REMARKS ON THE GEOMETRY OF THE EXTENDED SIEGEL–JACOBI UPPER HALF-PLANE

ELENA MIRELA BABALIC, STEFAN BERCEANU*

"Horia Hulubei" National R&D Institute for Physics and Nuclear Engineering,
Department of Theoretical Physics,
Reactorului 30, RO-077125, P.O.B. MG-6, Măgurele-Bucharest, Romania
E-mail: mbabalic@theory.nipne.ro
Corresponding author*: Berceanu@theory.nipne.ro

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Abstract. The real Jacobi group $G^J_1(\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}_1$, where H_1 denotes the 3-dimensional Heisenberg group, is parametrized by the S-coordinates (x,y,θ,p,q,κ). We show that the parameter η that appears passing from Perelomov's un-normalized coherent state vector based on the Siegel–Jacobi disk $\mathcal{D}^J_1$ to the normalized one is $\eta = q + \mathrm{i}p$. The two-parameter invariant metric on the Siegel–Jacobi upper half-plane $\mathcal{X}^J_1 = \frac{G^J_1(\mathbb{R})}{\mathrm{SO}(2)\times\mathbb{R}}$ is expressed in the variables $(x,y,\mathrm{Re}\ \eta,\mathrm{Im}\ \eta)$. It is proved that the five dimensional manifold $\tilde{\mathcal{X}}^J_1 = \frac{G^J_1(\mathbb{R})}{\mathrm{SO}(2)} \approx \mathcal{X}^J_1 \times \mathbb{R}$, called extended Siegel–Jacobi upper half-plane, is a reductive, non-symmetric, non-naturally reductive manifold with respect to the three-parameter metric invariant to the action of $G^J_1(\mathbb{R})$, and its geodesic vectors are determined.

Key words: Jacobi group, invariant metric, Siegel–Jacobi upper half-plane, extended Siegel–Jacobi upper half-plane, naturally reductive manifold, g. o. space, geodesic vector, coherent states.

1. INTRODUCTION

The Jacobi group is defined as the semi-direct product of the Heisenberg group and the symplectic group of appropriate dimension. The Jacobi group is intensively studied in Mathematics, Theoretical and Mathematical Physics [1–12]. We have studied the Jacobi group $G^J_n := \mathrm{H}_n \rtimes \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$, where $\mathrm{H_n}$ denotes the $(2n+1)$-dimensional Heisenberg group and $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}} := \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(n,n)$ [13, 14].

The real Jacobi group of degree n is defined as $G^J_n(\mathbb{R}) := \mathrm{Sp}(n,\mathbb{R}) \ltimes \mathrm{H}_n(\mathbb{R})$, where $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$ and G^J_n are isomorphic to $\mathrm{Sp}(n,\mathbb{R})$ and $G^J_n(\mathbb{R})$ respectively as real Lie groups, see [15, Proposition 2], [5, 10]. To simplify the notation we will denote in the following $\mathrm{H}_n(\mathbb{R})$ also with H_n.

The Siegel-Jacobi ball $\mathcal{D}^J_n$ is a G^J_n-homogeneous manifold, whose points are in $\mathbb{C}^n \times \mathcal{D}_n$ [13], where $\mathcal{D}_n \approx \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}/\mathrm{U}(n)$ denotes the Siegel (open) ball of degree n [16].

The Jacobi group is a unimodular, non-reductive, algebraic group of Harish-Chandra type [5, 17–21], and $\mathcal{D}^J_n$ is a reductive, non-symmetric manifold associated to the Jacobi group G_n^J by the generalized Harish-Chandra embedding [2, 3]. The holomorphic irreducible unitary representations of G_n^J based on $\mathcal{D}_n^J$ constructed in [5, 6, 22–24] are relevant to important areas of mathematics such as Jacobi forms, automorphic forms, L-functions and modular forms, spherical functions, the ring of invariant differential operators, theta functions, Hecke operators, Shimura varieties and Kuga fiber varieties.

The Jacobi group was investigated by mathematicians [25–28] in the context of coherent states (CS) [29–31]. Some CS systems based on $\mathcal{D}_n^J$ have been considered in the framework of quantum mechanics, geometric quantization, dequantization, quantum optics, squeezed states, quantum teleportation, quantum tomography, nuclear structure, signal processing, Vlasov kinetic equation [32–38].

The starting point in Perelomov's approach to CS is the triplet $(G,\pi,\mathfrak{H})$, where π is a unitary, irreducible representation of the Lie group G on a separable complex Hilbert space $\mathfrak{H}$ [31].

Two types of CS-vectors belonging to $\mathfrak{H}$ are locally defined on $M = G/H$: the normalized (un-normalized) CS-vector $\underline{e}_x$ (respectively, e_z)

$$\underline{e}_x = \exp(\sum_{\phi\in\Delta^+} x_\phi \boldsymbol{X}^+_\phi - \bar{x}_\phi \boldsymbol{X}^-_\phi)e_0, \quad e_z = \exp(\sum_{\phi\in\Delta^+} z_\phi \boldsymbol{X}^+_\phi)e_0, \tag{1}$$

where e_0 is the extremal weight vector of the representation π, Δ^+ is the set of positive roots of the Lie algebra $\mathfrak{g}$, and X^+_ϕ (X^-_ϕ) are the positive (respectively, negative) generators. For $X \in \mathfrak{g}$ we denoted in (1) $\boldsymbol{X} := \mathrm{d}\pi(X)$ [31, 39, 40].

In the standard procedure of CS, the G-invariant Kähler two-form on a $2n$-dimensional homogeneous manifold $M = G/H$ is obtained from the Kähler potential f *via* the recipe

$$-\mathrm{i}\omega_M = \partial\bar{\partial}f,\ f(z,\bar{z}) = \log K(z,\bar{z}),\ K(z,\bar{z}) := (e_z, e_z), \tag{2a}$$

$$\omega_M(z,\bar{z}) = \mathrm{i}\sum_{\alpha,\beta} h_{\alpha\bar{\beta}} \mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}_\beta,\ h_{\alpha\bar{\beta}} = \frac{\partial^2 f}{\partial z_\alpha \partial\bar{z}_\beta},\ h_{\alpha\bar{\beta}} = \bar{h}_{\beta\bar{\alpha}},\ \alpha,\beta = 1,\dots,n, \tag{2b}$$

where $K(z,\bar{z})$ is the scalar product of two un-normalized Perelomov's CS-vectors e_z at $z \in M$ [2, 13, 31].

It is well known, see [41, Theorem 4.17], [42, Proposition 20], [43, eq. (6), p. 156], that the condition $\mathrm{d}\omega = 0$ for a Hermitian manifold to have a Kähler structure is equivalent with the conditions

$$\frac{\partial h_{\alpha\bar{\beta}}}{\partial z_\gamma} = \frac{\partial h_{\gamma\bar{\beta}}}{\partial z_\alpha}, \quad \text{or} \quad \frac{\partial h_{\alpha\bar{\beta}}}{\partial z_\gamma} = \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z_{\bar{\beta}}}, \quad \alpha,\beta,\gamma = 1,\dots,n. \tag{3}$$

In accord with [41, p. 42], [42, Appendix B], [44, p. 28], the Riemannian metric associated with the Hermitian metric on the manifold M in local coordinates is

$$\mathrm{d}s^2_M(z,\bar{z}) = \sum_{\alpha,\beta} h_{\alpha\bar{\beta}} \mathrm{d}z_\alpha \otimes \mathrm{d}\bar{z}_\beta. \tag{4}$$

Using the CS approach, in [1] we have determined the Kähler invariant two-form $\omega_{\mathcal{D}^J_1}(w,z)$ on the Siegel–Jacobi disk $\mathcal{D}^J_1 = \frac{G^J_1}{\mathrm{U(1)}\times\mathbb{R}} \approx \mathcal{D}_1 \times \mathbb{C}$, where the Siegel disk $\mathcal{D}_1$ is realized as $\{w \in \mathbb{C} |\ |w| < 1\}$. In [1, 15, 45] we applied the partial Cayley transform to $\omega_{\mathcal{D}^J_1}(w,z)$ and we obtained the Kähler invariant two-form on the Siegel–Jacobi upper half-plane $\mathcal{X}^J_1 = \frac{G^J_1(\mathbb{R})}{\mathrm{SO(2)}\times\mathbb{R}} \approx \mathcal{X}_1 \times \mathbb{R}^2$, firstly determined by Kähler and Berndt [6, 46–50], where $\mathcal{X}_1$ denotes the Siegel upper half-plane, realized as $\{v \in \mathbb{C} | \operatorname{Im} v > 0\}$. The construction has been generalized in [13, 14] for the Jacobi group of degree n. In [2] we have underlined that the metric associated to the Kähler two-form on the Siegel–Jacobi ball $\mathcal{D}^J_n = \frac{G^J_n}{\mathrm{U(n)}\times\mathbb{R}}$ is a balanced metric [51–53].

In [3] we introduced a five-dimensional manifold $\tilde{\mathcal{X}}^J_1 = \frac{G^J_1(\mathbb{R})}{\mathrm{SO(2)}} \approx \mathcal{X}_1 \times \mathbb{R}^3$, called extended Siegel–Jacobi upper half-plane. Because in Berezin's approach to CS on $M = G/H$ traditionally are considered G-homogeneous Kähler metrics on M, we were interested in determining the invariant metrics as well on $\tilde{\mathcal{X}}^J_1$, so we had to abandon Berezin's procedure to obtain balanced metric *via* the CS approach based on homogeneous Kähler manifolds [54–57] and we applied in [3] Cartan's moving frame method [58–60], which allows to determine invariant metrics on odd or even dimensional manifolds.

Mathematicians consider the real Jacobi group $G^J_1(\mathbb{R})$ as subgroup of $\mathrm{Sp}(2,\mathbb{R})$. We followed this approach in [3, 4], while in [1, 13–15, 45, 46] the Jacobi group was investigated *via* the construction of Perelomov's CS. We adopt the notation from [6, 7] for the real Jacobi group $G^J_1(\mathbb{R})$, realized as submatrices of $\mathrm{Sp}(2,\mathbb{R})$ of the form

$$g = \begin{pmatrix} a & 0 & b & q \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -p \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \det M = 1, \tag{5}$$

where

$$Y := (p,q) = XM^{-1} = (\lambda,\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (\lambda d - \mu c, -\lambda b + \mu a) \tag{6}$$

is related to the Heisenberg group H_1 described by (λ,μ,κ). For coordinatization of the real Jacobi group we adopt the so called S-coordinates (x,y,θ,p,q,κ) [6].

The present investigation is a continuation of [3], where we have obtained invariant metrics for several homogeneous manifolds associated with the real Jacobi group. In particular, we have determined the 2 (3)–parameter invariant metric on

$\mathcal{X}_1^J$, (respectively, $\tilde{\mathcal{X}}_1^J$). We proved in [3] that $\mathcal{X}_1^J$ is a non-symmetric, not naturally reductive space with respect to the balanced metric.

Below we motivate our interest for naturally reductive spaces.

We denoted by FC [14] the change of variables $x \to z$ in formula (1) such that

$$\underline{e}_x = (e_z, e_z)^{-\frac{1}{2}} e_z; \quad z = FC(x). \tag{7}$$

In Remark 3 of the paper [61], devoted to coherent states with support on Hermitian symmetric spaces, we observed that

For symmetric manifolds the FC-transform gives geodesics (A)

i.e. for symmetric spaces $M = G/H$, the relation $\exp(tz(x)) = \exp(tFC(x))$ gives geodesics through the identity of M. Assertion (A) was verified by direct calculation for the complex Grassmann manifold $G_n(\mathbb{C}^{m+n}) = \frac{\mathrm{SU}(n+m)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(m))}$ and its noncompact dual $\frac{\mathrm{SU}(n,m)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(m))}$ [62]. Looking for a geometric meaning of the phase of the scalar product of two un-normalized Perelomov's CS-vectors [63, 64], in [63, Remark 1] we showed that assertion (A) is true for a larger class of manifolds verifying a technical condition which includes the naturally reductive spaces, a natural generalization of symmetric spaces.

So we have the following sequence of space inclusions

Hermitian symmetric $\subset$ symmetric $\subset$ naturally reductive $\subset$ g. o.

Indeed, in [65, 66] we observed that the Hermitian symmetric spaces are in particular naturally reductive spaces. Let $M = G/H$ be a reductive Riemannian homogeneous space [67]. We have the direct sum of non-intersecting vector spaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and geodesics on naturally reductive manifolds M are obtained just by taking the exponential of $\mathfrak{m}$ [67]. We recall that the g. o. spaces are Riemannian homogeneous spaces (M, g) with origin $p = \{H\}$ where all the geodesics are orbits of one parameter group of isometries $\exp(tZ)$, $Z \in \mathfrak{m}$. $X \in \mathfrak{g} \setminus \{0\}$ is a geodesic vector if the curve $\gamma(t) = \exp(tX)(p)$ is geodesic with respect to the Riemannian connection [68].

In [69, Lemma 3], [1, Lemma 6.11 and Remark 6.12] we proved that $(\mathbb{C}, \mathcal{D}_1) \ni (z, w) = FC(\eta, w)$, but in [3, Proposition 5.8] we showed that $\mathcal{X}_1^J$ is not naturally reductive with respect to the balanced metric. Consequently, the FC-transform on $\mathcal{D}_1^J$ does not generate geodesics as in (A), see also [70].

The paper is laid out as follows. In Section 2 we make several changes of coordinates in the Kähler two-form $\omega_{\mathcal{D}_1^J}$ which are used in Section 6. Section 3 (4), extracted from [3], gives information on the embedding of H_1 (respectively $\mathrm{SL}(2,\mathbb{R})$) in $\mathrm{Sp}(2,\mathbb{R})$. Minimal information on the real Jacobi group as a subgroup of $\mathrm{Sp}(2,\mathbb{R})$ is given in Section 5. Lemma 1, an enlarged and improved version of [3, Lemma 5.1], establishes the action of the real Jacobi group on some of its homogeneous spaces. In Proposition 2, an improved version of [3, Proposition 5.2], the fundamental vector fields (FVF) on homogeneous spaces associated to the Jacobi group are calculated. Comment 1, Proposition 4 and Proposition 5 enrich the corresponding assertions in [3]. In the last section, where geometric properties of the five dimensional manifold $\tilde{\mathcal{X}}^J_1$ are investigated. We also present very sketchy the definitions of the mathematical objects investigated here, see full details in [3] and the extended version [42]. The paper is concluded with Comment 2.

The new results of the present paper are contained in: Remark 1, where we emphasize that the S-variables $p,q \in \mathbb{R}$ have a "CS-meaning" given by the simple relation $\eta = q + \mathrm{i}p$; item d) in Lemma 1; item g) in Proposition 2; equation (61d) in Proposition 3; Proposition 4, which shows that the FVF determined in Proposition 2 f) are Killing vectors for the invariant metric on $\tilde{\mathcal{X}}^J_1$; Proposition 5, which shows that the FVF determined in Proposition 2 g) are Killing vectors for the invariant metric of $G^J_1(\mathbb{R})$. In Theorem 1, which summarises the main results of the present paper, we show that $\tilde{\mathcal{X}}^J_1$ is a non-symmetric, non-naturally reductive space with respect to the three-parameter invariant metric. In the same theorem we determine also the geodesic vectors on $\tilde{\mathcal{X}}^J_1$.

Notation:

We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. We denote the imaginary unit $\sqrt{-1}$ by i, the real and imaginary parts of a complex number $z \in \mathbb{C}$ by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ respectively, and the complex conjugate of z by $\bar{z}$. We denote by $|M|$ or by $\det(M)$ the determinant of matrix M. $M(n,m,\mathbb{F})$ denotes the set of $n \times m$ matrices with entries in the field $\mathbb{F}$. We denote by $M(n,\mathbb{F})$ the set $M(n,n,\mathbb{F})$. If $A \in M(n,\mathbb{F})$, then A^t denotes the transpose of A. We denote by d the differential. We use Einstein convention *i.e.* repeated indices are implicitly summed over. The scalar product of vectors in the Hilbert space $\mathfrak{H}$ is denoted $(\cdot,\cdot)$. The set of vector fields (1-forms) is denoted by $\mathfrak{D}^1$ (respectively $\mathfrak{D}_1$). If $\lambda \in \mathfrak{D}_1$ and $L \in \mathfrak{D}^1$, then $\langle \lambda \,|\, L \rangle$ denotes their pairing. If X_i, $i = 1,\ldots,n$ are vectors in a vector space V over the field $\mathbb{F}$, then $\langle X_1, X_2, \ldots, X_n \rangle_{\mathbb{F}}$ denotes their span over $\mathbb{F}$. If we denote with Roman capital letteres the Lie groups, then their associated Lie algebras are denoted with the corresponding lowercase letter.

2. INVARIANT KÄHLER TWO-FORMS ON THE SIEGEL–JACOBI UPPER HALF-PLANE

The next proposition is an improved and enlarged version of [3, Proposition 2.1]. Below $(w,z) \in (\mathcal{D}_1, \mathbb{C})$, $(v,u) \in (\mathcal{X}_1, \mathbb{C})$, and the parameters k and ν come from representation theory of the Jacobi group: k indexes the positive discrete series of $\mathrm{SU}(1,1)$, $2k \in \mathbb{N}$, while $\nu > 0$ indexes the representations of the Heisenberg group [1].

Proposition 1. *a) Let us consider the Kähler two-form*

$$-\mathrm{i}\omega_{\mathcal{D}^J_1}(w,z)=\frac{2k}{P^2}\mathrm{d}w\wedge\mathrm{d}\bar{w}+\nu\frac{A\wedge\bar{A}}{P},\ P:=1-|w|^2,\ A=A(w,z):=\mathrm{d}z+\bar{\eta}\mathrm{d}w, \tag{8}$$

G^J_0-invariant to the action on the Siegel–Jacobi disk $\mathcal{D}^J_1$

$$\mathrm{SU}(1,1)\times\mathbb{C}\ni(\left(\begin{array}{cc}\mathcal{P} & \mathcal{Q}\\ \bar{\mathcal{Q}} & \bar{\mathcal{P}}\end{array}\right),\alpha)\times(w,z)=(\frac{\mathcal{P}w+\mathcal{Q}}{\bar{\mathcal{Q}}w+\bar{\mathcal{P}}},\frac{z+\alpha-\bar{\alpha}w}{\bar{\mathcal{Q}}w+\bar{\mathcal{P}}}). \tag{9}$$

We have the change of variables $(w,z)\rightarrow(w,\eta)$

$$\mathrm{FC}\colon\ z=\eta-w\bar{\eta},\qquad \mathrm{FC}^{-1}\colon\ \eta=\frac{z+\bar{z}w}{P}, \tag{10}$$

and

$$\mathrm{FC}\colon\ A(w,z)\rightarrow\mathrm{d}\eta-w\mathrm{d}\bar{\eta}. \tag{11}$$

The complex two-form

$$\omega_{\mathcal{D}^J_1}(w,\eta):=FC^*(\omega_{\mathcal{D}^J_1}(w,z)) \tag{12}$$

is not a Kähler two-form.

The symplectic form corresponding to the FC-transform applied to Kähler two-form (8) *is invariant to the action $(g,\alpha)\times(w,\eta)=(w_1,\eta_1)$ of G^J_0 on $\mathbb{C}\times\mathcal{D}_1$*

$$\eta_1=\mathcal{P}(\eta+\alpha)+\mathcal{Q}(\bar{\eta}+\bar{\alpha}), \tag{13}$$

where $\mathcal{P},\mathcal{Q}$ appear in (9)*.*

b) Using the partial Cayley transform

$$\Phi^{-1}:v=\mathrm{i}\,\frac{1+w}{1-w},\ \ u=\frac{z}{1-w},\ \ w,z\in\mathbb{C},\ |w|<1; \tag{14a}$$

$$\Phi:w=\frac{v-\mathrm{i}}{v+\mathrm{i}},\ \ z=2\mathrm{i}\,\frac{u}{v+\mathrm{i}},\ \ v,u\in\mathbb{C},\ \mathrm{Im}\,v>0, \tag{14b}$$

we obtain

$$A\left(\frac{v-\mathrm{i}}{v+\mathrm{i}},\frac{2\mathrm{i}\,u}{v+\mathrm{i}}\right)=\frac{2\mathrm{i}}{v+\mathrm{i}}B(v,u),$$

where

$$B(v,u):=\mathrm{d}u-\frac{u-\bar{u}}{v-\bar{v}}\mathrm{d}v. \tag{15}$$

The Kähler two-form of Berndt–Kähler

$$-\mathrm{i}\omega_{\mathcal{X}^J_1}(v,u)=-\frac{2k}{(\bar{v}-v)^2}\mathrm{d}v\wedge\mathrm{d}\bar{v}+\frac{2\nu}{\mathrm{i}(\bar{v}-v)}B\wedge\bar{B}, \tag{16}$$

is $G^J(\mathbb{R})_0$-invariant to the action on the Siegel–Jacobi upper half-plane $\mathcal{X}^J_1$

$$\left(\mathrm{SL}(2,\mathbb{R})\times\mathbb{C}^2\ni\begin{pmatrix}a & b\\ c & d\end{pmatrix},\alpha\right)\times(v,u)=\left(\frac{av+b}{cv+d},\frac{u+nv+m}{cv+d}\right),\ \ \alpha=m+\mathrm{i}n. \tag{17}$$

We have the change of variables $\mathrm{FC}_1: \ (v,u) \rightarrow (v,\eta)$

$$\mathrm{FC}_1: \ 2\mathrm{i}u = (v+\mathrm{i})\eta - (v-\mathrm{i})\bar{\eta}, \qquad \mathrm{FC}_1^{-1}: \ \eta = \frac{u\bar{v}-\bar{u}v+\mathrm{i}(\bar{u}-u)}{\bar{v}-v}. \tag{18}$$

c) If

$$\mathbb{C} \ni u := pv+q, \ \ p,q \in \mathbb{R}, \quad \mathbb{C} \ni v := x+\mathrm{i}y, \ x,y \in \mathbb{R}, \ y > 0, \tag{19}$$

then

$$B(v,u) = \mathrm{d}u - p\mathrm{d}v, \tag{20}$$

and

$$B(v,u) = B(x,y,p,q) = v\mathrm{d}p + \mathrm{d}q = (x+\mathrm{i}y)\mathrm{d}p + \mathrm{d}q. \tag{21}$$

d) If we have (18)*,* (19) *and we write*

$$\mathbb{C} \ni \eta := \chi + \mathrm{i}\psi, \ \chi,\psi \in \mathbb{R}, \tag{22}$$

then we get the change of coordinates

$$(x,y,p,q) \rightarrow (x,y,\chi,\psi): \ \psi = p, \ \chi = q, \tag{23}$$

and

$$B(v,u) = B(x,y,\chi,\psi) = x\mathrm{d}\psi + \mathrm{d}\chi + \mathrm{i}y\mathrm{d}\psi. \tag{24}$$

We also have the relations

$$\eta = q+\mathrm{i}p, \ q = \frac{1}{2}(\eta+\bar{\eta}), \ p = \frac{1}{2\mathrm{i}}(\eta-\bar{\eta}). \tag{25}$$

Given (19) *and*

$$\mathbb{C} \ni u := \xi + \mathrm{i}\rho, \quad \xi,\rho \in \mathbb{R}, \tag{26}$$

we obtain the change of variables

$$(x,y,\xi,\rho) \rightarrow (x,y,p,q): \ \xi = px+q, \ \rho = py, \tag{27}$$

and

$$B(v,u) = \mathrm{d}u - \frac{\rho}{y}\mathrm{d}v = \mathrm{d}(\xi+\mathrm{i}\rho) - \frac{\rho}{y}\mathrm{d}(x+\mathrm{i}y). \tag{28}$$

If we have (19) *and* (22)*, then, with* (18)*, we have the change of coordinates*

$$(x,y,\xi,\rho) \rightarrow (x,y,\chi,\psi): \ \ \xi = \psi x + \chi, \ \ \rho = \psi y. \tag{29}$$

Proof. a) We determined in [1, 40] the scalar product $K(w,z) := (e_{wz}, e_{wz})$ of two Perelomov's CS states based on the Siegel–Jacobi disk. The associated Kähler potential on $\mathcal{D}_1^J$ is

$$f(w,z) = -2k\log(P) + \nu\frac{2|z|^2 + \bar{w}z^2 + w\bar{z}^2}{2P}. \tag{30}$$

In [1] we applied (2) to the potential (30) and we obtained the Kähler two-form on $\mathcal{D}^J_1$

$$-\mathrm{i}\omega_{\mathcal{D}^J_1}(w,z) = f_{z\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z} + f_{z\bar{w}}\mathrm{d}z\wedge\mathrm{d}\bar{w} - \bar{f}_{z\bar{w}}\mathrm{d}\bar{z}\wedge\mathrm{d}w + f_{w\bar{w}}\mathrm{d}w\wedge\mathrm{d}\bar{w}. \quad (31)$$

The matrix corresponding to the metric associated with the Kähler two-form (31) reads [40, (5.11)]

$$h(w,z) = \begin{pmatrix} f_{z\bar{z}} & f_{z\bar{w}} \\ \bar{f}_{z\bar{w}} & f_{w\bar{w}} \end{pmatrix} = \begin{pmatrix} \frac{\nu}{P} & \nu\frac{\eta}{P} \\ \nu\frac{\bar{\eta}}{P} & \frac{2k}{P^2}+\nu\frac{|\eta|^2}{P} \end{pmatrix}. \quad (32)$$

It is easy to verify that the matrix elements of (32) satisfy the conditions (3) and the Jacobi disk $\mathcal{D}^J_1$ is a Kähler manifold.

If we apply the non-holomorphic FC-transform to the Kähler two-form (8), then the complex two-form (12) in the variables (w,η) is not a Kähler two-form [71, Proposition 2, p. 50]. This fact can be directly verified: if we introduce in (32) the value of $\eta = \eta(w,z)$ given in (10), then the conditions (3) for a complex two-form to be a fundamental two-form are not satisfied.

For the invariance (13) see [14, (6.4)].

b) The Kähler two-form on the Siegel–Jacobi upper half-plane $\mathcal{X}^J_1$ was determined from $\omega_{\mathcal{D}^J_1}(w,z)$ using the partial Cayley transform in [1, 2, 40]. Note that in the Berndt–Kähler approach in [47] the Kähler potential (62) is just "guessed", see Comment 1.

We have

$$\frac{A\wedge\bar{A}}{P} = \frac{1}{y}B\wedge\bar{B}.$$

For (18) see [40, (3.4)]. □

Correlating (25) in Proposition 1 with the [1, Comment 6.12], (1), (7) and [40, Lemma 2], we make the following surprising remark giving a "CS - meaning" to the S-parameters p,q.

Remark 1. *The FC-transform* (10) *relates Perelomov's un-normalized CS-vector* e_{wz} *with the normalized one* $\underline{e}_{w\eta}$

$$\underline{e}_{w\eta} = (e_{wz},e_{wz})^{-\frac{1}{2}}e_{wz}, \quad w\in\mathcal{D}_1,\ z,\eta\in\mathbb{C},$$

and the S-variables p,q *are related to the parameter* η *by the simple relation*

$$\eta = q + \mathrm{i}p. \quad (33)$$

3. THE HEISENBERG GROUP EMBEDDED IN $\mathrm{Sp}(2,\mathbb{R})$

In this section, extracted from [3, Section 3], we summarize the parametrization of the Heisenberg group used in [6].

The composition law of the 3-dimensional Heisenberg group H_1 in (5) is

$$(\lambda,\mu,\kappa)(\lambda',\mu',\kappa') = (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\mu'-\lambda'\mu).$$

As in (5) with $M = \mathbb{1}_2$, we denote an element of H_1 embedded in $\mathrm{Sp}(2,\mathbb{R})$ by

$$\mathrm{H}_1 \ni \mathrm{g} = \left(\begin{array}{cccc} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{array}\right),\ \mathrm{g}^{-1} = \left(\begin{array}{cccc} 1 & 0 & 0 & -\mu \\ -\lambda & 1 & -\mu & -\kappa \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{34}$$

A basis of the Lie algebra $\mathfrak{h}_1 =< P,Q,R >_{\mathbb{R}}$ of the Heisenberg group H_1 in the realization (34) in the space $M(4,\mathbb{R})$ consists of the matrices

$$P = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right),\ Q = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),\ R = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

which verify the commutation relations

$$[P,Q] = 2R,\ [P,R] = [Q,R] = 0. \tag{35}$$

If we write

$$\mathrm{H}_1 \ni \mathrm{g}(\lambda,\mu,\kappa) = \mathbb{1}_4 + \lambda \mathrm{P} + \mu \mathrm{Q} + \kappa \mathrm{R},$$

then, using the formulas (36), see details in [42, Section 3]

$$g^{-1}\mathrm{d}g = P\lambda^p + Q\lambda^q + R\lambda^r,\ \ \mathrm{d}gg^{-1} = P\rho^p + Q\rho^q + R\rho^r, \tag{36}$$

we find the left-invariant one-forms and vector fields

$$\left\{\begin{array}{l} \lambda^p = \mathrm{d}\lambda \\ \lambda^q = \mathrm{d}\mu \\ \lambda^r = \mathrm{d}\kappa - \lambda\mathrm{d}\mu + \mu\mathrm{d}\lambda \end{array}\right. ; \qquad \left\{\begin{array}{l} L^p = \partial_\lambda - \mu\partial_\kappa \\ L^q = \partial_\mu + \lambda\partial_\kappa \\ L^r = \partial_\kappa \end{array}\right. .$$

4. THE $\mathrm{SL}(2,\mathbb{R})$ GROUP EMBEDDED IN $\mathrm{Sp}(2,\mathbb{R})$

In this section we extract from [3, Section 4] the minimum information we need to understand the embedding of $\mathrm{SL}(2,\mathbb{R})$ in the 4-dimensional matrix realization of $\mathrm{Sp}(2,\mathbb{R})$.

An element $M \in \mathrm{SL}(2,\mathbb{R})$ and its inverse are realized as elements of $\mathrm{Sp}(2,\mathbb{R})$ by the relations

$$M = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \mapsto g = \left(\begin{array}{cccc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \in G^J_1(\mathbb{R}),\ \ g^{-1} = \left(\begin{array}{cccc} d & 0 & -b & 0 \\ 0 & 1 & 0 & 0 \\ -c & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{37}$$

A basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) =< F,G,H >_{\mathbb{R}}$ consists of the matrices in $M(4,\mathbb{R})$

$$F = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$F, \; G, \; H$ verify the commutation relations (38)

$$[F,G] = H, \; [G,H] = 2G, \; [H,F] = 2F. \tag{38}$$

With the representation (37), we have

$$g^{-1}\mathrm{d}g = F\lambda^f + G\lambda^g + H\lambda^h, \; \mathrm{d}gg^{-1} = F\rho^f + G\rho^g + H\rho^h.$$

Using the parameterization (37) for $\mathrm{SL}(2,\mathbb{R})$, we find

$$\begin{cases} \lambda^f = d\mathrm{d}b - b\mathrm{d}d \\ \lambda^g = -c\mathrm{d}a + a\mathrm{d}c \\ \lambda^h = d\mathrm{d}a - b\mathrm{d}c = c\mathrm{d}b - a\mathrm{d}d \end{cases}; \qquad \begin{cases} \rho^f = -b\mathrm{d}a + a\mathrm{d}b \\ \rho^g = d\mathrm{d}c - c\mathrm{d}d \\ \rho^h = d\mathrm{d}a - c\mathrm{d}b \end{cases}.$$

The Iwasawa decomposition $M = NAK$ of an element M as in (37) reads

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \; y > 0.$$

We find

$$a = y^{1/2}\cos\theta - xy^{-1/2}\sin\theta, \quad b = y^{1/2}\sin\theta + xy^{-1/2}\cos\theta, \tag{39a}$$

$$c = -y^{-1/2}\sin\theta, \quad d = y^{-1/2}\cos\theta, \tag{39b}$$

and

$$x = \frac{ac+bd}{d^2+c^2}, \; y = \frac{1}{d^2+c^2}, \; \sin\theta = -\frac{c}{\sqrt{c^2+d^2}}, \; \cos\theta = \frac{d}{\sqrt{c^2+d^2}}. \tag{40}$$

We determined in [3] the left-invariant vector fields L^f, L^g, L^h on $\mathrm{SL}(2,\mathbb{R})$, dual orthogonal to the left-invariant one-forms $\lambda^f, \lambda^g, \lambda^h$. We introduced the left-invariant one-forms

$$\lambda_1 := \sqrt{\alpha}(\lambda^f + \lambda^g), \quad \lambda_2 := 2\sqrt{\alpha}\lambda^h, \quad \lambda_3 := \sqrt{\beta}(\lambda^f - \lambda^g). \tag{41}$$

In [3] we determined the left-invariant vector fields L^j such that $< \lambda_i | L^j > = \delta_{ij}, i,j = 1,2,3$, where

$$L^1 := \frac{1}{2\sqrt{\alpha}}(L^f + L^g), \quad L^2 := \frac{1}{2\sqrt{\alpha}}L^h, \quad L^3 := \frac{1}{2\sqrt{\beta}}(L^f - L^g). \tag{42}$$

5. THE JACOBI GROUP $G_1^J(\mathbb{R})$ EMBEDDED IN $\mathrm{Sp}(2,\mathbb{R})$

5.1. THE COMPOSITION LAW

The real Jacobi group of index one is the semidirect product of the real three-dimensional Heisenberg group H_1 with $\mathrm{SL}(2,\mathbb{R})$. The Lie algebra of the Jacobi group $G_1^J(\mathbb{R})$ is given by $\mathfrak{g}_1^J(\mathbb{R}) =< P,Q,R,F,G,H >_{\mathbb{R}}$, where the first three generators P,Q,R of $\mathfrak{h}_1$ verify the commutation relations (35), the generators F,G,H of $\mathfrak{sl}(2,\mathbb{R})$ verify the commutation relations (38) and the ideal $\mathfrak{h}_1$ in $\mathfrak{g}_1^J(\mathbb{R})$ is determined by the non-zero commutation relations

$$[P,F]=Q,\ [Q,G]=P,\ [P,H]=P,\ [H,Q]=Q. \tag{43}$$

Let $g:=(M,h)\in G_1^J(\mathbb{R})$, where M is as in (37), while $h:=(X,\kappa)\in \mathrm{H}_1$, $\mathrm{X}:=(\lambda,\mu)\in\mathbb{R}^2$ and similarly for $g':=(M',h')$. The composition law of $G_1^J(\mathbb{R})$ is

$$gg'=g_1,\ \text{where}\ M_1=MM',\ X_1=XM'+X',\ \kappa_1=\kappa+\kappa'+\begin{vmatrix} XM' \\ X'\end{vmatrix}, \tag{44}$$

i.e.

$$\begin{aligned} g_1 &= \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd'\end{pmatrix}, \\ (\lambda_1,\mu_1) &= (\lambda'+\lambda a'+\mu c', \mu'+\lambda b'+\mu d'), \\ \kappa_1 &= \kappa+\kappa'+\lambda q'-\mu p'. \end{aligned} \tag{45}$$

The inverse element of $g\in G_1^J(\mathbb{R})$ is given by

$$(M,X,\kappa)^{-1}=(M^{-1},-Y,-\kappa)\rightarrow\ g^{-1}=\begin{pmatrix} d & 0 & -b & -\mu \\ -p & 1 & -q & -\kappa \\ -c & 0 & a & \lambda \\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{46}$$

where Y was defined in (6) and similarly for Y', while g has the general form given in (5).

Using the notation in [6, p. 9], the *EZ-coordinates* (EZ - from Eichler & Zagier) of an element $g\in G_1^J(\mathbb{R})$ as in (5) are $(x,y,\theta,\lambda,\mu,\kappa)$, where M is related with (x,y,θ) by (39), (40).

The *S-coordinates* (S - from Siegel) of $g=(M,h)\in G_1^J(\mathbb{R})$ are (x,y,θ,p,q,κ), where (x,y,θ) are expressed as functions of $M\in\mathrm{SL}(2,\mathbb{R})$ by (39), (40).

5.2. THE ACTION

Let

$$\mathbb{C}\ni\tau:=x+\mathrm{i}y,\quad \mathbb{C}\ni z:=p\tau+q=\xi+\mathrm{i}\rho,\qquad x,y,p,q,\xi,\rho\in\mathbb{R}. \tag{47}$$

Let $\mathcal{X}_1^J \approx \mathcal{X}_1 \times \mathbb{R}^2$ be the Siegel–Jacobi upper half-plane, where $\mathcal{X}_1 = \{\tau \in \mathbb{C} |\ y := \operatorname{Im}\tau > 0\}$ is the Siegel upper half-plane, and $\tilde{\mathcal{X}}_1^J \approx \mathcal{X}_1^J \times \mathbb{R}$ denotes the extended Siegel–Jacobi upper half-plane. Simultaneously with the Jacobi group $G_1^J(\mathbb{R})$ consisting of elements (M, X, κ), we considered the group $G^J(\mathbb{R})_0$ of elements (M, X) [1, 3]. Then:

Lemma 1. *a) The action $G^J(\mathbb{R})_0 \times \mathcal{X}_1^J \to \mathcal{X}_1^J$ is given by*

$$(M,X) \times (\tau', z') = (\tau_1, z_1), \text{ where } \tau_1 = \frac{a\tau' + b}{c\tau' + d},\ z_1 = \frac{z' + n\tau' + m}{c\tau' + d}. \tag{48}$$

b) If $z' = p'\tau' + q'$, $\tau' = x' + \mathrm{i}\, y'$ as in (47)*, then the action*

$$(M,X) \times (x', y', p', q') = (x_1, y_1, p_1, q_1) \tag{49}$$

is given by the formula

$$(p_1, q_1) = (p, q) + (p', q') \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (p + dp' - cq', q - bp' + aq'). \tag{50}$$

c) The action $G_1^J(\mathbb{R}) \times \tilde{\mathcal{X}}_1^J \to \tilde{\mathcal{X}}_1^J$ is given by

$$\begin{aligned} &(M, X, \kappa) \times (\tau', z', \kappa') = (\tau_1, z_1, \kappa_1), \\ &(M, X, \kappa) \times (x', y', p', q', \kappa') = (x_1, y_1, p_1, q_1, \kappa_1), \\ &\kappa_1 = \kappa + \kappa' + \lambda q' - \mu p',\ (p', q') = (\frac{\rho'}{y'}, \xi' - \frac{x'}{y'}\rho'),\ (\lambda, \mu) = (p, q)M. \end{aligned} \tag{51}$$

d) The action $G_1^J(\mathbb{R}) \times G_1^J(\mathbb{R}) \to G_1^J(\mathbb{R})$ corresponding to the composition law (44)*, or equivalently* (45)*, is*

$$(M, X, \kappa) \times (x', y', \theta', p', q', \kappa') = (x_1, y_1, \theta_1, p_1, q_1, \kappa_1).$$

5.3. FUNDAMENTAL VECTOR FIELDS

We recall the notion of FVF, see [3, Appendix A.1], [16, p. 122] and [72, p. 51].

Let $M = G/H$ be a homogeneous n-dimensional manifold and let us suppose that the group G acts transitively on M from the *left*, $G \times M \to M :\ g \times x = y$, where $y = (y_1, \ldots, y_n)^t$. Then $g(t) \times x = y(t)$, where $g(t) = \exp(tX)$, $t \in \mathbb{R}$, generates a curve $y(t)$ in M with $y(0) = x$ and $\dot{y}(0) = X$. The *fundamental vector field* associated to $X \in \mathfrak{g}$ at $x \in M$ is defined as

$$X_x^* := \frac{\mathrm{d}}{\mathrm{d}t} y(t)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX) \times x)|_{t=0} = \sum_{i=1}^{n} (X_i^*)_x \frac{\partial}{\partial z_i},\ \ (X_i^*)_x = \frac{\mathrm{d}y_i(t)}{\mathrm{d}t}|_{t=0}.$$

With the action given in Lemma 1, we get the FVF on some homogeneous spaces associated to the real Jacobi group, see Proposition 5.2 in [3], reproduced below. Only the item g) is new.

Proposition 2. *a) The FVF expressed in the coordinates (τ, z) of the Siegel–Jacobi upper half-plane $\mathcal{X}_1^J$ on which the reduced Jacobi group $G^J(\mathbb{R})_0$ acts by* (48) *are given by the holomorphic vector fields*

$$F^* = \partial_\tau,\ G^* = -\tau^2\partial_\tau - z\tau\partial_z,\ H^* = 2\tau\partial_\tau + z\partial_z; \tag{52a}$$

$$P^* = \tau\partial_z,\ Q^* = \partial_z,\ R^* = 0. \tag{52b}$$

b) The real holomorphic FVF corresponding to $\tau := x + \mathrm{i}y,\ y > 0,\ z := \xi + \mathrm{i}\rho$ in the variables (x, y, ξ, ρ) are

$$F^* = F_1^*,\ G^* = G_1^* + (\rho y - \xi x)\partial_\xi - (\xi y + x\rho)\partial_\rho; \tag{53a}$$

$$H^* = H_1^* + \xi\partial_\xi + \rho\partial_\rho,\ P^* = x\partial_\xi + y\partial_\rho,\ Q^* = \partial_\xi,\ R^* = 0, \tag{53b}$$

where F_1^*, G_1^*, H_1^* are the FVF (54) *of the homogeneous manifold $\mathcal{X}_1$*

$$F_1^* = \frac{\partial}{\partial x},\ G_1^* = (y^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y},\ H_1^* = 2(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) \tag{54}$$

associated to the generators F, G, H of $\mathfrak{sl}(2, R)$ corresponding to the action (48) *of* $\mathrm{SL}(2, \mathbb{R})$ *on $\mathcal{X}_1$.*

c) If we express the FVF in the variables (x, y, p, q), where $\xi = px + q,\ \rho = py$, we find

$$F^* = F_1^* - p\partial_q,\ G^* = G_1^* - q\partial_p,\ H^* = H_1^* - p\partial_p + q\partial_q; \tag{55a}$$

$$P^* = \partial_p,\ Q^* = \partial_q,\ R^* = 0. \tag{55b}$$

d) If we consider the action (51) *of $G_1^J(\mathbb{R})$ on the points (τ, z, κ) of $\tilde{\mathcal{X}}_1^J$, we get instead of* (52) *the FVF in the variables (τ, z, p, q, κ)*

$$F^* = \partial_\tau,\ G^* = -\tau^2\partial_\tau - z\tau\partial_z,\ H^* = 2\tau\partial_\tau + z\partial_z; \tag{56a}$$

$$P^* = \tau\partial_z + q\partial_\kappa,\ Q^* = \partial_z - p\partial_\kappa,\ R^* = \partial_\kappa, p = \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)},\ q = \frac{\mathrm{Im}(\bar{z}\tau)}{\mathrm{Im}(\tau)}. \tag{56b}$$

e) Instead of (53)*, we get the FVF in $\tilde{\mathcal{X}}_1^J$ in the variables $(x, y, \xi, \rho, \kappa)$*

$$F^* = F_1^*,\ G^* = G_1^* + (\rho y - \xi x)\partial_\xi - (\xi y + x\rho)\partial_\rho; \tag{57a}$$

$$H^* = H_1^* + \xi\partial_\xi + \rho\partial_\rho,\ P^* = x\partial_\xi + y\partial_\rho + q\partial_\kappa,\ Q^* = \partial_\xi - p\partial_\kappa,\ R^* = \partial_\kappa. \tag{57b}$$

f) Instead of (55)*, we get the FVF in the variables (x, y, p, q, κ)*

$$F^* = F_1^* - p\partial_q,\ G^* = G_1^* - q\partial_p,\ H^* = H_1^* - p\partial_p + q\partial_q; \tag{58a}$$

$$P^* = \partial_p + q\partial_k,\ Q^* = \partial_q - p\partial_k,\ R^* = \partial_\kappa. \tag{58b}$$

g) The FVF in the S-variables corresponding to the composition law (44) *of the Jacobi group $G_1^J(\mathbb{R})$ are the same as in* (58)*, except G^**

$$F^* = F_1^* - p\partial_q,\ G^* = G_1^* - y\frac{\partial}{\partial\theta},\ H^* = H_1^* - p\partial_p + q\partial_q; \tag{59a}$$

$$P^* = \partial_p + q\partial_k,\ Q^* = \partial_q - p\partial_k,\ R^* = \partial_\kappa. \tag{59b}$$

6. INVARIANT METRICS

Using the formula

$$g^{-1}\mathrm{d}g = \lambda^F F + \lambda^G G + \lambda^H H + \lambda^P P + \lambda^Q Q + \lambda^R R,$$

where g is as in (5) and g^{-1} as in (46), we calculated in [3] the left-invariant one-forms on $G^J_1(\mathbb{R})$.

The left-invariant vector fields L^α for the real Jacobi group $G^J_1(\mathbb{R})$ are orthogonal with respect to the invariant one-forms λ^β,

$$< \lambda^\beta | L^\alpha >= \delta_{\alpha\beta},\ \ \alpha,\beta = F,G,H,P,Q,R.$$

Besides the formulas for $\lambda_1,\lambda_2,\lambda_3$ defined in (41), we have introduced in [3] the left-invariant one-forms

$$\lambda_4 := \sqrt{\gamma}\lambda^P,\ \lambda_5 := \sqrt{\gamma}\lambda^Q,\ \lambda_6 := \sqrt{\delta}\lambda^R,$$

where $\lambda^P,\lambda^Q,\lambda^R$ are defined in [3, p. 18]. Also, besides the left-invariant vector fields L^1,L^2,L^3 defined in (42), we have introduced the left-invariant one forms

$$L^4 := \frac{1}{\sqrt{\gamma}}L^P,\ L^5 := \frac{1}{\sqrt{\gamma}}L^Q,\ L^6 := \frac{1}{\sqrt{\delta}}L^R,$$

where L^P,L^Q,L^R are defined in [3, Proposition 5.3].

The vector fields L^i, $i = 1,\ldots,6$ verify the commutations relations

$$[L^1,L^2] = -\frac{\sqrt{\beta}}{\alpha}L^3 \qquad [L^2,L^3] = \frac{1}{2\sqrt{\beta}}L^1 \qquad [L^3,L^1] = \frac{1}{\sqrt{\beta}}L^2 \tag{60a}$$

$$[L^1,L^4] = -\frac{1}{2\sqrt{\alpha}}L^5 \qquad [L^1,L^5] = -\frac{1}{2\sqrt{\alpha}}L^4 \qquad [L^1,L^6] = 0 \tag{60b}$$

$$[L^2,L^4] = -\frac{1}{2\sqrt{\alpha}}L^4 \qquad [L^2,L^5] = \frac{1}{2\sqrt{\alpha}}L^5 \qquad [L^2,L^6] = 0 \tag{60c}$$

$$[L^3,L^4] = -\frac{1}{2\sqrt{\alpha}}L^5 \qquad [L^3,L^5] = \frac{1}{2\sqrt{\beta}}L^4 \qquad [L^3,L^6] = 0 \tag{60d}$$

$$[L^4,L^5] = \frac{2\sqrt{\delta}}{\gamma}L^6 \qquad [L^4,L^6] = 0 \qquad [L^5,L^6] = 0. \tag{60e}$$

Following [3, Proposition 5.4], we obtain from the Kähler two-form (16) in Proposition 1 the metric on $\mathcal{X}^J_1$ in the convention of (4), replacing $v \to \tau,\ u \to z,\ k \to 2c_1,\ \nu \to c_2$.

Only (61d) in the next proposition is new.

Proposition 3. *The two-parameter balanced metric on the Siegel–Jacobi upper half-plane* $\mathcal{X}_1^J$*, left-invariant to the action* (48), (49) *and* (50)*, respectively* (51) *of the reduced group* $G^J(\mathbb{R})_0$*, is given by the formulas*

$$\mathrm{d}s^2_{\mathcal{X}_1^J}(\tau,z) = -c_1\frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}}{(\tau-\bar{\tau})^2}+\frac{2\mathrm{i}c_2}{\tau-\bar{\tau}}(\mathrm{d}z-p\mathrm{d}\tau)\times cc,\;\; p=\frac{z-\bar{z}}{\tau-\bar{\tau}}, \tag{61a}$$

$$\begin{aligned}\mathrm{d}s^2_{\mathcal{X}_1^J}(x,y,p,q) &= c_1\frac{\mathrm{d}x^2+\mathrm{d}y^2}{4y^2}+\frac{c_2}{y}\left[(x^2+y^2)\mathrm{d}p^2+\mathrm{d}q^2+2x\mathrm{d}p\mathrm{d}q\right] \\ &= c_1\frac{\mathrm{d}x^2+\mathrm{d}y^2}{4y^2}+c_2\frac{x^2+y^2}{y}\left[(\mathrm{d}p+\frac{x}{x^2+y^2}\mathrm{d}q)^2+(\frac{y\mathrm{d}q}{x^2+y^2})^2\right],\end{aligned} \tag{61b}$$

$$\begin{aligned}\mathrm{d}s^2_{\mathcal{X}_1^J}(x,y,\xi,\rho) &= c_1\frac{\mathrm{d}x^2+\mathrm{d}y^2}{4y^2}+ \\ &+\frac{c_2}{y}\left[\mathrm{d}\xi^2+\mathrm{d}\rho^2+(\frac{\rho}{y})^2(\mathrm{d}x^2+\mathrm{d}y^2)-2\frac{\rho}{y}(\mathrm{d}x\mathrm{d}\xi+\mathrm{d}y\mathrm{d}\rho)\right],\end{aligned} \tag{61c}$$

$$\mathrm{d}s^2_{\mathcal{X}_1^J}(x,y,\chi,\psi) = c_1\frac{\mathrm{d}x^2+\mathrm{d}y^2}{4y^2}+\frac{c_2}{y}\left[(x\mathrm{d}\psi+\mathrm{d}\chi)^2+y^2\mathrm{d}\psi^2\right]. \tag{61d}$$

Proof. In the expression (15) of $B(v,u)$, $v\in\mathcal{X}_1$, $u\in\mathbb{C}$, we introduce the parametrizations of v,u appearing in Proposition 1 and then we pass from the Kähler two-form (16) of Kähler–Berndt to the associated Riemannian metric with the standard formula (4).

In Proposition 1 a) we have obtained from the Kähler two-form $\omega_{\mathcal{D}_1^J}$ (8) the Kähler two-form $\omega_{\mathcal{X}_1^J}$ (16), from which we get the metric (61a) with formula (4).

In formula (20) we introduce (19) and we get (21). With (4) we find (61b).

In formula (20) we introduce (26) and with (27), we get (28). With (4), we find (61c).

In (61c) we introduce (29), or in (61b) we introduce (23), and we get (61d).

Also, if in (21) we take into consideration (23), we get (24). We apply (4) to get (61d). □

Below we reproduce the Comment 5.5 in [3] with some completions:

Comment 1. *Berndt [47, p. 8] considered the closed two-form* $\Omega=\mathrm{d}\bar{\mathrm{d}}f'$ *of Siegel–Jacobi upper half-plane,* $G^J(\mathbb{R})_0$*-invariant to the action* (48)*, obtained from the Kähler potential*

$$f'(\tau,z) = c_1\log(\tau-\bar{\tau})-\mathrm{i}c_2\frac{(z-\bar{z})^2}{\tau-\bar{\tau}},\; c_1,c_2>0. \tag{62}$$

Formula (62) *is presented by Berndt as "communicated to the author by Kähler". Also in [47, p. 8] is given our equation* (61a)*, while our* (61b) *corrects two printing errors in Berndt's paper.*

Later, in [49, § 36], reproduced also in [50], Kähler argues how to choose the potential as in (62)*, see also [49, (9) § 37], where* $c_1 = -\frac{k}{2}$, $c_2 = \mathrm{i}\nu\pi$*, i.e.*

$$f'(\tau,z) = -\frac{k}{2}\log\frac{\tau-\bar{\tau}}{2\mathrm{i}} - \mathrm{i}\pi\nu\frac{(z-\bar{z})^2}{\tau-\bar{\tau}}. \tag{63}$$

Once the Kähler potential (63) *is known, we apply the recipe* (2b)

$$-\mathrm{i}\omega_{\mathcal{X}_1^J}(\tau,z) = f'_{\tau\bar{\tau}}\mathrm{d}\tau\wedge\mathrm{d}\bar{\tau} + f'_{\tau\bar{z}}\mathrm{d}\tau\wedge\mathrm{d}\bar{z} - \bar{f}'_{\tau\bar{z}}\mathrm{d}\bar{\tau}\wedge\mathrm{d}z + f'_{z\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

The metric (8) *in [49] differs from the metric* (61) *by a factor of two, since the Hermitian metric used by Kähler is* $\mathrm{d}s^2 = 2g_{i\bar{j}}\mathrm{d}z_i\otimes\mathrm{d}\bar{z}_j$*. If in* (63) *we take* $k/2\to k$*, we have*

$$f'_\tau = -k\frac{1}{\tau-\bar{\tau}} + \mathrm{i}\pi\nu\frac{(z-\bar{z})^2}{(\tau-\bar{\tau})^2},\ f'_{\tau\bar{\tau}} = -k\frac{1}{(\tau-\bar{\tau})^2} + 2\mathrm{i}\pi\nu\frac{(z-\bar{z})^2}{(\tau-\bar{\tau})^3},$$
$$f'_{\tau\bar{z}} = -2\mathrm{i}\pi\nu\frac{z-\bar{z}}{(\tau-\bar{\tau})^2},\ f'_z = -2\mathrm{i}\pi\nu\frac{z-\bar{z}}{\tau-\bar{\tau}},\ f'_{z\bar{z}} = 2\mathrm{i}\pi\nu\frac{1}{\tau-\bar{\tau}},$$

and we get (16)*. Relation* (16) *has been obtained by Berndt [48, p. 30], where the denominator of the first term is misprinted as* $v-\bar{v}$ *(or* $\tau-\bar{\tau}$ *in our notations). Equation* (61b) *appears also in [48, p. 30] and [6, p. 62].*

We also recall that in [1, (9.20)] we observed that the Kähler potential (63) *should correspond to a reproducing kernel*

$$K(\tau,z) = y^{-\frac{k}{2}}\exp(2\pi p^2 y). \tag{65}$$

In [5, (4.3)], see also [5, Proposition 4.1], we have presented a generalization of (65) *for* $\mathcal{X}_n^J$*, obtained by Takase in [22, §9].*

Yang calculated in [10] the metric on $\mathcal{X}_n^J$*, invariant to the action of* $G_n^J(\mathbb{R})_0$*. The equivalence of the metric of Yang with the metric obtained via CS on* $\mathcal{D}_n^J$ *and then transported to* $\mathcal{X}_n^J$ *via partial Cayley transform is underlined in [14]. In particular, the metric* (61c) *appears in [10, p. 99] for the particular values* $c_1 = 1$, $c_2 = 4$*. See also [9, 11, 12].*

We recall that a vector field X on a Riemannian manifold (M,g) is called an *infinitesimal isometry* or a *Killing vector field* if the local 1-parameter group of local transformations by X in a neighbourhood of each point of M consists of local isometries [72, Proposition 3.2], *i.e.*

$$L_X g = 0,\ \ X\in\mathfrak{D}^1(M), \tag{66}$$

where L_X is the Lie derivative on M. The condition (66) for a vector field (67)

$$X = \sum_{i=1}^n X^i\frac{\partial}{\partial x^i} \tag{67}$$

to be a Killing vector field amounts to its *contravariant* components to verify the equations [3, A.6]

$$X^\mu\partial_\mu g_{\lambda\chi}+g_{\mu\chi}\partial_\lambda X^\mu+g_{\lambda\mu}\partial_\chi X^\mu=0, \qquad \lambda,\chi,\mu=1,\dots,\dim M=n. \tag{68}$$

It is well known that *the fundamental vector field X^* on a Riemannian homogeneous manifold is a Killing vector*, see *e.g.* [73, p 4], [74, Proposition 2.2, p. 139] or [3, Remark A.4].

The next proposition completes [3, Proposition 5.6].

Proposition 4. *The three-parameter metric on the extended Siegel–Jacobi upper half -plane $\tilde{\mathcal{X}}^J_1$, in the S-coordinates (x,y,p,q,κ)*

$$\mathrm{d}s^2_{\tilde{\mathcal{X}}^J_1}=\mathrm{d}s^2_{\mathcal{X}^J_1}(x,y,p,q)+\lambda_6^2(p,q,\kappa) \tag{69a}$$

$$=\frac{\alpha}{y^2}(\mathrm{d}x^2+\mathrm{d}y^2)+[\frac{\gamma}{y}(x^2+y^2)+\delta q^2]\mathrm{d}p^2+(\frac{\gamma}{y}+\delta p^2)\mathrm{d}q^2+\delta\mathrm{d}\kappa^2$$
$$+2(\gamma\frac{x}{y}-\delta pq)\mathrm{d}p\mathrm{d}q+2\delta(q\mathrm{d}p\mathrm{d}\kappa-p\mathrm{d}q\mathrm{d}\kappa) \tag{69b}$$

is left-invariant with respect to the action of the Jacobi group $G^J_1(\mathbb{R})$ given in Lemma 1. *If*

$$X=X^1\partial_x+X^2\partial_y+X^3\partial_p+X^4\partial_q+X^5\partial_\kappa\in\mathfrak{g}^J_1(\mathbb{R})\ominus<L_3>,$$

then the fundamental vector fields (59a) *verify the following Killing equations* (68) *with respect to the metric* (69), *invariant to the action* (51)*:*

$$-X^2+y\partial_xX^1=0,$$
$$\partial_xX^2+\partial_yX^1=0,$$
$$(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_xX^3+\delta_q\partial_xX^5+(\gamma\frac{x}{y}-\delta pq)\delta_xX^4=0,$$
$$(\gamma\frac{x}{y}-\delta pq)\partial_xX^3+(\frac{\gamma}{y}+\delta p^2)\partial_xX^4+\frac{\alpha}{y^2}\partial_qX^2=0,$$
$$\delta pq\partial_xX^3-\delta p\partial_xX^4+\delta\partial_xX^5+\frac{\alpha}{y^2}\partial_\kappa X^1=0,$$
$$-X^2+y\partial_yX^2=0,$$
$$(\gamma\frac{x}{y}-\delta pq)\partial_yX^4+(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_yX^3+\delta_q\partial_\kappa X^5+\frac{\alpha}{y^2}\partial_pX^2=0,$$
$$(\gamma\frac{x}{y}-\delta pq)\partial_yX^3+(\frac{\gamma}{y}+\delta p^2)\partial_yX^4-\delta p\partial_yX^5+\frac{\alpha}{y^2}\partial_qX^2=0,$$
$$\delta q\partial_yX^3-\delta_p\partial_yX^4+\delta\partial_yX^5+\frac{\alpha}{y^2}\partial_kX^1=0,$$
$$2\gamma\frac{x}{y}X^1+\gamma(1-\frac{x^2}{y^2})X^2+2\delta qX^4+2(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_pX^3+2(\gamma\frac{x}{y}-\delta pq)\partial_pX^4$$

$$
\begin{aligned}
&\quad + 2\delta q \partial_p X^5 = 0,\\
&\gamma \frac{x}{y}(yX^1 - xX^2) - \delta(qX^3 + pX^4) + 2(\gamma\frac{x}{y} - \delta pq)\partial_p X^3 + (\frac{\gamma}{y} + \delta p^2)\partial_p X^4 - \delta p \delta_p \partial_p X^5\\
&\quad + \gamma(\frac{x^2+y^2}{y} + \delta q^2)\partial_q X^3 + (\gamma\frac{x}{y} - \delta pq)\partial_q X^4 + \delta_q \partial_q X^5 = 0,\\
&\delta X^4 + \delta q \partial_p X^3 - \delta p \partial_p X^4 + \delta \partial_p X^5 + (\gamma\frac{x^2+y^2}{y} + \delta q^2)\partial_\kappa X^3 + (\gamma\frac{x}{y} - \delta pq)\partial_\kappa X^4\\
&\quad + \delta q \partial_k X^5 = 0,\\
&-\frac{\gamma}{y^2}X^2 + 2\delta p X^3 + 2(\gamma\frac{x}{y} - \delta pq)\delta_q X^3 + 2(\frac{\gamma}{y} + \delta p^2)\partial_q X^4 - 2\delta pq \partial_q X^5 = 0,\\
&\delta X^3 + \delta q \partial_q X^3 - \delta p \partial_q X^4 + \delta \partial_q X^5 + (\gamma\frac{x}{y} - \delta pq)\partial_\kappa X^3 + (\frac{\gamma}{y} + \delta p^2)\partial_\kappa X^4 - \delta_p \partial_\kappa X^5 = 0,\\
&q\partial_\kappa X^3 - p\partial_k X^4 + \partial_k X^5 = 0.
\end{aligned}
$$

The next proposition is a completion of [3, Theorem 5.7].

Proposition 5. *The four-parameter left-invariant metric on the real Jacobi group $G_1^J(\mathbb{R})$ in the S-coordinates (x,y,θ,p,q,κ) is*

$$
\begin{aligned}
\mathrm{d}s^2_{G_1^J(\mathbb{R})} &= \sum_{i=1}^{6} \lambda_i^2\\
&= \alpha\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} + \beta(\frac{\mathrm{d}x}{y} + 2\mathrm{d}\theta)^2\\
&\quad + \frac{\gamma}{y}[\mathrm{d}q^2 + (x^2+y^2)\mathrm{d}p^2 + 2x\mathrm{d}p\mathrm{d}q] + \delta(\mathrm{d}\kappa - p\mathrm{d}q + q\mathrm{d}p)^2.
\end{aligned}
\tag{70}
$$

We have $<\lambda_i|L^j>=\delta_{ij}, i,j = 1,\ldots,6$, where the vector fields L^i, $i = 1,\ldots,6$ verify the commutation relations (60) *and are orthonormal with respect to the metric* (70).

If

$$
X = X^1\partial_x + X^2\partial_y + X^3\partial_\theta + X^4\partial_p + X^5\partial_q + X^6\partial_\kappa \in \mathfrak{g}_1^J(\mathbb{R}),
$$

then the fundamental vector fields (59) *in the S-variables, associated to the generators F,G,H,P,Q,R, corresponding to the left action* (44) *of the Jacobi group on himself are solution of the following Killing equations associated to the invariant metric* (70)

$$
\begin{aligned}
&-(\alpha+\beta)X^2 + (\alpha+\beta)y\partial_x X^1 + 2\beta y^2 \partial_x X^3 = 0,\\
&\alpha\partial_x X^2 + (\alpha+\beta)\partial_y X^1 + 2\beta y \partial_y X^3 = 0,\\
&-2\beta X^2 + 2\beta y \partial_x X^1 + 4\beta y^2 \partial_x X^3 + (\alpha+\beta)\partial_\theta X^1 + 2\beta y \partial_\theta X^3 = 0,
\end{aligned}
$$

$$(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_x X^4+(\gamma\frac{x}{y}-\delta pq)\partial_x X^5+\delta q\partial_x X^6+\frac{\alpha+\beta}{y^2}\partial_p X^1+2\frac{\beta}{y}\partial_p X^3=0,$$
$$(\gamma\frac{x}{y}-\delta pq)\partial_x X^4+(\frac{\gamma}{y}+\delta p^2)\partial_x X^6-\delta p\partial_x X^6+\frac{\alpha+\beta}{y^2}\partial_q X^1+2\frac{\beta}{y}\partial_q X^3=0,$$
$$\delta q\partial_x X^4-\delta p\partial_x X^5+\delta\partial_x X^6+\frac{\alpha+\beta}{y^2}\partial_\kappa X^3+2\frac{\beta}{y}\partial_\kappa X^3=0,$$
$$-X^2+y\partial_y X^2=0,$$
$$2\frac{\beta}{y}\partial_y X^1+4\beta\partial_y X^3+\frac{\alpha}{y^2}\partial_\theta X^2=0,$$
$$(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_y X^4+(\gamma\frac{x}{y}-\delta pq)\partial_y X^5+\delta q\partial_y X^6+\frac{\alpha}{y^2}\partial_p X^2=0,$$
$$(\gamma\frac{x}{y}-\delta pq)\partial_y X^4+(\frac{\gamma}{y}+\delta p^2)\partial_y X^5-\delta p\partial_y X^6+\frac{\alpha}{y^2}\partial_q X^2=0,$$
$$\delta q\partial_y X^4-\delta p\partial_y X^5+\delta\partial_y X^6+\frac{\alpha}{y^2}\partial_\kappa X^2=0,$$
$$\partial_\theta X^1+2y\partial_\theta X^3=0,$$
$$(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_\theta X^4+(\gamma\frac{x}{y}-\delta pq)\partial_\theta X^5+\delta q\partial_\theta X^6+2\frac{\beta}{y}\partial_p X^1+4\beta\partial_p X^6=0,$$
$$(\gamma\frac{x}{y}-\delta pq)\partial_\theta X^4+(\frac{\gamma}{y}+\delta p^2)\partial_\theta X^5-\delta p\partial_\theta X^6+2\frac{\beta}{y}\partial_q X^3+4\beta\partial_q X^3=0,$$
$$\delta q\partial_\theta X^4+(\frac{\gamma}{y}-\delta p)\partial_\theta X^5+\delta\partial_\theta X^6+4\beta\partial_\kappa X^3+2\frac{\beta}{y}\partial_\kappa X^1=0,$$
$$2\gamma\frac{x}{y}X^1+\gamma(1-\frac{x^2}{y^2})X^2+2\delta qX^5+2(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_p X^4$$
$$+2(\gamma\frac{x}{y}-\delta pq)\partial_p X^4+2\delta q\partial_p X^6=0,$$
$$\frac{\gamma}{y}X^1-\gamma\frac{x}{y^2}X^2-\delta(qX^4+pX^5)+(\gamma\frac{x}{y}-\delta pq)\partial_p X^4+(\frac{\gamma}{y}+\delta p^2)\partial_p X^5-\delta p\partial_p X^6$$
$$+(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_q X^4+(\gamma\frac{x}{y}-\delta pq)\partial_q X^5+\delta q\partial_q X^6=0,$$
$$\delta q\partial_p X^4-\delta p\partial_p X^5+\delta\partial_p X^6+(\gamma\frac{x^2+y^2}{y}+\delta q^2)\partial_\kappa X^4+(\gamma\frac{x}{y}-\delta pq)\partial_\kappa X^5+\delta q\partial_\kappa X^6=0,$$
$$-\frac{\gamma}{y^2}X^2+2\delta pX^4+2(\gamma\frac{x}{y}-\delta pq)\partial_q X^4+2(\frac{\gamma}{y}+\delta p^2)\partial_q X^4-2\delta p\partial_q X^6=0,$$
$$-\delta X^4+\delta q\partial_q X^4-\delta p\partial_q X^5+\delta\partial_q X^6+(\gamma\frac{x}{y}-\delta p\kappa)\partial_\kappa X^4+(\frac{\gamma}{y}+\delta p^2)\partial_\kappa X^5-\delta p\partial_\kappa X^6=0,$$
$$q\partial_\kappa X^6-p\partial_\kappa X^5+\partial_\kappa X^6=0.$$

7. NATURAL REDUCTIVITY AND GEODESIC VECTORS ON $\tilde{\mathcal{X}}_1^J$

We briefly recall the notions of natural reductivity and geodesic vectors. More references and details are given in [3, Appendix A].

In accord with Nomizu [67], a homogeneous space $M = G/H$ is *reductive* if the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an Ad(H)-invariant subspace $\mathfrak{m}$, that is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \mathfrak{h} \cap \mathfrak{m} = \emptyset, \tag{71a}$$

$$\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}. \tag{71b}$$

Condition (71b) implies

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \tag{71c}$$

and conversely, if H is connected, then (71c) implies (71b).

If the Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$ associated with the homogeneous manifold $M = G/H$ satisfy (71a), then a necessary and sufficient condition for M to be a *locally symmetric space* is

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{72}$$

If M is a complete, simply connected Riemannian locally symmetric space, then M is a Riemannian globally symmetric space [16, Theorem 5.6, p. 232].

We recall [75, Theorem 5.4], [3, Appendix A.4], [73, Proposition 1, p. 5], [43, Ch X, §3], [76, Theorem 6.2, p. 58].

Let (M, g) be a homogeneous Riemannian manifold. Then (M, g) is a naturally reductive Riemannian homogeneous space if and only if there exists a connected Lie subgroup G of $I(M)$ acting transitively and effectively on M and a reductive decomposition (71a) *such that one of the following equivalent statements holds*:

(i) *the following relation is verified*

$$g([X_1, X_3]_{\mathfrak{m}}, X_2) + g(X_1, [X_3, X_2]_{\mathfrak{m}}) = 0,\ \forall X_1, X_2, X_3 \in \mathfrak{m}; \tag{73}$$

(ii) $(*)$ *every geodesic in M is the orbit of a one-parameter subgroup of $I(M)$ generated by some $X \in \mathfrak{m}$.*

The natural reductivity is a special case of spaces with a more general property than $(*)$, see [3, Appendix A.7], [68]:

$(**)$ *Each geodesic of $(M = G/H, g)$ is an orbit of a one parameter group of isometries* $\exp(tZ)$, $Z \in \mathfrak{g}$.

A vector $X \in \mathfrak{g} \setminus \{0\}$ is called a *geodesic vector* if the curve $\gamma(t) = \exp(tX)(p)$ is a geodesic, cf. [68]. Riemannian homogeneous spaces with property (**) are called *g. o. spaces*. (g. o. = geodesics are orbits). All naturally reductive spaces are g. o. spaces.

Kowalski and Vanhacke [68] have proved that the condition to have a geodesic vector is expressed in the:

Geodesic Lemma: On homogeneous Riemannian manifolds $M = G/H$ a vector $X \in \mathfrak{g} \setminus \{0\}$ is geodesic if and only if

$$B([X,Y]_{\mathfrak{m}}, X_{\mathfrak{m}}) = 0, \forall Y \in \mathfrak{m}. \tag{74}$$

It is known [68] that: *Every simply connected Riemannian g. o. space $(G/H, g)$ of dimension $n \leq 5$ is a naturally reductive Riemannian manifold.*

The next theorem is a completion of [3, Proposition 5.8]

Theorem 1. *a) The Siegel–Jacobi upper half-plane, realized as homogeneous Riemannian manifold ($\mathcal{X}^J_1 = \frac{G^J_1(\mathbb{R})}{SO(2)\times\mathbb{R}}, g_{\mathcal{X}^J_1}$), is a reductive, non-symmetric manifold, not naturally reductive with respect to the balanced metric* (61b).

The Siegel–Jacobi upper half-plane $\mathcal{X}^J_1$ is not a g. o. space with respect to the balanced metric.

b) If

$$\mathfrak{g}^J_1(\mathbb{R}) \ni X = aL^1 + bL^2 + cL^3 + dL^4 + eL^5 + fL^6, \tag{75}$$

then the geodesic vectors of the homogeneous manifold $\mathcal{X}^J_1$ have one of the following expressions given in Table 1

Table 1

Components of the geodesic vectors (75) *on* $\mathcal{X}^J_1$.

N	a	b	c	d	e	f
1	0	0	c	0	0	f
2	a	b	0	0	0	f
3	rc	0	c	$\pm rc$	0	f
4	a	0	$-a$	0	$\epsilon\sqrt{r}a$	f
5	$\epsilon_1\epsilon_2\frac{1-r}{\sqrt{r}}e$	$\epsilon_1 e$	$-\frac{\epsilon_1\epsilon_2}{\sqrt{r}}e$	$\epsilon_2\sqrt{r}e$	e	f

where $r = \sqrt{\frac{\alpha}{\beta}}$, $\epsilon_1^2 = \epsilon_2^2 = \epsilon^2 = 1$.

c) The extended Siegel–Jacobi upper half-plane, realized as homogeneous Riemannian manifold ($\tilde{\mathcal{X}}^J_1 = \frac{G^J_1(\mathbb{R})}{SO(2)}, g_{\tilde{\mathcal{X}}^J_1}$), is a reductive, non-symmetric manifold, non-naturally reductive with respect to the metric (69).

The Siegel–Jacobi upper half-plane $\tilde{\mathcal{X}}^J_1$ is not a g. o. space with respect to the invariant metric (69).

d) If we take $X \in \mathfrak{g}^J_1$ as in (75)*, then the geodesic vectors on the extended Siegel–Jacobi manifold $\tilde{\mathcal{X}}^J_1$ are given in* Table 2 *for*

$$r > R_3 = [\frac{1}{2} + \frac{1}{6}(\frac{31}{3})^{\frac{1}{2}}]^{\frac{1}{3}} + [\frac{1}{2} - \frac{1}{6}(\frac{31}{3})^{\frac{1}{2}}]^{\frac{1}{3}} \approx 0.6823\ldots.$$

Table 2

Components of the geodesic vectors (75) *on* $\tilde{\mathcal{X}}_1^J$.

N	a	b	c	d	e	f
1	$\epsilon_1\epsilon_2(1-r)\sqrt{\frac{1+r^2}{rF_2}}e$	$\epsilon_2\sqrt{\frac{F_3}{r(r^2+1)}}e$	$-\epsilon_1\epsilon_2 r\sqrt{\frac{r}{(r^2+1)F_2}}e$	$\epsilon_1\sqrt{\frac{F_3}{F_2}}e$	$e\neq 0$	f
2	$\epsilon\sqrt{\frac{r}{r^2+2}}e$	0	$-\epsilon\sqrt{\frac{r}{r^2+2}}e$	$d=0$	$e\neq 0$	f
3	rc	0	$c\neq 0$	$\epsilon\sqrt{2+r^2}c$	0	f
4	0	0	c	0	0	f
5	0	b	0	0	0	f
6	a	0	0	0	0	f

The polynomials F_2, F_3 *are defined in* (76)

$$F_3(r) = r^3 + r - 1,\ F_2(r) = r^2 - r + 1,\ r \in \mathbb{R}. \tag{76}$$

Proof. a) This part has been proved in [3].

b) This part was also proved in [3]. Here we give more details which are used in the next parts of the theorem.

To find the geodesic vectors on the Siegel–Jacobi upper half-plane $\mathcal{X}_1^J$, we look for the solution (75) that verifies the condition (74).

Taking

$$\mathfrak{m} \ni Y = a_1L^1 + b_1L^2 + d_1L^4 + e_1L^5,$$

the condition (74) becomes

$$a_1\left(\frac{bc}{\sqrt{\beta}} + \frac{ed}{\sqrt{\alpha}}\right) + \frac{b_1}{2}\left[-\frac{ac}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha}}(d^2-e^2)\right] - \frac{d_1}{2\sqrt{\alpha}}(bd+ec+ae) + \frac{e_1}{2}\left[\frac{cd}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha}}(be-ad)\right] = 0,$$

and must be satisfied for every values of a_1, b_1, d_1, e_1, *i.e.* the coefficients of the geodesic vector (75) are solutions to the system of algebraic equations

$$rbc + de = 0, \tag{77a}$$

$$-rac + d^2 - e^2 = 0, \tag{77b}$$

$$bd + e(a+c) = 0, \tag{77c}$$

$$rcd + be - ad = 0. \tag{77d}$$

Now suppose that we have $de \neq 0$. We write (77c) and (77d) as

$$a + c = -\frac{bd}{e}, \tag{78a}$$

$$-a + rc = -\frac{be}{d}. \tag{78b}$$

We find from (78)

$$a = \frac{b}{de}\frac{e^2 - rd^2}{1+r}, \quad c = -\frac{b}{ed}\frac{d^2+e^2}{1+r}. \tag{79}$$

Introducing a and c from (79) into (77a) and (77b), we find the compatibility condition

$$d^2 = re^2. \tag{80}$$

If $ed \neq 0$, we get the first line in Table 1. The other situations are contained in the next lines of Table 1.

c) We consider

$$\mathfrak{m} = < F, G, H, P, Q >, \ \mathfrak{h} = < R >,$$

and with the commutation relations (35), (38), (43), we get $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, but $[\mathfrak{m}, \mathfrak{m}] \not\subseteq \mathfrak{h}$. This contradicts relation (72) satisfied by a symmetric manifold.

In order to verify the natural reductivity of the extended Siegel upper half-plane, we have to check relation (73). We take

$$X_i = a_i L^1 + b_i L^2 + c_i L^3 + d_i L^4 + e_i L^5 \subset \mathfrak{m}, \ i = 1,2,3. \tag{81}$$

Let us introduce the following notation

$$\gamma := \frac{1}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{\alpha}, \quad \zeta := \frac{\sqrt{\beta}}{\alpha} - \frac{1}{2\sqrt{\beta}}.$$

Then (73) reads

$$\begin{cases} \gamma(c_1 b_2 - b_1 c_2) = 0, \\ \zeta(c_1 a_2 - a_1 c_2) = 0, \\ \frac{3}{\sqrt{\beta}}(b_1 a_2 - a_1 b_2) + (\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}})(d_1 e_2 - e_1 d_2) = 0, \\ -a_1 e_2 - b_1 d_2 - c_1 e_2 + d_1 b_2 + e_1(a_2 + c_2) = 0, \\ \frac{d_1 a_2 - a_1 d_2 + b_1 e_2 - e_1 b_2}{\sqrt{\alpha}} + \frac{c_1 d_2 - d_1 c_2}{\sqrt{\beta}} = 0. \end{cases} \tag{82}$$

We write the system of algebraic equations (82) as

$$\sum_{j=1}^{5} A_{ij} x_j = 0, \quad i = 1, \ldots, 5, \tag{83}$$

where $x := (x_1, \ldots, x_5) = (a_1, \ldots, e_1)$.

Now we calculate matrix $A := \{A_{ij}\}_{i,j=1,\ldots,5}$ from (82)

$$A = \begin{pmatrix} 0 & \gamma c_2 & -\gamma b_2 & 0 & 0 \\ -\zeta c_2 & 0 & \zeta a_2 & 0 & 0 \\ -\frac{3b_2}{\sqrt{\beta}} & \frac{3a_2}{\sqrt{\beta}} & 0 & (\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}})e_2 & -(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}})d_2 \\ -e_2 & -d_2 & -e_2 & b_2 & a_2 + c_2 \\ -\frac{d_2}{\sqrt{\alpha}} & \frac{e_2}{\sqrt{\alpha}} & \frac{d_2}{\sqrt{\beta}} & \frac{a_2}{\sqrt{\alpha}} - \frac{c_2}{\sqrt{\beta}} & -\frac{b_2}{\sqrt{\alpha}} \end{pmatrix} \tag{84}$$

and, computing its determinant, we find $\det A = 0$ for any $X_2 \in \mathfrak{m}$. This means that we can find $X_1 \in \mathfrak{m}$ such that (73) be satisfied for any $X_2, X_3 \in \mathfrak{m}$, and thus $\tilde{\mathcal{X}}_1^J$ is not a naturally reductive space with respect to the metric (69).

d) The condition for a vector X (75) to be a geodesic vector on the extended Siegel–Jacobi upper half-plane is to verify (74). If we take

$$\mathfrak{m} \ni Y = a_1 L^1 + b_1 L^2 + c_1 L^3 + d_1 L^4 + e_1 L^5,$$

with the commutation relations (60), we find

$$\begin{aligned}
[X,Y] = & -\frac{1}{2\sqrt{\beta}}(cb_1 - c_1 b)L^1 + \frac{1}{\sqrt{\beta}}(ca_1 - c_1 a)L^2 - \frac{\sqrt{\beta}}{\alpha}(ab_1 - a_1 b)L^3 \\
& + [-\frac{1}{2\sqrt{\alpha}}(ae_1 - a_1 e) - \frac{1}{2\sqrt{\alpha}}(bd_1 - db_1) + \frac{1}{2\sqrt{\beta}}(ce_1 - c_1 e)]L^4 \\
& + [-\frac{1}{2\sqrt{\alpha}}(ad_1 - da_1) + \frac{1}{2\sqrt{\alpha}}(be_1 - b_1 e) - \frac{1}{2\sqrt{\alpha}}(cd_1 - c_1 d)]L^5 \\
& + 2(de_1 - ed_1)\frac{\sqrt{\delta}}{\gamma}L^6.
\end{aligned}$$

Condition (74) requires the components of the geodesic vector X to verify the system of algebraic equations:

$$(r + \frac{1}{r})bc + de = 0, \tag{85a}$$

$$-(r + \frac{2}{r})ac + d^2 - e^2 = 0, \tag{85b}$$

$$-rab + (1 - r)de = 0, \tag{85c}$$

$$bd + e(a + c) = 0, \tag{85d}$$

$$rcd + be - ad = 0. \tag{85e}$$

From (85d) and (85e) we get for a and c the expressions given in (79), which we introduce in (85a) and obtain

$$\frac{b^2}{d^2 e^2} = \frac{r(r+1)}{r^2+1}\frac{1}{d^2+e^2}. \tag{86}$$

We also introduce in (85c) the expressions for a and c given in (79) and we get

$$\frac{b^2}{d^2 e^2} = \frac{1-r^2}{r}\frac{1}{e^2 - rd^2}. \tag{87}$$

The compatibility of equations (86) and (87) imposes the following restriction:

$$\frac{d^2}{\mathrm{e}^2} = \frac{F_3(r)}{F_2(r)}. \tag{88}$$

The real root R_3 of equation $F_3(r) = 0$ is obtained with Cardano's formula as

$$R_3 = [\frac{1}{2} + \frac{1}{6}(\frac{31}{3})^{\frac{1}{2}}]^{\frac{1}{3}} + [\frac{1}{2} - \frac{1}{6}(\frac{31}{3})^{\frac{1}{2}}]^{\frac{1}{3}} \approx 0.6823\ldots.$$

Introducing (79) in (85b), we come back to condition (88).

□

In conclusion,

Comment 2. *In this paper we have investigated some geometric properties of the extended Siegel–Jacobi upper half-plane introduced in [3]. If the invariant metric on the four dimensional manifold $\mathcal{X}^J_1$ can be obtained with the CS methods, the invariant metric on the five dimensional manifold $\tilde{\mathcal{X}}^J_1$ can be obtained only with Cartan's moving frame method. Both manifolds $\mathcal{X}^J_1$ and $\tilde{\mathcal{X}}^J_1$ are reductive, non-symmetric, non-naturally reductive manifolds and consequently are not g. o. spaces.*

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