DARBOUX TRANSFORMATION AND HIGHER-ORDER SOLUTIONS OF
THE SASA-SATSUMA EQUATION

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Abstract. We derive a general n-fold Darboux transformation (DT) matrix $T_n$ for
the integrable Sasa-Satsuma equation. The elements of the matrix $T_n$ are expressed in
compact determinant forms. Through the detailed analysis of the linear instability phe-
nomenon, different kinds of solitons on a continuous wave background are obtained by
using the DT, including $W$-shaped first-order and second-order rational solutions, co-
eexisting $W$-shaped second-order rational solutions, first-order semi-rational solutions,
first-order and second-order periodic solutions, first-order half-periodic solutions, and
second-order breather-positon solutions.

Key words: Sasa-Satsuma equation, Darboux transformation, $W$-shaped
rational solitons, Semi-rational solitons, Periodic solutions.

1. INTRODUCTION

The study of rational soliton solutions, especially, rogue (freak) waves, has at-
ttracted a lot of attention during the past years. The Peregrine soliton of the nonlinear
Schrödinger equation (NLSE), which describes an extreme (“giant”) wave has been
studied in many different physical systems [1–14]. The generic NLSE, which con-
tains only the terms accounting for the group-velocity dispersion and the self-phase
modulation effect is a very good dynamical model to describe doubly localized high
amplitude wave wells, with duration of the order of a few picoseconds. However, the
study of propagation in optical fibers of ultrashort pulses with duration shorter than
100 fs requests the use of generalized NLSEs such as the Sasa-Satsuma equation
(SSE) [15–17]:

$$i\dot{q} - q_{xx} - 2|q|^2q + i[6(|q|^2)_{x} - 3(|q|^2)_{xx}q + q_{xxx}] = 0.$$  (1)

The SSE (1) is a natural extention of the NLSE. It contains the third-order dispersion
and self-steepening terms and describes the propagation of subpicosecond pulses in
optical fibers.

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Because the SSE (1) is completely integrable, different categories of exact solutions have been obtained through distinct mathematical methods, such as the inverse scattering method [18–20], the Darboux transformation (DT) method [21] and the binary DT method [22], the dressing operator method [23–26], and the Hirota bilinear method [27–30]. In particular, Ref. [21] provides the one-fold DT method through a deep reduction of the Darboux transformation to a $3 \times 3$ spectral problem.

Recently, anti-dark solitons [31] and special types of rational solutions, such as $W$-shaped rational solutions [32–36] have been studied. In this paper, we investigate new types of soliton solutions of the SSE (1) and illustrate their dynamical behavior. We mainly address the following issues:

I) Can we find the determinant representation $T_n$ of the $n$-fold DT for the SSE without the use of quasi-determinants [22]? This issue is a natural extension of the one-fold DT given in Ref. [21]. In recent years, the determinant representation $T_n$ is a convenient and effective tool to yield rogue (freak) waves (i.e., special rational solutions which are doubly localized in both time and space) and other types of solutions in a plethora of soliton equations [37–47]; see also Refs. [48–62] for recent works on rogue waves and other types of localized structures in a series of relevant physical settings. Moreover, the quasi-determinant of matrices is mainly used to study noncommutative rings [63], hence it is not necessary to use this relatively new and complicated tool to investigate a commutative mathematical object such as the soliton solution of SSE. So it is an interesting problem to give a simple representation of the soliton solution without the use of the quasi-determinant.

II) Except for $W$-shaped rational solutions and other solutions given in Refs. [32–34], are there other soliton solutions showing new dynamical behavior? In this paper, by using the DT, we obtain for the generic SSE different kinds of solitons on a continuous wave background, such as $W$-shaped first-order and second-order rational soliton solutions, coexisting $W$-shaped second-order rational soliton solutions, first-order semi-rational soliton solutions, first-order and second-order periodic solutions, first-order semi-periodic solutions, and second-order breather-positon solutions.

This paper is organized as follows: in Sec. 2, we construct the $n$-fold DT for the SSE, which is a simple and compact expression in the determinant form. In Sec. 3, through the linear stability analysis of the SSE, different kinds of solitons on a continuous wave background are obtained by using the DT and we investigate two categories of $W$-shaped rational soliton solutions according to the choices of Eq. (19) and the root conditions of Eq. (21). Finally, a discussion of the obtained results and our conclusions are given in Sec. 4.
2. DARBOUX TRANSFORMATION

Equation (1) is the compatibility condition of the following Lax pair:

\[ \Psi_x = U \Psi, \quad \Psi_t = V \Psi, \]

where

\[ U = \lambda U_0 + U_1, \quad V = \lambda^3 U_0 + \lambda^2 U_1 + \lambda U_2 + U_3, \]

with

\[ U_0 = \begin{pmatrix} -\frac{2i}{3} & 0 & 0 \\ 0 & \frac{i}{3} & 0 \\ 0 & 0 & \frac{i}{3} \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & -e^{-i\theta} q & -e^{i\theta} q^* \\ e^{i\theta} q^* & 0 & 0 \\ e^{-i\theta} q & 0 & 0 \end{pmatrix}, \]

where \( \theta = -\frac{1}{3} x + \frac{2}{27} t \) and

\[ U_2 = -\frac{1}{3} U_0 + [U_{1x}, U_0] - U_1[U_0, U_1], \]

\[ U_3 = -\frac{1}{3} U_1 + [U_{1x}, U_1] - U_{1xx} + 2U_1^3. \]

Here, \([x, y] = xy - yx\) and the symbol * stands for complex conjugation.

From the knowledge of the Lax pair, the Darboux transformation has been proved to be an efficient tool to construct solitons, breathers, and rogue waves. In order to apply the DT method, it is crucial to find a gauge transformation \( T \) such that

\[ \Psi_x^{[1]} = U^{[1]} \Psi^{[1]}, \quad \Psi_t^{[1]} = T \Psi^{[1]} \]

By cross-differentiating Eqs. (6), we obtain

\[ U_t^{[1]} - V_x^{[1]} + [U^{[1]}, V^{[1]}] = T(U_t - V_x + [U, V])T^{-1}. \]

This implies that, if the SSE is invariant under the gauge transformation \( T \), then \( U^{[1]} \) and \( V^{[1]} \) have the same form as \( U \) and \( V \), and the seed solutions \( (q, q^*) \) are mapped to the new solutions \( (q^{[1]}, q^{[1]*}) \).

2.1. ONE-FOLD DT

We consider the one-fold Darboux transformation in the following form [39, 40, 46]:

\[ T_1 = \lambda E + A, \]

where \( E \) is an identity matrix and \( A = (a_{ij})_{3\times3} \) is a undetermined function matrix whose entries can be expressed in terms of members in the kernel of the matrix \( T \).
Substituting Eq. (8) into the first of equations (6) and comparing the coefficients of $\lambda$

$$U_1^{[1]} - U_1 = [A, U_0],$$

(9)

one obtains the relation between the new solution and the “seed” one:

$$q^{[1]} = q - ie^\theta a_{12}. \tag{10}$$

We assume that $\Psi_1 = (f_1(\lambda, x, t), g_1(\lambda, x, t), h_1(\lambda, x, t))^T \triangleq (f_1, g_1, h_1)^T$ (here $T$ represents the transpose) is a solution of the Lax pair Eq. (2) corresponding to the spectral parameter $\lambda_1$, and we choose three members of $T$’s kernel as following:

$$\lambda_1 \leftrightarrow \Psi_1 = \begin{pmatrix} f_1 \\ g_1 \\ h_1 \end{pmatrix}, \lambda_2 = \lambda_1^* \leftrightarrow \Psi_2 = \begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix} = \begin{pmatrix} -g_1^* \\ f_1^* \\ 0 \end{pmatrix},$$

(11)

$$\lambda_3 = \lambda_1^* \leftrightarrow \Psi_3 = \begin{pmatrix} f_3 \\ g_3 \\ h_3 \end{pmatrix} = \begin{pmatrix} -h_1^* \\ 0 \\ f_1^* \end{pmatrix}.$$ 

Hence, by solving the algebraic equations $T_1(\lambda)|_{\lambda=\lambda_k} \Psi_k = 0, (k = 1, 2, 3)$, the entries in the matrix $T_1$ are obtained by Cramer’s rule. Finally we get,

$$T_1 = \begin{pmatrix} (\hat{T}_1)_{11} & (\hat{T}_1)_{12} & (\hat{T}_1)_{13} \\ (\hat{T}_1)_{21} & (\hat{T}_1)_{22} & (\hat{T}_1)_{23} \\ (\hat{T}_1)_{31} & (\hat{T}_1)_{32} & (\hat{T}_1)_{33} \end{pmatrix}$$

(12)

where

$$(\hat{T}_1)_{11} = \begin{vmatrix} 1 & 0 & 0 & -\lambda \\ D_1 & \eta_1 & 0 \end{vmatrix}, (\hat{T}_1)_{12} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ D_1 & \eta_1 & 0 \end{vmatrix}, (\hat{T}_1)_{13} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ D_1 & \eta_1 & 0 \end{vmatrix},$$

$$(\hat{T}_1)_{21} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ D_1 & \eta_2 & 0 \end{vmatrix}, (\hat{T}_1)_{22} = \begin{vmatrix} 0 & 1 & 0 & -\lambda \\ D_1 & \eta_2 & 0 \end{vmatrix}, (\hat{T}_1)_{23} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ D_1 & \eta_2 & 0 \end{vmatrix},$$

$$(\hat{T}_1)_{31} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ D_1 & \eta_3 & 0 \end{vmatrix}, (\hat{T}_1)_{32} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ D_1 & \eta_3 & 0 \end{vmatrix}, (\hat{T}_1)_{33} = \begin{vmatrix} 0 & 0 & 1 & -\lambda \\ D_1 & \eta_3 & 0 \end{vmatrix},$$

and

$$D_1 = \begin{pmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{pmatrix}, \eta_1 = \begin{pmatrix} -\lambda_1 f_1 \\ -\lambda_2 f_2 \\ -\lambda_3 f_3 \end{pmatrix}, \eta_2 = \begin{pmatrix} -\lambda_1 g_1 \\ -\lambda_2 g_2 \\ -\lambda_3 g_3 \end{pmatrix}, \eta_3 = \begin{pmatrix} -\lambda_1 h_1 \\ -\lambda_2 h_2 \\ -\lambda_3 h_3 \end{pmatrix}.$$

Hence, the explicit form of $q^{[1]}$ is given by

$$q^{[1]} = q + i \frac{(\lambda_1 - \lambda_1^*) g_1^* f_1}{|f_1|^2 + |g_1|^2 + |h_1|^2} \exp(i\theta) \tag{13}$$
2.2. n-FOLD DT

According to the form of the one-fold Darboux matrix, we set the \( n \)-fold Darboux matrix

\[
T_n = T_n(\lambda) = \lambda^n E + \sum_{k=0}^{n-1} t_k \lambda^k,
\]

where \( t_k \) are undetermined matrices depending on the members of \( T_n \)'s kernel. Considering the \( n \)-fold transformation

\[
\Psi[n] = U[n] \Psi, U[n] T_n = T_{nx} + T_n U,
\]

and comparing the coefficient of \( \lambda^{n-1} \), we obtain the relation between the \( n \)-th order new solution and the “seed” solution:

\[
q[n] = q - i(t_{n-1}) \exp(i\theta).
\]

Similar to the one-fold DT, we choose the kernel of \( T_n \) as the following reduction condition:

\[
\lambda_k \leftrightarrow \Psi_k = \begin{pmatrix} f_k \\ g_k \\ h_k \end{pmatrix}, \quad \lambda_{k+1} \leftrightarrow \Psi_{k+1} = \begin{pmatrix} f_{k+1} \\ g_{k+1} \\ h_{k+1} \end{pmatrix} = \begin{pmatrix} -g_k^* \\ \lambda_k^* \\ 0 \end{pmatrix},
\]

\[
\lambda_{k+2} = \lambda_k^* \leftrightarrow \Psi_{k+2} = \begin{pmatrix} f_{k+2} \\ g_{k+2} \\ h_{k+2} \end{pmatrix} = \begin{pmatrix} -h_k^* \\ 0 \\ f_k^* \end{pmatrix},
\]

with \( k = 3l - 2, l = 1, 2, \ldots, n \). Solving the algebraic equations \( T_n(\lambda)|_{\lambda=\lambda_j} \Psi_j = 0(j = 1, 2, \ldots, 3n) \), we can obtain the determinant representation of \( T_n \).

**Theorem 1** The \( n \)-fold DT of the SSE can be expressed as

\[
T_n = \frac{\begin{pmatrix} (T_n)_{11} & (T_n)_{12} & (T_n)_{13} \\ (T_n)_{21} & (T_n)_{22} & (T_n)_{23} \\ (T_n)_{31} & (T_n)_{32} & (T_n)_{33} \end{pmatrix}}{|D_n|}
\]

\[
(\hat{T}_n)_{11} = \begin{vmatrix} \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & 0 & \cdots & 1 & 0 & 0 \\ & & & D_n & & & & -\lambda^n \hat{\chi}_1 \end{vmatrix},
\]

\[
(\hat{T}_n)_{12} = \begin{vmatrix} 0 & \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & 0 & \cdots & 0 & 1 & 0 & 0 \\ & & & D_n & & & & \hat{\chi}_1 \end{vmatrix},
\]

\[
(\hat{T}_n)_{13} = \begin{vmatrix} 0 & 0 & \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & \cdots & 0 & 0 & 1 & 0 \\ & & & D_n & & & & \hat{\chi}_1 \end{vmatrix},
\]

\[
(\hat{T}_n)_{21} = \begin{vmatrix} \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & 0 & \cdots & 1 & 0 & 0 & 0 \\ & & & D_n & & & & \hat{\chi}_2 \end{vmatrix},
\]

\[
(\hat{T}_n)_{22} = \begin{vmatrix} 0 & \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & 0 & \cdots & 0 & 1 & 0 & -\lambda^n \\ & & & D_n & & & & \hat{\chi}_2 \end{vmatrix},
\]

\[
(\hat{T}_n)_{31} = \begin{vmatrix} \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & 0 & \cdots & 1 & 0 & 0 & 0 \\ & & & D_n & & & & \hat{\chi}_2 \end{vmatrix},
\]

\[
(\hat{T}_n)_{32} = \begin{vmatrix} 0 & \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & 0 & \cdots & 0 & 1 & 0 & -\lambda^n \\ & & & D_n & & & & \hat{\chi}_2 \end{vmatrix},
\]

\[
(\hat{T}_n)_{33} = \begin{vmatrix} \lambda^{n-1} & 0 & 0 & \lambda^{n-2} & 0 & \cdots & 1 & 0 & 0 & 0 \\ & & & D_n & & & & \hat{\chi}_2 \end{vmatrix}.
\]
Considering the reduction relation given by Eq. (15), we derive the relation between \(\hat{\chi}_1\), \(\hat{\chi}_2\), and \(\hat{\chi}_3\) as follows:

\[
\chi_1 = (-\lambda_1^n f_1, -\lambda_2^n f_2, -\lambda_3^n f_3, \ldots, -\lambda_{3n-2}^n f_{3n-2}, -\lambda_{3n-1}^n f_{3n-1}, -\lambda_{3n}^n f_{3n})^T, \\
\chi_2 = (-\lambda_1^n g_1, -\lambda_2^n g_2, -\lambda_3^n g_3, \ldots, -\lambda_{3n-2}^n g_{3n-2}, -\lambda_{3n-1}^n g_{3n-1}, -\lambda_{3n}^n g_{3n})^T, \\
\chi_3 = (-\lambda_1^n h_1, -\lambda_2^n h_2, -\lambda_3^n h_3, \ldots, -\lambda_{3n-2}^n h_{3n-2}, -\lambda_{3n-1}^n h_{3n-1}, -\lambda_{3n}^n h_{3n})^T.
\]

Corollary 1 The \(n\)-th order solution \(q^{[n]}\) is derived compactly by

\[
q^{[n]} = q - i \frac{F}{G} e^{i\theta},
\]  

where \(G\) is the determinant:

\[
\begin{vmatrix}
\lambda_1^n f_1 & \lambda_1^n g_1 & \lambda_1^n h_1 & \ldots & f_1 & g_1 & h_1 \\
-(\lambda_1^n g_1)^* & (\lambda_1^n f_1)^* & 0 & \ldots & -g_1^* & f_1^* & 0 \\
-(\lambda_1^n h_1)^* & 0 & (\lambda_1^n f_1)^* & \ldots & -h_1^* & 0 & f_1^* \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-(\lambda_{3n-2}^n g_{3n-2})^* & (\lambda_{3n-2}^n f_{3n-2})^* & 0 & \ldots & -g_{3n-2}^* & f_{3n-2}^* & 0 \\
-(\lambda_{3n-2}^n h_{3n-2})^* & 0 & (\lambda_{3n-2}^n f_{3n-2})^* & \ldots & -h_{3n-2}^* & 0 & f_{3n-2}^*
\end{vmatrix}
\]
and $F$ is constructed from $G$ through the replacing of the second column by a special vector $\hat{\eta}$:

$$\hat{\eta} = (-\lambda_1^{n+1} f_1, \lambda_1^{n+1} g_1, \lambda_1^{n+1} h_1, \cdots, -\lambda_{3n-2}^{n+1} f_{3n-2}, \lambda_{3n-2}^{n+1} g_{3n-2}, \lambda_{3n-2}^{n+1} h_{3n-2})^T.$$ 

Note that $G$ is obtained from $|D_n|$ by applying the reduction condition given by Eq. (15).

3. TWO KINDS OF W-SHAPED RATIONAL SOLUTIONS WITH NON-VANISHING BOUNDARY CONDITIONS

In this Section, we start to consider different kinds of solutions with non-vanishing boundary conditions. The modulation instability (MI) phenomenon as a rogue wave generation mechanism has resulted in the obtaining of rogue wave solutions for SSE [23, 25, 26], and the modulation stability (MS) on the continuous wave (cw) background has resulted in the emerging of W-shaped solitons for SSE [32–34]. However, other types of soliton solutions such as semi-periodic solutions, coexisting W-shaped second-order rational soliton solutions, and second-order breather-positon solutions have not been studied before, to the best of our knowledge. Here, in order to obtain such new types of soliton solutions, we first review the linear stability analysis of the cw for SSE. Let us start with the cw background $q = c \exp[i(ax + bt)]$, $b = a^3 - 6ac^2 + a^2 - 2c^2$. A perturbed nonlinear background can be written as $\hat{q} = [c + p(x, t)\epsilon] \exp[i(ax + bt)]$, where $\epsilon$ is a small parameter and $p$ satisfies a linear equation. Whenever $p$ is $x$-periodic with frequency $Q$, i.e., $p = r(t)e^{iQx} + s(t)e^{-iQx}$, the linear equation reduces to a $2 \times 2$ linear differential equation $\eta' = iM\eta$, where $\eta = (r, s^*)^T$ (here the prime denotes differentiation with respect to $t$) and

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$ 

Here

$$M_{11} = Q^3 + 3Q^2a + 3Qa^2 - 9Qc^2 - 6ac^2 + Q^2 + 2Qa - 2c^2,$$

$$M_{12} = -3Qc^2 - 6ac^2 - 2c^2,$$

$$M_{21} = -3Qc^2 + 6ac^2 + 2c^2,$$

$$M_{22} = Q^3 - 3Q^2a + 3Qa^2 - 9Qc^2 + 6ac^2 - Q^2 + 2Qa + 2c^2.$$ 

The linear dispersion relation between $Q$ and the eigenvalues $\tilde{m}_{1,2}$ of the matrix $M$ is

$$\tilde{m}_{1,2} = \left(Q^2 + 3a^2 - 9c^2 + 2a \pm \sqrt{(9a^2 + 6a + 1)Q^2 - c^2M_1} \right)Q$$ (18)
with
\[ M_1 = (6a + 2 + 3c)(6a + 2 - 3c) \] (19)

From Eqs. (18), when \( M_1 > 0 \), MI occurs. However, in the present work, we focus our attention on the MS condition, \( i.e., M_1 \leq 0 \).

Substituting the cw “seed” solution \( q = c \exp \{ i(ax + (a^3 - 6ac^2 + a^2 - 2c^2)t) \} \) into the Lax pair (2), we obtain the following forms of the three fundamental Bloch eigenfunctions
\[ \psi_k = R \left( \frac{1}{3c(i(\lambda + 3a + 1) + 3\xi_k)} \exp(\omega_k) \right), k = 1, 2, 3, \] (20)

where \( \xi_k \) are the three roots of the cubic equation:
\[ \xi^3 + \left( \frac{1}{9} + \frac{\lambda^2}{3} + a^2 + \frac{2a}{3} + 2c^2 \right) \xi + \frac{2i}{27}(9a^2 - 9c^2 - \lambda^2 + 6a + 1)\lambda = 0, \] (21)

and
\[ R = \text{diag}\{1, \exp[i(\theta - ax - bt)], \exp[-i(\theta - ax - bt)]\}, \]
\[ \omega_k = \xi_k x + [i\lambda \xi_k^2 + \frac{1}{9}(9a^2 - 36c^2 + 6\lambda^2 + 6a - 2)\xi_k \]
\[ + \frac{2i\lambda^3}{9} + \frac{2i}{27}(9a^2 + 18c^2 + 6a + 1)\lambda]t \]

Because of linearity of the Lax pair given by Eq. (2), we combine the three fundamental functions as
\[ \Psi = K_1 \psi_1 + K_2 \psi_2 + K_3 \psi_3, \] (23)

where \( K_j (j = 1, 2, 3) \) are arbitrary complex constants. \( \Psi \) is an eigenfunction solution of the Lax pair (2) corresponding to the spectral parameter \( \lambda \).

Remark According to Eq. (9), there is another solution, namely, \( q^{[1]} = q - ia_{31}e^{i\theta} = q + i(\lambda_1 - \lambda_1^*)h^{[*]}_{f_1}, \) which implies that \( g_1 f_1 = h f_1^* \). In particular, we always guarantee \( f_1 = -f_1^*, h_1 = -g_1^*, \) \( i.e., \lambda_1 \) is pure imaginary and \( \xi_k \) are real roots in Eq. (20) by choosing some specific spectral parameters. Furthermore, through simply analyzing the root conditions of Eq. (21), we find that when \( M_1 = 0 \), that equation has a triple real root whereas it has two roots when \( M_1 < 0 \).

3.1. THE CASE WHEN \( M_1 = 0 \)

In the Subsection, we consider the case when \( M_1 = 0 \). Thus Eq. (21) has a triple real root, \( i.e., \lambda_1 = \lambda_2 = \lambda_3 = \lambda_0 \). In order to obtain the higher-order rational solutions, we need to modify the \( n \)-fold DT for the \( n \)-th order solution \( q^{[n]} \) in Eq. (17) as in the following Theorem.
Theorem 2 Let $\Psi(\lambda_0) = (f(\lambda_0), g(\lambda_0), h(\lambda_0))^T$ given by Eq. (23) be the eigenfunction solution of the Lax pair (2) with the spectral parameter $\lambda_0$, then the $n$-th order $W$-shaped rational solutions of SSE is given by

$$q_r^{[n]} = q - i \frac{F'}{G'} e^{i\theta}. \quad (24)$$

Here $q$ is a “seed” solution of SSE,

$$F' = \frac{\partial^{3(n_i-1)}}{\partial \epsilon^3(n_i-1)} |_{\epsilon=0} (F)_{ij}(\lambda_0 + \epsilon^3)_{3n \times 3n},$$

$$G' = \frac{\partial^{3(n_i-1)}}{\partial \epsilon^3(n_i-1)} |_{\epsilon=0} (G)_{ij}(\lambda_0 + \epsilon^3)_{3n \times 3n},$$

and $n_i = \lfloor \frac{i+2}{3} \rfloor$, where $\lfloor i \rfloor$ denotes the floor function of integer $i$.

For simplicity, we set $a = \frac{1}{6}, c = 1$ in this Subsection. We choose the following combination forms of

$$F_1 = \psi_1 + \psi_2 + \psi_3,$$

$$F_2 = \frac{1}{\epsilon}(\psi_1 + w\psi_2 + w^2\psi_3),$$

$$F_3 = \frac{1}{\epsilon^2}(\psi_1 + w^2\psi_2 + w\psi_3),$$

with $w = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$. By further combination of the above functions with $\epsilon$-dependent coefficients, the eigenfunction solutions $\Psi_k$ of the Lax pair (2) associated with $\lambda_k$ are

$$\Psi_k = (mF_1 + nF_2 + sF_3)|_{\lambda = \lambda_k}, \quad k = 1, 4, 7, \ldots, 3n - 2, \quad (26)$$

with

$$m = m_1 + m_2\epsilon^3 + m_3\epsilon^6 + \cdots + m_j\epsilon^{3(j-1)}, \quad n = n_1 + n_2\epsilon^3 + n_3\epsilon^6 + \cdots + n_j\epsilon^{3(j-1)},$$

$$s = s_1 + s_2\epsilon^3 + s_3\epsilon^6 + \cdots + s_j\epsilon^{3(j-1)},$$

where $m_j, n_j, s_j$ are complex parameters.

3.1.1. The first-order solutions

When $\lambda_0 = \frac{3\sqrt{3}i}{2}$, according to different choices of the parameters $m_1, n_1, s_1$, we classify two categories of the first-order solutions involving a second-order polynomial and a fourth-order one, respectively.

Case 1. When $m_1 = 0, n_1 = 1, s_1 = 0$, the solution involving a second-order polynomial is a $W$-shaped rational soliton as follows

$$q_r^{[1]} = \frac{1}{2} \tilde{F}_1 e^{\frac{26\epsilon - 64\epsilon^3}{216}}, \quad (28)$$
with
\[
\begin{align*}
\tilde{F}_1 &= 10609 i \sqrt{3} t^2 - 2472 i \sqrt{3} x + 144 i \sqrt{3} x^2 + 3708 i t - 432 i x + 412 \sqrt{3} t \\
&\quad - 48 \sqrt{3} x - 10609 t^2 + 2472 t x - 144 x^2 + 192, \\
\tilde{G}_1 &= 824 \sqrt{3} t - 360 \sqrt{3} x + 10609 t^2 - 2472 t x + 144 x^2 + 96.
\end{align*}
\]  
(29)

When \(103t - 12x = -4\sqrt{3}\), the soliton \(|q^{[1]}|\) has a maximum value 2. When \(103t - 12x = \pm 12 - 4\sqrt{3}\), the soliton \(|q^{[1]}|\) has a minimum value \(\frac{\sqrt{10}}{4}\). And \(|q^{[1]}|\) goes to a constant background 1 as \(x \to \infty, t \to \infty\). Its dynamical evolution is shown in Fig. 1(a).

The trajectory of the peak’s center of \(|q^{[1]}|\), i.e. the dependence of the peak’s coordinate, \(T_p\), on \(t\) is \(T_p = \frac{103 \sqrt{3}}{12} + \frac{\sqrt{3}}{3}\), while the coordinates \(T_v\) of the centers of the two soliton’s valleys are given by \(T_v = \frac{103 \sqrt{3}}{12} + \frac{\sqrt{3}}{3} \pm 1\). So we define the soliton speed \(v_{\text{speed}} = \frac{103 \sqrt{3}}{12}\), and the soliton width, which is defined as the distance between the two valleys, is given by \(t_{\text{width}} = \frac{24 \sqrt{10753}}{10753}\). The plots of the soliton’s trajectory is shown in Fig. 1(b).

**Case 2.** When \(s_1 \neq 0\), the soliton solution involves a fourth-order polynomial. There are two different dynamical behaviours for distinct choices of the parameters \(m_1\) and \(n_1\). (I) We set \(n_1 = 0\) and \(s_1 = 1\), and if we fix the spatial coordinate \(x = 0\), then when \(m_1\) goes from \(-\infty\) to \(+\infty\), the two \(W\)-shaped rational soliton solutions gradually fuse to a single anti-dark soliton illustrated by the blue dashed lines in Fig. 2(a). (II) We set \(m_1 = 0\) and \(s_1 = 1\), and we fix \(x = -20\). We see that when \(n_1\) goes from \(-\infty\) to \(+\infty\), the two \(W\)-shaped rational solutions, illustrated by the red-dotted lines, fuse to a single anti-dark soliton and finally, two \(W\)-shaped rational solutions emerge, which are illustrated by the blue-dashed lines in Fig. 2(b).
3.1.2. The second-order solutions

Because of the awkward expressions of the second-order solution $q_r^{[2]}$ in Eq. (24), we only use a graphical method to illustrate their dynamical behaviours.

**Case 3.** We choose $n_1 = 1$ and we consider that the other parameters are all zero. This solution displays a complex dynamical behavior, as shown in Fig. 3(a), where we see two $W$-shaped solitons in the negative half plane, while there exist three $W$-shaped solitons in the positive half plane. In addition, their profiles can be clearly seen by making cut plots at a certain spatial location $x$. The soliton profiles at $x = -4$ (green solid line) and at $x = 7$ (red dashed line) clearly show this rather complex dynamics, see Figs. 3(b-c).

**Case 4.** When we consider the parameter $s_1 = 20$ and other parameters are equal to zero, we see a more complex dynamical behavior in Fig. 4(a), where there...
Eq. (23) with \( \lambda \) observed by plotting them at a certain spatial location shaped solitons in the positive half plane. In addition, their profiles can be clearly 

W are three 

are three \( W \)-shaped solitons in the negative half plane, while there exist four \( W \)-shaped solitons in the positive half plane. In addition, their profiles can be clearly observed by plotting them at a certain spatial location \( x \). The cut plots at \( x = -4 \) (green solid line) and at \( x = 5 \) (red dashed line) clearly show this complex dynamical behavior, see Figs. 4(b-c).

3.2. THE CASE WHEN \( M_1 < 0 \)

In this subsection, we consider the case when \( M_1 < 0 \). Equation (21) has two roots, i.e., \( \lambda_1 = \lambda_{0a} \) and \( \lambda_2 = \lambda_{0b} \). In order to obtain the higher-order rational solutions of SSE, we need to modify the \( n \)-fold DT for the \( n \)-th order solution \( q^{[n]} \) in Eq. (17) as in the following Theorem.

**Theorem 3** Let \( \Phi(\lambda_{0a}(\lambda_{0b})) = (f(\lambda_{0a}(\lambda_{0b})), g(\lambda_{0a}(\lambda_{0b})), h(\lambda_{0a}(\lambda_{0b}))^T \) given by Eq. (23) with \( K_3 = 0 \) be two distinct eigenfunction solutions of the Lax pair (2) with the spectral parameter \( \lambda_{0a}(\lambda_{0b}) \), and corresponding Taylor expansion to \( m_i \) order \((m_i = 0, 2, \cdots ; l = 1, 2)\), which satisfies the constraint condition \( n = 2 + \sum_{i=1,2} m_i \), then the \( n \)-th order rational solution of SSE is given by

\[
q^{[n]} = q - i \frac{F'}{G'} e^{i \theta}.
\]

(30)

Here \( q \) is a “seed” solution of SSE,

\[
\begin{align*}
F' &= \left( \frac{\partial^3 (n_{ij})}{\partial \lambda^3} \right)_{\lambda=0} (F)_{ij}(\lambda_0 + \epsilon^2)_{3n \times 3n}, \\
G' &= \left( \frac{\partial^3 (n_{ij})}{\partial \lambda^3} \right)_{\lambda=0} (G)_{ij}(\lambda_0 + \epsilon^2)_{3n \times 3n}, \quad i = 1, 2, \cdots, 3\left( \frac{m_1}{2} + 1 \right),
\end{align*}
\]

\[
\begin{align*}
F' &= \left( \frac{\partial^3 (n_{ij})}{\partial \lambda^3} \right)_{\lambda=0} (F)_{ij}(\lambda_0 + \epsilon^2)_{3n \times 3n}, \\
G' &= \left( \frac{\partial^3 (n_{ij})}{\partial \lambda^3} \right)_{\lambda=0} (G)_{ij}(\lambda_0 + \epsilon^2)_{3n \times 3n}, \quad i = 3(n - \frac{m_2}{2}) + 1, \cdots, 3n,
\end{align*}
\]

and \( n_j = \left[ \frac{i+2}{3} \right] \), where \([i]\) denotes the floor function of integer \( i \).
If the eigenfunctions given by Eq. (23) have an exponent function associated with the third root of Eq. (21), the constraint $K_3 = 0$ in Theorem 3 is removed, then Eq. (30) in Theorem 3 can produce semi-rational solutions of SSE.

**Corollary 2** If $K_3 \neq 0$ in Theorem 3, then Eq. (30) yields the semi-rational solutions $q_{1\text{semi}}^{[n]}$ of SSE.

In what follows, for convenience we set $a = \frac{1}{3}$, $c = \frac{5}{3}$, $\lambda_{a} = 2\sqrt{6}i$, and $\lambda_{b} = \frac{9}{2}i$ to discuss different kinds of solutions according to Theorem 3 and Corollary 2.

### 3.2.1. W-shaped rational solution

We set $K_3 = 0$ in Eq. (23), and we choose different parameters $K_1$ and $K_2$. We obtain the following two functions:

$$G_1 = \psi_1 + \psi_2, \quad G_2 = \frac{1}{\epsilon} (\psi_1 - \psi_2).$$

(31)

By further combination of the above functions with $\epsilon$-dependent coefficients, the eigenfunction solutions $\Psi_k$ of the Lax pair (2) and the associated $\lambda_k$ are

$$\Psi_k = mG_1 + nG_2|_{\lambda = \lambda_k}, \quad k = 1, 4, 7, \cdots , 3n - 2,$$

(32)

with

$$m = m_1 + m_2 \epsilon^2 + m_3 \epsilon^4 + \cdots + m_j \epsilon^{2(j-1)}, \quad n = n_1 + n_2 \epsilon^2 + n_3 \epsilon^4 + \cdots + n_j \epsilon^{2(j-1)},$$

(33)

where $m_j$ and $n_j$ are complex parameters. Thus by setting distinct eigenfunctions of the Lax pair (2) with distinct parameters $\lambda$, we can obtain higher-order rational solutions.

**A. The first-order rational solutions**

**Case 5.** When $m_1 = 0, n_1 = 1, n = 1, \lambda = \lambda_{a}$, Eq. (30) gives a $W$-shaped first-order rational soliton $q_{r}^{[1]}$ whose dynamical evolution plot is shown in Fig. 5(a).

Through simple calculations, we get the maximum value of the wave $|q_{r}^{[1]}|_{\text{max}} = \frac{7}{3}$, and the minimum value $|q_{r}^{[1]}|_{\text{min}} = \sqrt{\frac{7}{3}}$. Similar to Case 1, we plot the intensity of the rational soliton $|q_{r}^{[1]}|$ and we get the trajectory of the peak’s center, i.e., the dependence of $T_p$ on $t$ as $T_p = 19t + \frac{\sqrt{6}}{2}$, while the coordinates $T_v$ of the centers of the two valleys are $T_v = 19t + \frac{\sqrt{6}}{2} \pm \frac{3\sqrt{2}}{4}$. So we get the soliton’s speed: $v_{\text{speed}} = 19$ and the soliton’s width that is defined as the distance between the corresponding two soliton’s valleys: $t_{\text{width}} = \frac{3\sqrt{181}}{362}$.

**Case 6.** When $m_1 = 0, n_1 = 1, n = 1, \lambda = \lambda_{b}$, we obtain another type of first-order $W$-shaped rational soliton $q_{r}^{[1]}$ whose dynamical evolution plot is shown in Fig. 5(b).

The maximum value of the wave intensity is $|q_{r}^{[1]}|_{\text{max}} = \frac{13}{9}$, whereas the minimum value is $|q_{r}^{[1]}|_{\text{min}} = \frac{\sqrt{7}}{9}$. As in the Case 5, the trajectory of the center of the intensity
peak, i.e., the dependence of \( T_p \) on \( t \) is given by \( T_p = 29t + \frac{1}{4} \), while the coordinates \( T_v \) of the centers of two valleys of the soliton are given by \( T_v = 29t + \frac{1}{4} \pm \frac{\sqrt{3}}{4} \). So we get the soliton speed \( v_{\text{speed}} = 29 \), and the soliton width that is defined as the distance between the two soliton valleys is given by \( t_{\text{width}} = \frac{\sqrt{2920}}{1684} \). Comparing the Case 5 with the Case 6, we see from Figs. 5(a, b) that these two first-order \( W \)-shaped rational solitons are indeed different.

**B. The second-order rational solutions**

**Case 7.** If we choose in Theorem 3 an eigenfunction \( \Phi(\lambda_0a) \) and its corresponding derivative order \( \tilde{m}_1 = 2 \), we can then obtain a \( W \)-shaped second-order rational soliton with parameters \( m_1 = 0, n_1 = 1, m_2 = 0, n_2 = 0 \) by means of a two-fold DT. Its dynamical evolution is shown in Fig. 6(a). However, if we choose the eigenfunction \( \Phi(\lambda_0b) \) and its corresponding derivative order \( \tilde{m}_2 = 2 \), then we also get another \( W \)-shaped second-order rational soliton with the same parameters \( m_1 = 0, n_1 = 1, m_2 = 0, n_2 = 0 \) in Eq. (30); Fig. 6(b) displays its dynamical behavior. Furthermore, we see that the two soliton profiles are so different by plotting them at \( x = 0 \), as clearly seen in Fig. 6(c) and Fig. 6(d).

**Case 8.** In Theorem 3, choosing two distinct eigenfunctions \( \Phi(\lambda_0a) \) with parameters \( m_1 = 0, n_1 = 1 \) and \( \Phi(\lambda_0b) \) with parameters \( m_1 = 0, n_1 = 1 \), and the corresponding derivative orders \( \tilde{m}_1 = 0, \tilde{m}_2 = 0 \), we can obtain a novel type of rational solution. Its dynamical evolution plotted in Fig. 6(e) clearly shows the coexistence of the two distinct \( W \)-shaped second-order rational solitons, plotted in Fig. 6(c) and Fig. 6(d).

![Fig. 5](image-url) – (Color online) Dynamical evolution of different first-order \( W \)-shaped rational soliton solutions: (a) \( m_1 = 0, n_1 = 1, n = 1, \lambda = \lambda_{0a} \) in Case 5; (b) \( m_1 = 0, n_1 = 1, n = 1, \lambda = \lambda_{0b} \) in Case 6.

### 3.2.2. Semi-rational solution

If \( K_3 \neq 0 \) in Eq. (23), we modify the combination of eigenfunctions given in Eq. (31) as follows:

\[
G_1 = K_2\psi_2 + K_3\psi_3, \quad G_2 = \frac{1}{\epsilon}(K_1\psi_1 + K_2\psi_2)
\]  (34)
Fig. 6 – (Color online) Dynamical evolution plots of two different second-order rational solitons. (a) $\lambda_{0a} = 2\sqrt{6}i, m_1 = 0, n_1 = 1, m_2 = 0, n_2 = 0$ in Case 7; (b) $\lambda_{0b} = \frac{3}{2}i, m_1 = 0, n_1 = 1, m_2 = 0, n_2 = 0$ in Case 7; (c) and (d) are the section plots at $x = 0$ for (a) and (b), respectively; (e) coexistence of the above two $W$-shaped second-order rational solitons in Case 8.

Fig. 7 – (Color online) Dynamical evolution plots of different first-order semi-rational solutions. (a) $m_1 = 1, n_1 = 0, K_2 = 1, K_3 = 1, n = 1, \lambda = \lambda_{0a}$ in Case 9; (b) $m_1 = 1, n_1 = 1, K_1 = I, K_2 = 1, K_3 = 1, n = 1, \lambda = \lambda_{0b}$ in Case 10; (c) the profile of the wave in (b) at $t = -0.5$ (red, solid) and $t = 0.5$ (blue, dash).
Fig. 8 – (Color online) Plots of the first-order periodic solutions. (a) $K_1 = K_2 = K_3 = 1, \lambda_0 = \frac{5i}{2}$ in Case 11. (b) $K_1 = K_2 = K_3 = 1, \lambda_0 = -\frac{5i}{2}$ in Case 11. (c) $K_1 = K_2 = 1, K_3 = 0, \lambda_0 = \frac{3i}{2}$ in Case 12. (d) $K_1 = K_2 = 1, K_3 = 0, \lambda_0 = \frac{3i}{2}$ in Case 12.

Fig. 9 – (Color online) (a) The dynamical evolution plot of the second-order periodic solutions at $\lambda_1 = \frac{5i}{2}$ and $\lambda_3 = \sqrt{7}i$ in Case 13; (b) The dynamical evolution plot of the second-order breather-positon (“bpositon”) solution at $\lambda_1 = \frac{5i}{2}$ in Case 14.
By further combination of the above functions with $\epsilon$-dependent coefficients, the eigenfunction solutions $\Psi_k$ of the Lax pair (2) and the associated $\lambda_k$ are

$$\Psi_k = mG_1 + nG_2|_{\lambda=\lambda_k}, k = 1, 4, 7, \cdots, 3n-2,$$

with

$$m = m_1 + m_2\epsilon^2 + m_3\epsilon^4 + \cdots + m_j\epsilon^{2(j-1)}, n = n_1 + n_2\epsilon^2 + n_3\epsilon^4 + \cdots + n_j\epsilon^{2(j-1)},$$

where $m_j$ and $n_j$ are complex parameters.

According to Corollary 2, we take the spectral parameter $\lambda_0 = \lambda_{0a}$ as an example to show below two different first semi-rational solution by using one-fold DT.

**Case 9.** When $m_1 = 1, K_2 = 1, K_3 = 1, n_1 = 0$, Eq. (30) yields a first-order anti-dark soliton solution $q_{\text{anti}}^{[1]}$. Its dynamical evolution is shown in Fig. 7(a).

**Case 10.** When $m_1 = 1, n_1 = 1, K_1 = I, K_2 = 1, K_3 = 1$, Eq. (30) gives a semi-rational solution $q_{\text{semi}}^{[1]}$, namely, the interaction between an anti-dark soliton and a W-shaped rational solution; see Fig. 7(b). Furthermore, we explicitly show the profile of this solution at different values of time in Fig. 7(c).

### 4. PERIODIC SOLUTIONS

In this Section, through choosing different combinations of functions given by Eq. (20) and selecting specific spectral parameters, we obtain different types of periodic solutions of SSE.

**C. The-first order periodic solutions**

**Case 11.** When $K_1 = K_2 = K_3 = 1$ in Eq. (23) and $\lambda_1 = \frac{5i}{2}$, then Eq. (17) with $n = 1$ yields a first-order half-periodic solution $q_{\text{half}}^{[1]}$ in time and space shown in Fig. 8(a). If we take $\lambda_1 = -\frac{5i}{2}$, then the half-periodicity property will occur in the other semiplane as illustrated in Fig. 8(b); see also Ref. [34], where a two-fold DT was used to construct such types of soliton solutions.

**Case 12.** When $K_1 = K_2 = 1, K_3 = 0$ in Eq. (23) and $\lambda_1 = \frac{5i}{2}$, in contrast to the above case, Eq. (17) with $n = 1$ gives a fully-periodic solution $q_{[1]}^{[1]}$ in the whole plane of time and space coordinates whose dynamical evolution is shown in Fig. 8(c). However, if we change the value of $\lambda_1$ to $\lambda_1 = \frac{3i}{2}$, we get another type of periodic solution as shown in Fig. 8(d).

**D. The second-order periodic solutions**

**Case 13.** When choosing two distinct spectral parameters $\lambda_1 = \frac{5i}{2}$ and $\lambda_4 = \sqrt{7}i$ and their corresponding eigenfunctions $\psi_1$ and $\psi_4$ given in Eq. (23) with $K_1 = K_2 = K_3 = 1$, then Eq. (17) with $n = 2$ yields a general second-order periodic solution $q_{[2]}^{[2]}$ as shown in Fig. 9(a).

**Case 14.** When we choose $\lambda_1 = \frac{5i}{2}$ and let $\lambda_4 \rightarrow \lambda_1$, similar to the DT in Theorem 2,
i.e., just changing the special spectral parameter $\lambda_0$ into $\lambda_1$, we obtain a second-order breather-positon (“bpositon”) solution $q_{\text{bpos}}^{[2]}$ from Eq. (24) with $n = 2$. Its dynamical evolution plot is displayed in Fig. 9(b). These special types of periodic solutions of other relevant soliton equations have been recently studied in Refs. [47, 64, 65].

5. CONCLUSION AND DISCUSSION OF THE RESULTS

In this paper, we have constructed a general $n$-fold DT for the SSE (1) (see Theorem 1 and Corollary 1) that was expressed by a compact determinant representation in contrast to the representation obtained by using the dressing operator method in Refs. [23, 34]. Especially, one can directly compare Eq. (8) in the present work with Eq. (8) in Ref. [34] and with Eq. (4) in Ref. [23]. According to the value of $M_1$ in Eq. (19), i.e., the multiple root condition of Eq. (21), we have considered in this work two distinct situations: I) when $M_1=0$, Eq. (21) has a triple real root, and we have obtained $W$-shaped rational solutions up to second order according to Theorem 2; II) when $M_1 < 0$, Eq. (21) possesses two distinct roots and according to Theorem 3 and Corollary 2 we have obtained many types of soliton solutions such as $W$-shaped first-order rational solutions, two distinct types of second-order rational solutions and the coexistence phenomenon of these two kinds of second-order rational solutions, first-order semi-rational solutions, first-order half-periodic solutions, and second-order periodic solutions including breather-positon (“bpositon”) solutions. In Section 4, we find different types of periodic solutions of the SSE (1). Though we have derived solutions up to second order in Section 4, Eqs. (17), (24), and (30) can be also used to analyze higher-order solutions, and our method could be extended to other integrable soliton models.

Comparing to the results reported in Refs. [22, 23, 33, 34], we list below the new results obtained in this paper:

- We have obtained a compact determinant representation $T_n$ of the $n$-fold DT for the SSE (1) without the use of quasi-determinant [22].

- $W$-shaped second-order rational solitons (see the Case 4 and the Case 7) and the coexistence of two second-order rational solitons, which are different in shape, are first reported (see the Case 8).

- The combination between an anti-dark soliton and a $W$-shaped soliton is obtained by using one-fold DT. This result is different from that illustrated in Fig. 8 of Ref. [33] where a two-fold DT was used.

- The semi-periodic solutions in Case 11 are obtained by using a one-fold DT whereas a similar result was derived in Ref. [34] by using a two-fold DT. Different types of periodic solutions were illustrated in Cases 12, 13, 14, in particular,
the second-order breather-soliton (“bpositon”) solution $q_{bp}^{[2]}$ in Case 14 has not been reported before, to the best of our knowledge. We believe that the compact determinant form $T_n$ (see Theorem 1 and Corollary 1) can be applied to generate much more complicated patterns of mixed waveforms involving other types of rational solutions of the Sasa-Satsuma equation. Also, it can be applied to generate interesting wave patterns for other physically relevant soliton equations. We note that the $n$-fold DT matrix $T_n$ given in this paper cannot generate rogue wave structures. Thus in order to get other interesting mixed solutions associated with rogue waves, breathers, and different types of rational solitons, we have to find a new form of DT matrix, an issue that will be addressed elsewhere.

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