_{ae} \rangle) = \left[ \sum_{c \in I_2} f'_{dca} f'_{ceb} - (a \leftrightarrow b) \right] + \sum_{c \in I_1} f_{abc} f_{dec} + \sum_{c \in I_2} f_{abc} f'_{dec} \quad (38)$$

2)  $a, b, d \in I_2, e \in I_3$

$$m_a T_b g_{ad} - m_b T_a g_{bd} = \sum_{c \in I_2} (m_c f_{abc} g_{cd} - m_a f'_{dcb} g_{ac} + m_b f'_{dca} g_{bc})$$

3)  $a, b \in I_2, d, e \in I_3$

$$T_a T_b - T_b T_a - G_{ab} = - \sum_{c \in I_1 \cup I_2} f_{abc} T_c \quad (39)$$

4)  $a \in I_1, b, d \in I_2, e \in I_3$

$$T_a g_{bd} = - \sum_{c \in I_2} (f_{abc} g_{cd} + f_{adc} g_{bc}) \quad (40)$$

5)  $a \in I_1, b \in I_2, d, e \in I_3$

$$T_a T_b - T_b T_a = - \sum_{c \in I_1 \cup I_2} f_{abc} T_c \quad (41)$$

6)  $a \in I_1, b, d, e \in I_3$

$$\sum_{c \in I_1} f_{abc} f_{dec} = 0 \quad (42)$$

This is the only non-trivial case requiring some computations.



7)  $a, b \in I_1, d, e \in I_2$

$$\left[ \sum_{c \in I_2} f_{dca} f_{ceb} - (a \leftrightarrow b) \right] + \sum_{c \in I_1} f_{abc} f_{dec} + \sum_{c \in I_2} f_{abc} f'_{dec} = 0 \quad (43)$$

8)  $a, b \in I_1, d \in I_2, e \in I_3$

$$\sum_{c \in I_2} f_{abc} g_{cd} = 0 \quad (44)$$

9)  $a, b \in I_1, d, e \in I_3$

$$T_a T_b - T_b T_a = - \sum_{c \in I_1 \cup I_2} f_{abc} T_c \quad (45)$$

### 3. THE STANDARD MODEL

We consider the following particular case relevant for the electro-weak sector of the standard model. The Lie algebra is real and isomorphic to  $u(1) \times su(2)$  and we have  $I_1 = \{0\}, I_2 = \{1, 2, 3\}$ . The non-zero constants  $f_{abc}$  are:

$$f_{210} = \sin \theta, \quad f_{321} = \cos \theta \quad (46)$$

with  $\cos \theta > 0$  and the other constants determined through the anti-symmetry property;  $\theta$  is the *Weinberg angle*. It is interesting to see that for a four-dimensional Lie algebra, the Jacobi identity is trivially verified. So there are two cases: only one of the structure constants  $f_{012}, f_{023}, f_{031}$  is non-zero (and we end up with the case above after some re-scalings) and the case when at least two of the preceding structure constants are non-zero. The last case leads to the equality of all masses and it is not interesting from the physical point of view.

1) We consider first order gauge invariance:

- From (29) we obtain

$$m_1 = m_2 \quad (47)$$

- From the other identities we get:

$$\begin{aligned} f'_{231} = f'_{312} &= -\cos \theta \frac{m_3}{2m_1} \\ f'_{123} &= -\cos \theta \left( 1 - \frac{m_3^2}{2m_1^2} \right) \\ f'_{120} &= -\sin \theta \end{aligned} \quad (48)$$

$$\begin{aligned}
 f'_{121} = f'_{131} = f'_{122} = f'_{232} = f'_{133} = f'_{233} = f'_{130} = f'_{230} = 0 \\
 f'_{0bc} = 0, \forall b, c \\
 f'_{j10} = f'_{j20} = f'_{j30} = 0, \forall j \in I_3
 \end{aligned} \tag{49}$$

$$\begin{aligned}
 f'_{jab} = m_a g_{jab}, \\
 g_{jab} = g_{jba}, \quad \forall j \in I_3, \forall a, b = 1, 2, 3.
 \end{aligned} \tag{50}$$

2) Second order gauge invariance. We rewrite the relations 1)-9) from the preceding Section defining by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  the scalar product and respectively the norm from  $\mathbb{R}^{|I_3|}$ . Then we have from (34) the following cases:

1)  $a, b, d, e \in I_2$

$$m_1^2 (\langle g_{11}, g_{22} \rangle - |g_{12}|^2) = 1 - \cos^2 \theta \frac{3m_3^2}{4m_1^2} \tag{51}$$

$$m_1^2 (\langle g_{22}, g_{33} \rangle - |g_{23}|^2) = \cos^2 \theta \frac{m_3^2}{4m_1^2} \tag{52}$$

$$m_1^2 (\langle g_{11}, g_{33} \rangle - |g_{13}|^2) = \cos^2 \theta \frac{m_3^2}{4m_1^2} \tag{53}$$

$$\langle g_{11}, g_{23} \rangle - \langle g_{12}, g_{13} \rangle = 0 \tag{54}$$

$$\langle g_{22}, g_{13} \rangle - \langle g_{12}, g_{23} \rangle = 0 \tag{55}$$

$$\langle g_{12}, g_{33} \rangle - \langle g_{13}, g_{23} \rangle = 0 \tag{56}$$

2)  $a, b, d \in I_2, e \in I_3$

$$m_1 (T_2 g_{11} - T_1 g_{12}) = -\frac{3}{2} m_3 \cos \theta g_{13} \tag{57}$$

$$m_1 (T_2 g_{12} - T_1 g_{22}) = -\frac{3}{2} m_3 \cos \theta g_{23} \tag{58}$$

$$m_1 (T_2 g_{13} - T_1 g_{23}) = m_3 \cos \theta \left( \frac{1}{2} g_{11} + \frac{1}{2} g_{22} - g_{33} \right) \tag{59}$$

$$T_3 g_{12} - \frac{m_3}{m_1} T_2 g_{13} = \cos \theta \left[ -g_{11} + \left( 1 - \frac{m_3^2}{2m_1^2} \right) g_{22} + \frac{m_3^2}{2m_1^2} g_{33} \right] \tag{60}$$

$$T_3 g_{22} - \frac{m_3}{m_1} T_2 g_{23} = -\cos \theta \left( 2 - \frac{m_3^2}{2m_1^2} \right) g_{12} \tag{61}$$

$$T_3 g_{23} - \frac{m_3}{m_1} T_2 g_{33} = -\cos\theta \left(1 + \frac{m_3^2}{2m_1^2}\right) g_{13} \quad (62)$$

$$T_3 g_{11} - \frac{m_3}{m_1} T_1 g_{13} = \cos\theta \left(2 - \frac{m_3^2}{2m_1^2}\right) g_{12} \quad (63)$$

$$T_3 g_{13} - \frac{m_3}{m_1} T_1 g_{33} = \cos\theta \left(1 + \frac{m_3^2}{2m_1^2}\right) g_{23} \quad (64)$$

$$T_3 g_{12} - \frac{m_3}{m_1} T_1 g_{23} = \cos\theta \left[ -\left(1 - \frac{m_3^2}{2m_1^2}\right) g_{11} + g_{22} - \frac{m_3^2}{2m_1^2} g_{33} \right] \quad (65)$$

3)  $a, b \in I_2, d, e \in I_3$

$$T_1 T_2 - T_2 T_1 - G_{12} = \sin\theta T_0 + \cos\theta T_3 \quad (66)$$

$$T_2 T_3 - T_3 T_2 - G_{23} = \cos\theta T_1 \quad (67)$$

$$T_3 T_1 - T_1 T_3 - G_{23} = \cos\theta T_2 \quad (68)$$

4)  $a \in I_1, b, d \in I_2, e \in I_3$

$$T_0 g_{11} = 2\sin\theta g_{12} \quad (69)$$

$$T_0 g_{12} = \sin\theta (-g_{11} + g_{22}) \quad (70)$$

$$T_0 g_{22} = -2\sin\theta g_{12} \quad (71)$$

$$T_0 g_{13} = \sin\theta g_{23} \quad (72)$$

$$T_0 g_{23} = -\sin\theta g_{13} \quad (73)$$

$$T_0 g_{33} = 0. \quad (74)$$

5)  $a \in I_1, b \in I_2, d, e \in I_3$

$$T_0 T_1 - T_1 T_0 = \sin\theta T_2 \quad (75)$$

$$T_0 T_2 - T_2 T_1 = -\sin\theta T_1 \quad (76)$$

$$T_0 T_3 - T_3 T_0 = 0 \quad (77)$$

6)  $a \in I_1, b, d, e \in I_2$

We obtain identities.

Not all of the preceding identities are independent. From the relations (66) - (68) and (75) - (77) we see that the expressions  $T_a$  are a representation, up to the cocycle  $G_{ab}$ .

#### 4. SOLUTION OF THE SYSTEM

In the space  $\mathbb{R}^{|I_3|}$  we identify the subspace  $V$  generated by applying polynomials in  $I, T_0, \dots, T_3$  on the vectors  $g_{ab}$ . It is obvious that this space is left invariant by the operators  $T_0, \dots, T_3$ . Moreover because of the antisymmetry property:

$$\langle x, T_a y \rangle = - \langle T_a x, y \rangle, \quad \forall x, y \in \mathbb{R}^{|I_3|} \quad (78)$$

of the matrices  $T_a$  it follows that the orthogonal subspace  $V^\perp$  (with respect to the scalar product  $\langle \cdot, \cdot \rangle$ ) is also left invariant by the operators  $T_0, \dots, T_3$  and the restriction of these operators to  $V^\perp$  is a representation of the algebra  $su(1) \times su(2)$ .

So we must divide the analysis in two parts corresponding to the two subspaces  $V$  and  $V^\perp$ . We immediately prove that we cannot have  $V = \{0\}$  so we start the analysis in the subspace  $V$ . First we have

**Theorem 4.1** *The system of vectors  $g_\pm \equiv \frac{1}{2} (g_{11} \pm g_{22})$ ,  $g_{12}$ ,  $g_{13}$ ,  $g_{23}$  is orthonormal.*

**Proof:** We can rewrite the system (69) - (74) as:

$$T_0 g_- = 2 \sin \theta g_{12} \quad (79)$$

$$T_0 g_{12} = -2 \sin \theta g_- \quad (80)$$

$$T_0 g_{13} = \sin \theta g_{23} \quad (81)$$

$$T_0 g_{23} = -\sin \theta g_{13} \quad (82)$$

$$T_0 g_+ = 0. \quad (83)$$

We use the antisymmetry property (78) of the matrix  $T_0$ . It is known that for such a matrix we can find a basis  $E_0^j$  and  $E_A^k, F_A^k$  such that

$$T_0 E_0^j = 0 \quad (84)$$

and

$$T_0 E_A^k = -\lambda_A F_A^k, \quad T_0 F_A^k = \lambda_A E_A^k \quad (85)$$

where  $\lambda_A > 0$  are distinct positive eigenvalues of the operator  $-(T_0)^{1/2}$ . It is now easy to prove that the basis above is an orthonormal system. Moreover we note that:

$$|E_A^j| = |F_A^j|. \quad (86)$$

Now we see that the vector  $g_+$  is associated with the eigenvalue 0, the vectors  $g_{13}, g_{23}$  with the eigenvalue 1 and the vectors  $g_-, g_{12}$  with the eigenvalue 2 and this leads to the result. ■

The eigenvalue subspaces of  $T_0$  are more precisely described as follows.

**Theorem 4.2** *Let us define*

$$A \equiv -(T_0)^2. \quad (87)$$

*Then:*

- (i) *The eigenvalues of A are  $\geq 0$ .*
- (ii) *If  $E \in V_{\lambda^2}$  then*

$$T_1 E, T_2 E \in V_{(\lambda+\sin\theta)^2} \oplus V_{(\lambda-\sin\theta)^2}.$$

*Moreover*

$$T_3 V_{\lambda^2} \subset V_{\lambda^2}.$$

(iii) *The subspace V can be written as*

$$V = \bigoplus_{m \geq 0} V_m \quad (88)$$

*where*

$$A|_{V_m} = m^2 \sin^2 \theta \cdot I \quad (89)$$

**Proof:** Because of the antisymmetry of the matrix  $T_0$  we consider, as above, a basis  $E_\lambda^j, F_\lambda^j \in V_{\lambda^2}$  such that

$$T_0 E_\lambda^j = -\lambda F_\lambda^j, \quad T_0 F_\lambda^j = \lambda E_\lambda^j \quad (90)$$

for the values  $\lambda > 0$ .

From (75) we obtain

$$T_2 = \frac{1}{\sin \theta} (T_0 T_1 - T_1 T_0) \quad (91)$$

and if we substitute in (76) we get:

$$(T_0)^2 T_1 + T_1 (T_0)^2 - 2 T_0 T_1 T_0 = -\sin^2 \theta T_1 \quad (92)$$

From here:

$$\begin{aligned} (A + \lambda^2 - \sin^2 \theta) T_1 E_\lambda^j - 2 \lambda T_0 T_1 F_\lambda^j &= 0 \\ (A + \lambda^2 - \sin^2 \theta) T_1 F_\lambda^j + 2 \lambda_j T_0 T_1 E_\lambda^j &= 0 \end{aligned} \quad (93)$$

It follows from here:

$$\begin{aligned} [(A + \lambda^2 - \sin^2 \theta)^2 - 4 \lambda^2 A] T_1 E_\lambda^j &= 0 \\ [(A + \lambda^2 - \sin^2 \theta)^2 - 4 \lambda^2 A] T_1 F_\lambda^j &= 0 \end{aligned} \quad (94)$$

Because the preceding square bracket is

$$[\dots] = [A - (\lambda + \sin \theta)^2] [A - (\lambda - \sin \theta)^2] \quad (95)$$

it follows that  $(\lambda \pm \sin \theta)^2$  can be also eigenvalues of  $A$  and this gives (ii) of the theorem.

We also note that

$$g_+ \in V_0, \quad g_{13}, g_{23} \in V_1, \quad g_-, g_{12} \in V_2. \quad (96)$$

So if we use (ii) of the theorem we can generate from the vectors  $g_{ab}$  only vectors from  $V_n$ . ■

We need a convenient basis in the real vector space  $V$ . This basis is obtained in a similar way to the complex case where one uses raising and lowering operators. However, working in real vector spaces is more difficult. The basic trick is used for the proof of the following result:

**Theorem 4.3** *Suppose that  $V_0 \neq \{0\}$  and let  $\{E_0^j\}_{j \in J_0}$  be a orthonormal basis (here  $J_0$  is some finite index set) i.e.*

$$\langle E_0^j, E_0^k \rangle = \delta_{jk}, \quad \forall j, k \in J_0 \quad (97)$$

and also

$$T_0 E_0^j = 0, \quad \forall j \in J_0 \quad (98)$$

Then there exists the chain of subsets of  $J_0$

$$J_N \subset J_{N-1} \subset \dots \subset J_1 \subset J_0$$

and the system of orthonormal vectors  $\{E_n^j, F_n^j\}_{j \in J_n}$  in  $V_n$  such that

$$T_0 E_n^j = -n \sin \theta F_n^j, \quad T_0 F_n^j = n \sin \theta E_n^j, \quad \forall j \in J_n. \quad (99)$$

Let us denote for  $n = 0, \dots, N$

$$\alpha_n^j \equiv |E_n^j|^2 = |F_n^j|^2, \quad \forall j \in J_n \quad (100)$$

and for  $n = 1, \dots, N$

$$A_n^j \equiv \alpha_n^j / \alpha_{n-1}^j, \quad \forall j \in J_n. \quad (101)$$

Then we have

(i) for  $\forall j \in J_0$

$$T_1 E_0^j = E_1^j, \quad T_2 E_0^j = -F_1^j, \quad (102)$$

(ii) for  $\forall j \in J_1$

$$\begin{aligned} T_1 E_1^j &= -\alpha_1^j E_0^j + E_2^j \\ T_1 F_1^j &= F_2^j \\ T_2 E_1^j &= -F_2^j \\ T_2 F_1^j &= \alpha_1^j E_0^j + E_2^j \end{aligned} \quad (103)$$

(iii) and for  $n = 2, \dots, N$  and  $\forall j \in J_n$

$$\begin{aligned} T_1 E_n^j &= -A_n^j E_{n-1}^j + E_{n+1}^j \\ T_1 F_n^j &= -A_n^j F_{n-1}^j + F_{n+1}^j \\ T_2 E_n^j &= -A_n^j F_{n-1}^j - F_{n+1}^j \\ T_2 F_n^j &= A_n^j E_{n-1}^j + E_{n+1}^j \end{aligned} \quad (104)$$

where we make the convention that

$$\begin{aligned} E_n^j &= 0, \quad F_n^j = 0, \quad \forall j \in J_{n-1} - J_n \\ E_{N+1}^j &= 0, \quad F_{N+1}^j = 0 \end{aligned} \quad (105)$$

**Proof:** (i) We make the choice of the basis  $\{E_0^j\}_{j \in J_0}$  such that we have (97) and (98) and then apply the previous theorem and get  $T_1 E_0^j \in V_1$  so we define  $E_1^j \in V_1$  according to

$$T_1 E_0^j = E_1^j, \quad \forall j \in J_0. \quad (106)$$

Next, we define  $F_1^j \in V_1$  through

$$T_0 E_1^j = -\sin \theta F_1^j, \quad \forall j \in J_0 \quad (107)$$

and by direct computations, using (75) and (76) we obtain

$$T_2 E_0^j = -F_1^j, \quad \forall j \in J_0 \quad (108)$$

and

$$T_0 F_1^j = \sin \theta E_1^j, \quad \forall j \in J_0. \quad (109)$$

Of course, we might have

$$\alpha_1^j = 0 \quad (110)$$

for some values of  $j \in J_0$  (and in this case  $E_1^j = F_1^j = 0$ ) so we define

$$J_1 = \{j \in J_0 \mid \alpha_1^j \neq 0\}. \quad (111)$$

(ii) Now we apply again the preceding theorem and get that  $T_1 E_1^j, T_1 F_1^j \in V_0 \oplus V_2$ . We choose a basis  $\{E_2^j, F_2^j\}$  in  $V_2$  such that we have (99) for  $n = 2$  and we

have  $\forall j \in J_1$  the following generic formulas:

$$\begin{aligned} T_1 E_1^j &= a_{jk} E_0^k + c_{jk} E_2^k + d_{jk} F_2^k \\ T_1 F_1^j &= \tilde{a}_{jk} E_0^k + \tilde{c}_{jk} E_2^k + \tilde{d}_{jk} F_2^k \end{aligned} \quad (112)$$

If we use the antisymmetry of the operator  $T_1$  we obtain a more precise form:

$$\begin{aligned} T_1 E_1^j &= -\alpha_1^j E_0^j + c_{jk} E_2^k + d_{jk} F_2^k \\ T_1 F_1^j &= \tilde{c}_{jk} E_2^k + \tilde{d}_{jk} F_2^k. \end{aligned} \quad (113)$$

Next, we use (75) to obtain the expression of  $T_2$ :

$$\begin{aligned} T_2 E_1^j &= (\tilde{c} + 2d)_{jk} E_2^k + (\tilde{d} - 2c)_{jk} F_2^k \\ T_1 F_1^j &= \alpha_1^j E_0^j + (2\tilde{d} - c)_{jk} E_2^k - (d + 2\tilde{c})_{jk} F_2^k. \end{aligned} \quad (114)$$

Now if we use (76) we obtain the restrictions

$$\tilde{c} = -d, \quad \tilde{d} = c. \quad (115)$$

If we define

$$\tilde{E}_2^j \equiv c_{jk} E_2^k + d_{jk} F_2^k, \quad \tilde{F}_2^j \equiv -d_{jk} E_2^k + c_{jk} F_2^k, \quad \forall j \in J_1 \quad (116)$$

we have

$$T_0 \tilde{E}_2^j = -2 \sin \theta \tilde{F}_2^j, \quad T_0 \tilde{F}_2^j = 2 \sin \theta E_2^j, \quad \forall j \in J_1. \quad (117)$$

and

$$\begin{aligned} T_1 E_1^j &= -\alpha_1^j E_0^j + \tilde{E}_2^j \\ T_1 F_1^j &= \tilde{F}_2^j \\ T_2 E_1^j &= -\tilde{F}_2^j \\ T_2 F_1^j &= \alpha_1^j E_0^j + \tilde{E}_2^j \end{aligned} \quad (118)$$

so if we redefine

$$\tilde{E}_2^j \rightarrow E_2^j, \quad \tilde{F}_2^j \rightarrow F_2^j \quad (119)$$

we get (103). Of course, we might have

$$\alpha_2^j = 0 \quad (120)$$

for some values of  $j \in J_1$  (and in this case  $E_2^j = F_2^j = 0$ ) so we define

$$J_2 = \{j \in J_1 \mid \alpha_2^j \neq 0\}. \quad (121)$$

We have obtained the convenient basis in  $V_2$ .

(iii) Next, we use the preceding theorem and have  $T_1 E_2^j, T_1 F_2^j \in V_1 \oplus V_3$ . We choose a basis  $\{E_3^j, F_3^j\}$  in  $V_3$  such that we have (99) for  $n = 3$  and we have  $\forall j \in J_2$



the following generic formulas:

$$\begin{aligned} T_1 E_2^j &= a_{jk} E_1^k + b_{jk} F_1^k + c_{jk} E_3^k + d_{jk} F_3^k \\ T_1 F_2^j &= \tilde{a}_{jk} E_1^k + \tilde{b}_{jk} F_1^k + \tilde{c}_{jk} E_3^k + \tilde{d}_{jk} F_3^k. \end{aligned} \quad (122)$$

If we use the antisymmetry of the matrix  $T_1$  we obtain a simplified form:

$$\begin{aligned} T_1 E_2^j &= -A_2^j E_1^j + c_{jk} E_3^k + d_{jk} F_3^k \\ T_1 F_2^j &= -A_2^j F_1^j + \tilde{c}_{jk} E_3^k + \tilde{d}_{jk} F_3^k. \end{aligned} \quad (123)$$

As at (ii) we use (75) to obtain the expression of  $T_2$ :

$$\begin{aligned} T_2 E_2^j &= -A_2^j F_1^j + (2\tilde{c} + 3d)_{jk} E_3^k + (2\tilde{d} - 3c)_{jk} F_2^k \\ T_1 F_2^j &= A_2^j E_1^j + (3\tilde{d} - 2c)_{jk} E_3^k - (2d + 3\tilde{c})_{jk} F_2^k. \end{aligned} \quad (124)$$

If we use (76) we obtain as above  $\tilde{c} = -d$ ,  $\tilde{d} = c$  so after a redefinition of the type (119) we obtain (104) for  $n = 3$ . It is a straightforward exercise to extend, by induction, the mechanism above for all  $n = 3, \dots, N$ . ■

We still do not have a complete basis in  $V$ . The reason is that the vectors  $E_1^j, F_1^j$ ,  $j \in J_1$  might span only a subspace of  $V_1$ . We have to add  $E_1^j, F_1^j$ ,  $j \in K_1$  such that  $E_1^j, F_1^j$ ,  $j \in J_1 \cup K_1$  is an orthonormal basis in  $V_1$ . Then we have to repeat the procedure from the preceding theorem and obtain a chain

$$K_P \subset J_{P-1} \subset \dots \subset K_2 \subset K_1$$

and corresponding vectors  $\{E_n^j, F_n^j\}_{j \in K_n}$  in  $V_n$ ,  $n \geq 2$  such that we have (99). Then we determine the expressions of  $T_1, T_2$ . These can be easily guessed from the statement of the preceding theorem and one can repeat the proof to get for  $\forall j \in K_1$

$$\begin{aligned} T_1 E_1^j &= E_2^j \\ T_1 F_1^j &= F_2^j \\ T_2 E_1^j &= -F_2^j \\ T_2 F_1^j &= E_2^j \end{aligned} \quad (125)$$

and for  $n = 2, \dots, P$  and  $\forall j \in K_n$

$$\begin{aligned} T_1 E_n^j &= -A_n^j E_{n-1}^j + E_{n+1}^j \\ T_1 F_n^j &= -A_n^j F_{n-1}^j + F_{n+1}^j \\ T_2 E_n^j &= -A_n^j F_{n-1}^j - F_{n+1}^j \\ T_2 F_n^j &= A_n^j E_{n-1}^j + E_{n+1}^j. \end{aligned} \quad (126)$$

The procedure must be repeated if  $E_2^j, F_2^j$ ,  $j \in J_2 \cup K_2$  do not span the whole subspace  $V_2$ . We have to add  $E_2^j, F_2^j$ ,  $j \in L_2$  such that  $E_2^j, F_2^j$ ,  $j \in J_2 \cup K_2 \cup L_2$  is

an orthonormal basis in  $V_2$ . Then we have to repeat the procedure from the preceding theorem and obtain a chain

$$L_Q \subset L_{Q-1} \subset \cdots \subset L_3 \subset L_2$$

and corresponding vectors  $\{E_n^j, F_n^j\}_{j \in L_n}$  in  $V_n$ ,  $n \geq 3$  such that we have (99). The expressions of  $T_1, T_2$  are  $\forall j \in L_2$

$$\begin{aligned} T_1 E_2^j &= E_3^j \\ T_1 F_2^j &= F_3^j \\ T_2 E_2^j &= -F_3^j \\ T_2 F_2^j &= E_3^j \end{aligned} \quad (127)$$

and for  $n = 3, \dots, Q$  and  $\forall j \in L_n$

$$\begin{aligned} T_1 E_n^j &= -A_n^j E_{n-1}^j + E_{n+1}^j \\ T_1 F_n^j &= -A_n^j F_{n-1}^j + F_{n+1}^j \\ T_2 E_n^j &= -A_n^j F_{n-1}^j - F_{n+1}^j \\ T_2 F_n^j &= A_n^j E_{n-1}^j + E_{n+1}^j. \end{aligned} \quad (128)$$

The procedure described above stops after a finite number of steps.

Next we determine the expression of the operator  $T_3$  on the basis obtained above and this is done using formula (66) and the expressions of  $T_0, T_1, T_2$  already obtained. We leave the derivation of these formulas as an exercise. Finally we impose the relations (67) and (68); in fact only one is enough, the other follows by abstract algebra. One derives some identities which we leave, again, as an exercise. We give for illustration only one such consistency relation:

$$(1 - \alpha_1^j + 2A_2^j) F_1^j - G_{12} E_1^j - \cos\theta G_{13} E_0^j + F_1 G_{12} E_0^j = 0, \quad \forall j \in J_0. \quad (129)$$

Now we have all elements to analyze the rest of the equations (57) - (65); in fact only (57), (63), (64) and (65) should be investigated, the others following by abstract algebra.

We make some notations:

$$\begin{aligned} a_j &\equiv \langle E_1^j, g_{13} \rangle, & b_j &\equiv \langle F_1^j, g_{13} \rangle \\ c_j &\equiv \langle E_2^j, g_{12} \rangle, & d_j &\equiv \langle F_2^j, g_{12} \rangle \\ \beta_j &\equiv \langle E_0^j, g_+ - g_{33} \rangle \end{aligned} \quad (130)$$

and

$$\gamma \equiv \frac{m_3 \cos\theta}{m_1}. \quad (131)$$

The well-known interaction Lagrangean of the standard model assumes that

there is only one Higgs scalar field *i.e.*  $|J_3| = 1$  and one can derive that  $\gamma = 1$ . However, in the general case we do not make this assumption and we will get new solutions, as explained in the Introduction. We give the results of these computations. From (57), (63), (64) and (65) we obtain three sets of relations: a) in the first sector (with indices in  $J_n$ ):

$$\begin{aligned}
a_j &= 0, \quad \forall j \in J_1 \\
a_j &= \frac{1}{2} \gamma \beta_j, \quad \forall j \in J_1 \\
\beta_j &= 0, \quad \forall j \in J_0 - J_1 \\
c_j &= 0, \quad \forall j \in J_2 \\
\alpha_1^j < E_{0,g+}^j > &= \frac{3}{4} \gamma^2 \beta_j, \quad \forall j \in J_1 - J_2 \\
\alpha_1^j < E_{0,g+}^j > - 2d_j &= \frac{3}{4} \gamma^2 \beta_j, \quad \forall j \in J_2 \\
\left( 2 - \frac{1}{2} \gamma^2 - 2A_2^j + 2A_3^j - 2m_1^2 |g_-|^2 \right) d_j + \frac{1}{2} \gamma^2 A_2^j \beta_j &= 0, \quad \forall j \in J_2 \\
\frac{1}{2} \left( 1 + \frac{1}{2} \gamma^2 - \alpha_1^j + 2A_2^j - m_1^2 |g_{13}|^2 \right) \beta_j + \alpha_1^j < E_{0,g_{33}}^j > &= 0, \quad \forall j \in J_2 \quad (132)
\end{aligned}$$

b) in the second sector (with indices in  $K_n$ ):

$$\begin{aligned}
c_k &= 0, \quad \forall k \in K_1 - K_2 \\
c_k &= -\frac{3}{4} \gamma a_k, \quad \forall k \in K_2 \\
d_k &= 0, \quad \forall k \in K_1 - K_2 \\
d_k &= -\frac{3}{4} \gamma b_k, \quad \forall k \in K_2, \\
\left( 1 + \frac{1}{2} \gamma^2 + 2A_2^k - m_1^2 |g_{13}|^2 \right) a_k &= 0, \quad \forall k \in K_2 \\
\left( 1 + \frac{1}{2} \gamma^2 + 2A_2^k - m_1^2 |g_{13}|^2 \right) b_k &= 0, \quad \forall k \in K_2 \\
\left( 2 - \frac{1}{2} \gamma^2 - \frac{10}{3} A_2^k + 2A_3^k - m_1^2 |g_-|^2 \right) a_k &= 0, \quad \forall k \in K_2 \\
\left( 2 - \frac{1}{2} \gamma^2 - \frac{10}{3} A_2^k + 2A_3^k - m_1^2 |g_-|^2 \right) b_k &= 0, \quad \forall k \in K_2 \quad (133)
\end{aligned}$$

c) in the third sector (with indices in  $L_n$ ):

$$\begin{aligned} \left(2 - \frac{1}{2} \gamma^2 + 2A_3^l - m_1^2 |g_-|^2\right) a_k &= 0, \quad \forall l \in L_2 \\ \left(2 - \frac{1}{2} \gamma^2 + 2A_3^l - m_1^2 |g_-|^2\right) b_k &= 0, \quad \forall l \in L_2. \end{aligned} \quad (134)$$

Now we use the consistency relation (129) and obtain:

$$(1 - \alpha_1^j + 2A_2^j) \alpha_1^j \delta_{jk} = \frac{3}{4} m_1^2 \gamma^2 \beta_j \beta_k, \quad \forall j \in J_0, \quad \forall k \in J_1 \quad (135)$$

In particular we have:

$$\beta_j \beta_k = 0, \quad \forall j \in J_0, \quad \forall k \in J_1, \quad j \neq k \quad (136)$$

so there is at most one value  $j_0 \in J_1$  such that  $\beta_{j_0} \neq 0$ .

Finally we have all the elements to determine the solutions of our problem from Section 3. We have two cases.

**Theorem 4.4** *Suppose that*

$$\beta_j = 0, \quad \forall j \in J_0. \quad (137)$$

*Then there are two solutions of the problem from Section 3.*

A) *The space  $V$  is one-dimensional, it is generated by the vector  $E_0 \in V_0$ ,  $|E_0| = 1$  and we have:*

$$\begin{aligned} g_{11} = g_{22} = g_{33} &= \frac{1}{2m_1} E_0 \\ g_{ab} &= 0, \quad \forall a \neq b \end{aligned} \quad (138)$$

*Moreover we have*

$$\gamma = 1 \iff m_1 = m_3 \cos \theta \quad (139)$$

*and*

$$T_a = 0, \quad a = 0, \dots, 3. \quad (140)$$

B) *The space  $V$  is generated by the vector  $E_0 \in V_0$ ,  $|E_0| = 1$  and the vectors  $E_n, F_n \in V_n$ ,  $n = 2, \dots, N$ . such that we have (99). It corresponds to  $|J_0| = 1$ ,  $J_1 = J_2 = \dots = \emptyset$  and  $|K_2| = \dots = |K_N| = 1$  so we can omit the indices  $j, k, \dots$*

Moreover we have:

$$\begin{aligned}
g_+ = g_{33} &= \frac{\gamma}{2m_1} E_0 \\
g_{12} &= E_2, \quad g_- = F_2 \\
\gamma^2 = N &\implies \gamma > 1 \\
|E_2|^2 = |F_2|^2 &\equiv \alpha_2 = \frac{\gamma^2 - 1}{2m_1^2} \\
A_n \equiv \alpha_n / \alpha_{n-1} &= \frac{n}{4} (\gamma^2 - n + 1), \quad n = 3, \dots, N.
\end{aligned} \tag{141}$$

The expressions of the the operators  $T_a$ ,  $a = 0, \dots, 3$  are

$$T_a E_0 = 0, \quad a = 0, \dots, 3 \tag{142}$$

$$\begin{aligned}
T_1 E_2 &= E_3, \quad T_1 F_2 = F_3, \\
T_2 E_2 &= -F_3, \quad T_2 F_2 = E_3
\end{aligned} \tag{143}$$

and for  $n = 3, \dots, N$  and  $\forall j \in L_n$

$$\begin{aligned}
T_1 E_n &= -A_n E_{n-1} + E_{n+1}, \quad T_1 F_n = -A_n F_{n-1} + F_{n+1}, \\
T_2 E_n &= -A_n F_{n-1} - F_{n+1}, \quad T_2 F_n = A_n E_{n-1} + E_{n+1}.
\end{aligned} \tag{144}$$

Also

$$T_3 E_n = -\lambda_n F_n, \quad T_3 F_n = \lambda_n E_n, \quad n = 2, \dots, N \tag{145}$$

with

$$\lambda_n \equiv n \cos \theta - \frac{\gamma^2}{2 \cos \theta}. \tag{146}$$

The other case is:

**Theorem 4.5** *Let us suppose that  $\exists j_0 \in J_0$  such that  $\beta_{j_0} \neq 0$ . Then the space  $V$  is generated by the vectors  $E_0, E'_0 \in V_0$  and  $E_1, F_1 \in V_1$  and we have  $\gamma < 1$ . The norms are*

$$\begin{aligned}
|E_0| &= 1, \quad |E'_0| = \frac{1}{m_1^2 (3\gamma^2 + 1)} \\
|E_1|^2 &= |F_1|^2 \equiv \alpha = \frac{3\gamma^2 + 1}{4}.
\end{aligned} \tag{147}$$

The expressions of the operators  $T_a$ ,  $a = 0, \dots, 3$  are:

$$\begin{aligned}
T_1 E_0 &= E_1, \quad T_2 E_0 = -F_1, \quad T_3 E_0 = 0 \\
T_1 E_1 &= -\alpha E_0, \quad T_1 F_1 = 0, \quad T_2 E_1 = 0, \quad T_2 F_1 = \alpha E_0 \\
T_3 E_1 &= -\lambda F_1, \quad T_3 F_1 = \lambda E_1
\end{aligned} \tag{148}$$

with

$$\lambda = \frac{1}{\cos \theta} \left( \sin^2 \theta - \frac{\gamma^2 + 1}{2} \right) \quad (149)$$

and

$$T_a E'_0 = 0, \quad a = 0, \dots, 3 \quad (150)$$

We also have:

$$\begin{aligned} g_+ &= \frac{3\gamma^2\beta}{4\alpha} E_0 + E'_0, & g_{33} &= -\frac{\beta}{4\alpha} E_0 + E'_0 \\ g_{13} &= \frac{\gamma\beta}{2\alpha} F_1, & g_{23} &= \frac{\gamma\beta}{2\alpha} E_1 \end{aligned} \quad (151)$$

with

$$\beta \equiv \frac{3}{4m_1^2} \frac{(3\gamma^2 + 1)(1 - \gamma^2)}{\gamma^2}. \quad (152)$$

The proofs of the preceding two theorems are long but straightforward and will be omitted. Let us note that if we make  $\gamma \rightarrow 1$  in the last theorem we obtain case A from the preceding theorem. We have obtained three case corresponding to  $\gamma = 1, \gamma > 1, \gamma < 1$ .

We still have to analyze the problem from Section 3 in the orthogonal supplement  $V^\perp$ .

Let us suppose for the moment that  $V^\perp = \emptyset$ .

The first case  $\gamma = 1$  corresponds then to the usual standard model. We work out the second line of the interaction Lagrangean (20). Because we have only one Higgs field *i.e.*  $|I_3| = 1$  we can take  $I_3 = \{H\}$  and the non-zero expressions  $f'_{abc}$  from the second line of the interaction Lagrangean (20) are:

$$\begin{aligned} f'_{321} = -f'_{312} &= \frac{1}{2}, & f'_{123} &= -\frac{\cos 2\theta}{2\cos \theta}, & f'_{210} &= \sin \theta \\ f'_{H11} = f'_{H22} &= \frac{1}{2}, & f'_{H33} &= \frac{1}{2\cos \theta} \end{aligned} \quad (153)$$

and those following from the antisymmetry property in the first two indices. As a result we have the scalar + Yang-Mills interaction:

$$\begin{aligned} T_{s+YM} &= \sin \theta [(\Phi_2 \phi_{1\mu} - \Phi_1 \phi_{2\mu}) v_0^\mu + m_1 (\Phi_2 \tilde{u}_1 - \Phi_1 \tilde{u}_2) u_0 \\ &+ \frac{1}{2} [(\Phi_3 \phi_{2\mu} - \Phi_2 \phi_{3\mu}) v_1^\mu + (m_2 \Phi_3 \tilde{u}_2 - m_3 \Phi_2 \tilde{u}_3) u_1 \\ &+ (\Phi_1 \phi_{3\mu} - \Phi_3 \phi_{1\mu}) v_2^\mu + (m_3 \Phi_1 \tilde{u}_3 - m_1 \Phi_3 \tilde{u}_1) u_2] \end{aligned}$$

$$\begin{aligned}
& + \frac{\cos 2\theta}{2 \cos \theta} (\Phi_2 \phi_{1\mu} - \Phi_1 \phi_{2\mu}) v_3^\mu + m_1 (\Phi_2 \tilde{u}_1 - \Phi_1 \tilde{u}_2) u_3 \\
& + \frac{1}{2} [(\Phi_H \phi_{1\mu} - \Phi_1 \partial_\mu \Phi_H) v_1^\mu + m_1 \Phi_H \tilde{u}_1 u_1 \\
& + (\Phi_H \phi_{2\mu} - \Phi_2 \partial_\mu \Phi_H) v_2^\mu + m_1 \Phi_H \tilde{u}_2 u_2] \\
& + \frac{1}{2 \cos \theta} [(\Phi_H \phi_{3\mu} - \Phi_3 \partial_\mu \Phi_H) v_3^\mu + m_3 \Phi_H \tilde{u}_3 u_3].
\end{aligned} \tag{154}$$

In the second case corresponding to  $\gamma > 1$  we can take  $I_3 = \{H, H_2, K_2, \dots, H_N, K_N\}$  so beside the Higgs field  $\Phi_H$  there are some other (real) scalar fields  $\Phi_{H_n}, \Phi_{K_n}$ ,  $n = 2, \dots, N$ .

The non-zero expressions  $f'_{abc}$  from the second line of the interaction Lagrangean (20) are:

$$\begin{aligned}
f'_{321} = -f'_{312} &= \frac{\gamma}{2}, & f'_{123} &= -\frac{2 \cos^2 \theta - \gamma^2}{2 \cos \theta}, & f'_{210} &= \sin \theta, \\
f'_{H11} = f'_{H22} &= \frac{\gamma}{2}, & f'_{H33} &= \frac{\gamma^2}{2 \cos \theta}, \\
f'_{K_2 11} = -f'_{K_2 22} &= f'_{H_2 12} = \frac{\gamma^2 - 1}{2 m_1}, \\
f'_{H_{n+1} H_n 1} = f'_{K_{n+1} H_n 1} &= f'_{H_{n+1} K_n 2} = -f'_{K_{n+1} H_n 2} = \alpha_{n+1} \\
f'_{K_n H_n 3} &= \frac{\gamma^2 - 2n \cos \theta}{a \cos \theta} \alpha_n
\end{aligned} \tag{155}$$

The scalar + Yang-Mills part of the Lagrangian  $T_{s+YM}$  has three parts: One is generalizing the preceding expression (154) with some minor change of the coefficients:

$$\begin{aligned}
T_{s+YM}^{(1)} &= \sin \theta [(\Phi_2 \phi_{1\mu} - \Phi_1 \phi_{2\mu}) v_0^\mu + m_1 (\Phi_2 \tilde{u}_1 - \Phi_1 \tilde{u}_2) u_0 \\
& + \frac{\gamma}{2} [(\Phi_3 \phi_{2\mu} - \Phi_2 \phi_{3\mu}) v_1^\mu + (m_2 \Phi_3 \tilde{u}_2 - m_3 \Phi_2 \tilde{u}_3) u_1 \\
& + (\Phi_1 \phi_{3\mu} - \Phi_3 \phi_{1\mu}) v_2^\mu + (m_3 \Phi_1 \tilde{u}_3 - m_1 \Phi_3 \tilde{u}_1) u_2] \\
& + \frac{2 \cos^2 \theta - \gamma^2}{2 \cos \theta} [(\Phi_2 \phi_{1\mu} - \Phi_1 \phi_{2\mu}) v_3^\mu + m_1 (\Phi_2 \tilde{u}_1 - \Phi_1 \tilde{u}_2) u_3] \\
& + \frac{\gamma}{2} [(\Phi_H \phi_{1\mu} - \Phi_1 \partial_\mu \Phi_H) v_1^\mu + m_1 \Phi_H \tilde{u}_1 u_1 \\
& + (\Phi_H \phi_{2\mu} - \Phi_2 \partial_\mu \Phi_H) v_2^\mu + m_1 \Phi_H \tilde{u}_2 u_2] \\
& + \frac{\gamma^2}{2 \cos \theta} [(\Phi_H \phi_{3\mu} - \Phi_3 \partial_\mu \Phi_H) v_3^\mu + m_3 \Phi_H \tilde{u}_3 u_3].
\end{aligned} \tag{156}$$

The other two parts of  $T_{s+YM}$  contain the new scalar fields and will be not given in detail here but can be obtained from the expressions  $f'_{abc}$  listed above.

In the third case, corresponding to  $\gamma < 1$  we have  $I_3 = \{H, K, H_1, K_1\}$

The non-zero expressions  $f'_{abc}$  from the second line of the interaction Lagrangean (20) are:

$$\begin{aligned}
f'_{321} = -f'_{312} &= \frac{\gamma}{2}, & f'_{123} &= -\frac{2\cos^2\theta - \gamma^2}{2\cos\theta}, & f'_{210} &= \sin\theta, \\
f'_{H11} = f'_{H22} &= \frac{3\gamma^2\beta m_1}{4}, & f'_{H33} &= \frac{\beta m_3}{4\alpha}, \\
f'_{K11} = f'_{K22} &= \frac{1}{m_1(3\gamma^2 + 1)}, & f'_{K33} &= \frac{m_3}{m_1^2(3\gamma^2 + 1)} \\
f'_{K13} = f'_{H123} &= \frac{\gamma\beta m_1}{2}, & f'_{H_1H_1} = f'_{K_1H_2} &= \alpha, & f'_{K_1H_13} &= \lambda
\end{aligned} \tag{157}$$

The scalar + Yang-Mills part of the Lagrangian  $T_{s+YM}$  has three parts: One is generalizing the preceding expression with some minor change of the coefficients:

$$\begin{aligned}
T_{s+YM}^{(1)} &= \sin\theta [(\Phi_2 \phi_{1\mu} - \Phi_1 \phi_{2\mu}) v_0^\mu + m_1 (\Phi_2 \tilde{u}_1 - \Phi_1 \tilde{u}_2) u_0 \\
&\quad + \frac{\gamma}{2} [(\Phi_3 \phi_{2\mu} - \Phi_2 \phi_{3\mu}) v_1^\mu + (m_2 \Phi_3 \tilde{u}_2 - m_3 \Phi_2 \tilde{u}_3) u_1 \\
&\quad \quad + (\Phi_1 \phi_{3\mu} - \Phi_3 \phi_{1\mu}) v_2^\mu + (m_3 \Phi_1 \tilde{u}_3 - m_1 \Phi_3 \tilde{u}_1) u_2] \\
&\quad + \frac{2\cos^2\theta - \gamma^2}{2\cos\theta} [(\Phi_2 \phi_{1\mu} - \Phi_1 \phi_{2\mu}) v_3^\mu + m_1 (\Phi_2 \tilde{u}_1 - \Phi_1 \tilde{u}_2) u_3] \\
&\quad \quad + \frac{3\gamma^2\beta m_1}{4\alpha} [(\Phi_H \phi_{1\mu} - \Phi_1 \partial_\mu \Phi_H) v_1^\mu + m_1 \Phi_H \tilde{u}_1 u_1 \\
&\quad \quad \quad + (\Phi_H \phi_{2\mu} - \Phi_2 \partial_\mu \Phi_H) v_2^\mu + m_1 \Phi_H \tilde{u}_2 u_2] \\
&\quad \quad \quad + \frac{\beta m_3}{4\alpha} [(\Phi_H \phi_{3\mu} - \Phi_3 \partial_\mu \Phi_H) v_3^\mu + m_3 \Phi_H \tilde{u}_3 u_3] \\
&\quad \quad + \frac{1}{m_1(3\gamma^2 + 1)} [(\Phi_K \phi_{1\mu} - \Phi_1 \partial_\mu \Phi_K) v_1^\mu + m_1 \Phi_K \tilde{u}_1 u_1 \\
&\quad \quad \quad + (\Phi_K \phi_{2\mu} - \Phi_2 \partial_\mu \Phi_K) v_2^\mu + m_2 \Phi_K \tilde{u}_2 u_2] \\
&\quad \quad + \frac{m_3}{m_1^2(3\gamma^2 + 1)} [(\Phi_K \phi_{3\mu} - \Phi_3 \partial_\mu \Phi_K) v_3^\mu + m_3 \Phi_K \tilde{u}_3 u_3].
\end{aligned} \tag{158}$$

The other two parts of  $T_{s+YM}$  contain the new scalar fields and will be not given in detail here but can be obtained from the expressions  $f'_{abc}$  listed above.

### 5. SYMPLECTIC REPRESENTATIONS OF $so(3)$

As we have said in previous Section, in the orthogonal supplement  $V^\perp$  the operators  $T_a, a = 0, \dots, 3$  give a representation of the Lie algebra  $u(1) \times su(2)$ ; indeed we make  $G_{ab} \rightarrow 0$  in the relations (51) - (77) and we get something non-trivial only



from (66) - (68) and (75) - (77):

$$\begin{aligned}
T_1 T_2 - T_2 T_1 &= \sin \theta T_0 + \cos \theta T_3 \\
T_2 T_3 - T_3 T_2 &= \cos \theta T_1 \\
T_3 T_1 - T_1 T_3 &= \cos \theta T_2 \\
T_0 T_1 - T_1 T_0 &= \sin \theta T_2 \\
T_0 T_2 - T_2 T_1 &= -\sin \theta T_1 \\
T_0 T_3 - T_3 T_0 &= 0
\end{aligned} \tag{159}$$

If we define

$$\begin{aligned}
S_0 &\equiv \cos \theta T_0 - \sin \theta T_3 \\
S_1 &\equiv T_1, \quad S_2 \equiv T_2 \\
S_3 &\equiv \sin \theta T_0 + \cos \theta T_3
\end{aligned} \tag{160}$$

then we obtain the standard form for a representation of  $u(1) \times su(2)$ :

$$\begin{aligned}
[S_0, S_j] &= 0, \quad j = 1, 2, 3 \\
[S_j, S_k] &= \epsilon_{jkl} S_l
\end{aligned} \tag{161}$$

which we will analyze in the following. We start with the second relation, *i.e.* we determine the real, antisymmetric, irreducible finite dimensional representations of the algebra  $su(2) \simeq so(3)$ .

Proceeding in analogy with theorem 4.2 we obtain the following result:

**Theorem 5.1** *Let  $S_a$  be a real, antisymmetric, irreducible finite dimensional representations of the algebra  $so(3)$  in the real vector space  $V$ . Then there are three cases:*

a) *there exists  $\lambda_0 \in (0, 1)$ ,  $\lambda_0 \neq 1/2$  and the vector space  $V$  can be written as*

$$V = \left( \bigoplus_{m=0}^M V_m \right) \oplus \left( \bigoplus_{n=0}^N W_n \right) \tag{162}$$

*and we have the orthonormal basis  $(E_m^j, F_m^j)$  in  $V_m$  and  $(G_n^k, H_n^k)$  in  $W_n$  such that*

$$\begin{aligned}
S_3 E_m^j &= -(\lambda_0 + m) F_m^j, & S_3 F_m^j &= (\lambda_0 + m) E_m^j, \\
S_3 G_n^k &= -(\lambda'_0 + n) H_n^k, & S_3 H_n^k &= (\lambda'_0 + n) G_n^k,
\end{aligned} \tag{163}$$

where

$$\lambda'_0 \equiv 1 - \lambda_0. \tag{164}$$

b) *We have*

$$V = \bigoplus_{m=0}^M V_m \tag{165}$$

*and we have the orthonormal basis  $E_0^j$  in  $V_0$  and  $(E_m^j, F_m^j)$  in  $V_m$  such that*

$$\begin{aligned}
S_3 E_0^j &= 0, \\
S_3 E_m^j &= -m F_m^j, & S_3 F_m^j &= m E_m^j,
\end{aligned} \tag{166}$$

c) We have

$$V = \bigoplus_{m=0}^M V_m \quad (167)$$

and we have the orthonormal basis  $(E_m^j, F_m^j)$  in  $V_m$  such that

$$S_3 E_m^j = -\left(\frac{1}{2} + m\right) F_m^j, \quad S_3 F_m^j = \left(\frac{1}{2} + m\right) E_m^j. \quad (168)$$

**Proof:** As in the proof of theorem 4.2, we consider the operator  $-(S_3)^2$  with eigenvalues  $\geq 0$  and denote by  $V_{\lambda^2}$  the eigenspaces. Because of the antisymmetry of the matrix  $S_3$  it is easy to find a basis  $E_j^\lambda, F_j^\lambda \in V_{\lambda^2}$  such that

$$S_3 E_j^\lambda = -\lambda F_j^\lambda, \quad S_3 F_j^\lambda = \lambda E_j^\lambda \quad (169)$$

for the values  $\lambda > 0$ . Then we prove, as in theorem 4.2 that if  $E \in V_{\lambda^2}$  then

$$S_1 E, S_2 E \in V_{(\lambda+1)^2} \oplus V_{(\lambda-1)^2}.$$

Let us define  $\lambda_0 \in [0, 1)$  the minimal value for the eigenvalues  $\lambda$ . The three cases from the statement correspond to: a)  $\lambda_0 \in (0, 1)$ ,  $\lambda_0 \neq \frac{1}{2}$ ; b)  $\lambda_0 = 0$ ; c)  $\lambda_0 = \frac{1}{2}$ .

■

Now if we repeat the proof of theorem 4.3 we get after many computations that case a) from the preceding theorem is not possible. The cases b) and c) are described in the following two theorems and they correspond to “integer” and “half-integer” spin.

**Theorem 5.2** *In the case b) from the preceding theorem, suppose that the representation is irreducible. Then the representations are of the type  $\mathcal{D}^M$ ,  $M = 0, 1, \dots$  acting in the space  $V = \bigoplus_{m=0}^M V_m$  with  $\dim(V_0) = 1$  and  $\dim(V_m) = 2$ ,  $m = 1, 2, \dots, M$  with the basis  $E_0 \in V_0$ ,  $|E_0| = 1$  and  $E_m, F_m \in V_m$  such that*

$$\begin{aligned} S_3 E_0 &= 0 \\ S_3 E_m &= -m \sin \theta F_m, \quad S_3 F_m = m \sin \theta E_m, \quad m = 1, 2, \dots, M \end{aligned} \quad (170)$$

Moreover we have

$$S_1 E_0 = E_1, \quad S_2 E_0 = -F_1 \quad (171)$$

$$\begin{aligned} S_1 E_1 &= -\alpha_1 E_0 + E_2 \\ S_1 F_1 &= F_2 \\ S_2 E_1 &= -F_2 \\ S_2 F_1 &= \alpha_1 E_0 + E_2 \end{aligned} \quad (172)$$

and for  $m = 2, \dots, M$

$$\begin{aligned} S_1 E_m &= -A_m E_{m-1} + E_{m+1} \\ S_1 F_m &= -A_m F_{m-1} + F_{m+1} \\ S_2 E_m &= -A_m F_{m-1} - F_{m+1} \\ S_2 F_m &= A_m E_{m-1} + E_{m+1} \end{aligned} \quad (173)$$

where we make the convention that

$$E_{M+1} = 0, \quad F_{M+1} = 0 \quad (174)$$

and we have used the notations:

$$\alpha_m = |E_m|^2 = |F_m|^2, \quad m = 1, \dots, M \quad (175)$$

and

$$A_m = \alpha_m / \alpha_{m-1}, \quad m = 2, \dots, M. \quad (176)$$

One can explicitly determine these parameters:

$$\alpha_1 = \frac{1}{2} M(M+1) \quad (177)$$

and for  $m = 2, \dots, M$

$$A_m = \frac{1}{4} (M^2 + M - m^2 + m) \quad (178)$$

Moreover we have

$$S_1^2 + S_2^2 + S_3^2 = -M(M+1) \cdot \mathbf{I}. \quad (179)$$

**Theorem 5.3** *In the case c) from theorem 5.1, suppose that the representation is irreducible. Then the representations are of the type  $\mathcal{D}^{M+1/2}$ ,  $M = 0, 1, \dots$  acting in the space  $V = \bigoplus_{m=0}^M V_m$  with  $\dim(V_m) = 4$ ,  $m = 1, 2, \dots, M$  with the basis  $E_m^\epsilon, F_m^\epsilon$ ,  $\epsilon = \pm$  in  $V_n$  such that*

$$S_3 E_m^\epsilon = -\left(m + \frac{1}{2}\right) \sin \theta F_m^\epsilon, \quad S_3 F_m^\epsilon = \left(m + \frac{1}{2}\right) \sin \theta E_m^\epsilon, \quad (180)$$

$$m = 0, 1, \dots, M$$

Moreover we have

$$\begin{aligned} S_1 E_0^\epsilon &= \epsilon \frac{M+1}{2} E_0^{-\epsilon} + E_1^\epsilon, & S_1 F_0^\epsilon &= -\epsilon \frac{M+1}{2} F_0^{-\epsilon} + F_1^\epsilon, \\ S_2 E_0^\epsilon &= -\epsilon \frac{M+1}{2} F_0^{-\epsilon} - F_1^\epsilon, & S_2 F_0^\epsilon &= -\epsilon \frac{M+1}{2} E_0^{-\epsilon} + E_1^\epsilon \end{aligned} \quad (181)$$

and for  $m = 1, \dots, M$

$$\begin{aligned} S_1 E_m^\epsilon &= -A_m E_{m-1}^\epsilon + E_{m+1}^\epsilon, & S_1 F_m^\epsilon &= -A_m F_{m-1}^\epsilon + F_{m+1}^\epsilon, \\ S_2 E_m^\epsilon &= -A_m F_{m-1}^\epsilon - F_{m+1}^\epsilon, & S_2 F_m^\epsilon &= A_m E_{m-1}^\epsilon + E_{m+1}^\epsilon \end{aligned} \quad (182)$$

where we make the convention that

$$E_{M+1}^\epsilon = 0, \quad F_{M+1}^\epsilon = 0 \quad (183)$$

and we have used the notations:

$$\alpha_m = |E_m|^\epsilon = |F_m|^\epsilon \quad (184)$$

and

$$A_m = \alpha_m / \alpha_{m-1}. \quad (185)$$

One can explicitly determine these parameters:

$$\alpha_0 = 1 \quad (186)$$

and for  $m = 1, \dots, M$

$$A_m = \frac{1}{4} [(M+1)^2 - m^2] \quad (187)$$

Moreover we have

$$S_1^2 + S_2^2 + S_3^2 = - \left( M + \frac{1}{2} \right) \left( M + \frac{3}{2} \right) \cdot \mathbf{I}. \quad (188)$$

The proofs of these two theorems goes along the general lines of theorem 4.2. We notice an interesting fact concerning the irreducible representations  $\mathcal{D}^{M/2+1}$ ,  $M = 0, 1, \dots$  from the preceding theorem. It is known that Schur lemma is not valid for real irreducible representations: according to [12], pg. 153 this lemma is valid only on vector spaces over a division field which is algebraically closed. This is not the case for the real numbers and indeed we can find a non-trivial operator commuting with this representations.

$$\begin{aligned} S_0 E_n^+ &= -\lambda F_n^+ & S_0 F_n^+ &= \lambda E_n^+ \\ S_0 E_n^- &= \lambda F_n^- & S_0 F_n^- &= -\lambda E_n^- \end{aligned} \quad (189)$$

for any  $\lambda > 0$ .

Finally we extend the results obtained above for representations of the algebra  $S_a$ ,  $a = 0, \dots, 3$  from (161). Suppose that we have an real, antisymmetric, irreducible representation of this algebra. The operators  $S_a$ ,  $a = 1, 2, 3$  given then a real, anti-symmetric, but not necessarily irreducible representation of the  $so(3)$  algebra so it must fall in the cases b) and c) of theorem 5.1. In the case b) one can prove rather easily that there are two types of irreducible representations:

- 1) Of the type  $\mathcal{D}^M$  with  $S_0 = 0$ ;

2) Of the type  $\mathcal{D}^M \oplus \mathcal{D}^M$  with the basis  $E_0^\epsilon \in V_0$  and  $E_m^\epsilon, F_m^\epsilon \in V_m$  ( $m = 1, \dots, M$ ) and the actions of the representation is for  $S_0$  given by

$$\begin{aligned} S_0 E_0^\epsilon &= -\epsilon\mu E_0^{-\epsilon} \\ S_0 E_n^\epsilon &= \epsilon\mu E_n^{-\epsilon}, \quad S_0 F_n^\epsilon = \epsilon\mu F_n^{-\epsilon} \end{aligned} \quad (190)$$

and for  $S_a$ ,  $a = 1, \dots, 3$  is given by the expressions from theorem 5.2 for both values of  $\epsilon$ .

3) In case c) the irreducible representations are  $\mathcal{D}^{M/2+1}$ ,  $M = 0, 1, \dots$  extended by (189).

So, the general solution of the problem from Section 3 is given by a direct sum between one of the three solutions from theorems 4.4 and 4.5 and some direct sum of representations of type 1), 2) and 3) described above. *A priori* there is no way to discriminate between them. However, such a criterion is given by the super-renormalizability condition recently proposed in [6]. After some tedious computations one can prove that only the first solution from theorem 4.4 is admissible *i.e.* the usual solution of the standard model, but we have to add some real and anti-symmetric representation  $S_a$ ,  $a = 0, \dots, 3$  of the Lie algebra  $su(1) \times so(3)$  and this means that we must add to the Lagrangean (154) an extra piece of the form

$$T = (S_a)_{jk} \Phi_j \partial_\mu \Phi_k v_a^\mu. \quad (191)$$

This means that there are some extra Higgs fields in the nature, beside the usual one. The solution for the representation  $S_a$ ,  $a = 0, \dots, 3$  is not unique.

## 6. CONCLUSIONS

We have seen that gauge invariance in the first two orders of the perturbation theory for Yang-Mills models leads to a problem of classification of symplectic representations *i.e.* real, antisymmetric (and irreducible) representations of the Lie algebra relevant for the model; in our case  $u(1) \otimes su(2) \simeq u(1) \otimes so(3)$ .

The theory of real representations of real Lie algebras is known in the literature [7, 8], but we think that the particular case of antisymmetric representations deserves a special treatment in analogy with the complex case and we provided this here for the algebra considered above.

From the physical point of view it follows from above that in the case of a single Higgs field *i.e.*  $|I_3| = 1$  we get exactly the Yang-Mills Lagrangean of the standard model. In the general case we have much more solutions and a way to select the physical ones is to use the condition of super-renormalizability. This condition reduces the arbitrariness of this representation but does not fix it uniquely.

In forthcoming papers we will investigate if this condition can be implemented for some extensions of the standard model. Also it is interesting to see if the condition

of super-renormalizability applied to higher orders of perturbation theory reduces even further the arbitrariness of  $S$  from (191).

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