A NEW FAMILY OF WOODS-SAXON POTENTIALS WITH COMPLEX POLES

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In theory and practical nuclear evaluations an important part is the potential. We introduce a Woods-Saxon Potential with Complex Poles (WSPCP) implying four parameters. The residue theorem permits an analytical evaluation of matrix elements for eigenvalues determination, which is equivalent to a reduction of the computation time. We have applied this theory to determine the ground state energy for the He nucleus.

Key words: Woods-Saxon potential; Ground state energy; He nucleus; Complex poles.

1. INTRODUCTION

For nuclear structure calculation are used nuclear potentials of different types. A very famous one is the central Woods-Saxon potential (W-S) [1]. The W-S potentials are used to solve the Schrödinger equation [4, 5] for different applications such as: alpha decay reaction [6, 7], Padé approximations of W-S [8–10] potential, nuclear energy [11], fission [12], fusion [13], etc. A class of thimble potentials with poles [14] was also developed.

2. THE WOODS-SAXON POTENTIAL WITH COMPLEX POLES

We define the family member of a modified nuclear potential of Woods-Saxon type:

$$V_M^k\ (r) = \frac{V_M}{1+\left(\dfrac{r}{b_M}\right)^k \exp\left(\dfrac{r-R_M}{a_M}\right)};\ r \geq 0 \tag{1}$$

where V_M – the potential strength, R_M – the nuclear radius, a_M – the diffuseness, b_M – a parameter and *k* is a positive integer number.

The classical Woods-Saxon potential is given by [1]:

$$V(r) = \frac{V}{1 + \exp\left(\frac{r - R}{a}\right)}; \quad r \geq 0 \tag{2}$$

where V, R and *a* are the potential strength, the nuclear radius and the diffuseness, respectively.

We construct the functionals:

$$J_M = \int_0^\infty \left[V_M^k(r) - V(r)\right]^2 \mathrm{dr}, \tag{3}$$

$$J = \int_0^\infty \left[V(r)\right]^2 \mathrm{dr}. \tag{4}$$

If the condition, $V_M(0) = V(0)$ must be satisfied, then

$$V_M(0) = \frac{V}{1 + \exp\left(-\frac{R}{a}\right)} \tag{5}$$

and the value of V_M is determined.

From minimizing condition of functional (3)

$$\frac{\mathrm{d}J_M}{\mathrm{d}R_M} = 0; \ \frac{\mathrm{d}J_M}{\mathrm{d}a_M} = 0; \ \frac{\mathrm{d}J_M}{\mathrm{d}b_M} = 0 \tag{6}$$

one obtains a system of nonlinear equations for determining the parameters a_M, b_M, R_M.

Expanding the integrals of the system (6) we observed that we need to evaluated only two integrals of the type:

$$\frac{\partial J_M}{\partial b_M} = \frac{\partial}{\partial b_M}\left\{\int_0^\infty \left[V_M^k(r)\right]^2 \mathrm{dr}\right\} - 2\frac{\partial}{\partial b_M}\left\{\int_0^\infty V_M^k(r) V(r) \mathrm{dr}\right\} \tag{7}$$

using the residues theorem.

In order to speed up the integral evaluation, the term $\left[V_M^k(r)\right]^2$ in the right hand of equation (7) is replaced with:

$$V_M^k(r) \quad \frac{V_M}{1+\dfrac{\varepsilon}{n^2}+\left(\dfrac{r}{b_M}\right)^k \exp\left(\dfrac{r-R_M}{a_M}\right)} \tag{8}$$

with ε << 1 and n big.

Using this replacement, the multiple pole of two order was substituted by two simple poles. The distance between these two simple poles is small, being determined by the election of ε << 1 and n big. In this way one realizes a more rapid evaluation of the first integral.

By analyzing the last equation of the system (6) for the modified Woods-Saxon nuclear potential, corresponding to k = 2 in (1), we found the following singularities:

$$a_M \mathrm{Lambert}W\left[-\frac{b_M}{a_M}\exp\left(\frac{R_M}{a_M}\right)\right] \tag{9}$$

$$a_M \mathrm{Lambert}W\left[-\frac{\left(n^2+\varepsilon\right)b_M}{n^2 a_M}\exp\left(\frac{R_M}{a_M}\right)\right] \tag{10}$$

and

$$R+\pi a i,\ \ a_M \mathrm{Lambert}W\left[-\frac{b_M}{a_M}\exp\left(\frac{R_M}{a_M}\right)\right] \tag{11}$$

where LambertW means Lambert W function [15].

Similarly for the other eqs. in (6) we repeat the procedure. The singularities are:

$$R_M+\left\{2\mathrm{Lambert}W\left[(-1)^j\frac{ib_M}{2a_M}\exp\left(\frac{R_M}{2a_M}\right)\right]-\frac{R_M}{a_M}\right\}a_M \tag{12}$$

$$R_M+\left\{2\mathrm{Lambert}W\left[(-1)^j\frac{ib_M}{2a_M}\sqrt{\frac{n^2+\varepsilon}{n^2}}\exp\left(\frac{R_M}{2a_M}\right)\right]-\frac{R_M}{a_M}\right\}a_M \tag{13}$$

and

$$R+\pi a i,\qquad R_M+\left\{2\mathrm{Lambert}W\left[(-1)^j\frac{ib_M}{2a_M}\exp\left(\frac{R_M}{2a_M}\right)\right]-\frac{R_M}{a_M}\right\}a_M \tag{14}$$

where $j\in\{1;\,2\}$ and $i=\sqrt{-1}$.

The system of nonlinear equations (6) may be solved whatever the modified nuclear potential Woods-Saxon of type (1). Just specify the value of *k*.

3. RESULTS AND CONCLUSIONS

The ground state energy of the nuclear Hamiltonian

$$H_M^k = \left(T + V_M^k\right) \tag{15}$$

can be determined analytically with significant precision by variational method. The energy of the system is defined as:

$$E_0 = \int_0^\infty \psi^* H_M^k \psi \mathrm{d}r, \tag{16}$$

where for simplicity we use an oscillator wave function of type [16]:

$$\psi_n(\xi) = \sqrt{\frac{\alpha}{2^n n! \sqrt{n}}} \exp\left(-\frac{\xi^2}{2}\right) H_n(\xi) \tag{17}$$

where,

$$\xi = \alpha r,\ \alpha = (1/1.01)\ A^{-1/6}\mathrm{fm}^{-1}.$$

$H_n(\xi)$ are the Hermite polynomials defined by the recurrent formula:

$$H_0(\xi) = 1;\ H_1(\xi) = 2\xi;\ H_{n+1}(\xi) - 2\xi H_n(\xi) + 2nH_{n-1}(\xi). \tag{18}$$

So, by residue theorem the ground state is analytically evaluated.
Oscillator frequency is [1]:

$$\hbar\omega_0^{N,Z} = 41A^{-\frac{1}{3}}\left(1 \pm \frac{1}{3}\frac{N-Z}{A}\right). \tag{19}$$

We have considered, as example, the helium nucleus. For the helium nucleus, the following parameters values are available [1]: $A = 4$; $V_0 = -50$ Mev; $a_M = a =$ $= 0.5$ fm; $R_M = R = 1.1A^{1/3}$ fm. Also, n = 2 and $\varepsilon = 0.01$.

For parameter ε and eqs. (9, 10) we obtain singularities: 1.5284+1.2316 i and 1.5294+1.2317 i, respectively.

For b_M =1.27 fm, the global error of the new nuclear potential (1) is given by the ratio of functionals (3) and (4): $J_M /J = 0.000927$.

The condition $V_M(0) = V(0)$ conserves for this new nuclear potential three independent parameters from four. We define the relative global error for the ground state as follows:

$$\mathrm{Re}\, lGlobErr = abs\left(\frac{E_{0new} - E_0}{E_0}\right) \tag{20}$$

where E_{0new} is the ground state energy for the new potential (see 16). Increasing the oscillator base from one to two functions reduce relative global error from 0.0149 to 0.008.

So we obtain a rapid analytical convergence in the evaluation of ground state energy. The four parameters of the this new Woods-Saxon potential with complex poles (WSPCP) are for He: $V_M = -48.5232$ MeV, $R_M = 1.7461$ fm, $a_M = 0.5$ fm, $b_M = 1.27$ fm.

Appendix: Lambert W function

The function $W(z)$ satisfying the equation

$$W(z)e^{W(z)} = z$$

was named the **Lambert W function**. Sometimes, it is called the **omega function** or the **product log function.**

So, the Lambert W function is defined as a multivalued inverse of the function $z \to we^{w}$.

In the complex plane, the Lambert W function has many branches.

In the real domain, there are two solutions of the above equation for $-1/e \le x < 0$ and a single solution if $x \ge 0$. In other words there two real branches of $W(x)$ denoted with $W_0(x)$ if $W(x) \ge -1$ and $W_{-1}(x)$ if $W(x) < -1$. $W_0(x)$ is named as the principal branch of the $W(x)$.

Also, the Lambert W function has the following series expansion:

$$W(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-2}}{(n-1)!} x^n .$$

The Lambert W function has many applications in physics, statistical mechanics, quantum chemistry, combinatorics, enzyme kinetics, physiology of vision, engineering of thin films, hydrology, analysis of algorithms and solar wind.

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