A NOTE ON PROPER CONFORMAL VECTOR FIELDS OF STATIC SPHERICALLY SYMMETRIC SPACE-TIMES IN f(G) THEORY WITH PERFECT FLUID SOURCE

FIAZ HUSSAIN[1], GHULAM SHABBIR[2,*], F. M. MAHOMED[3], MUHAMMAD RAMZAN[1]

[1] Department of Mathematics, The Islamia University of Bahawalpur, Pakistan
[2] Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology, Topi, Swabi, KPK, Pakistan
[3] DST-NRF Centre of Excellence in Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa
[*] Corresponding author: *E-mail*: shabbir@giki.edu.pk

Received April 6, 2020

Abstract. The aim of this paper is to investigate proper conformal vector fields of static spherically symmetric space-times in the f(G) theory. Firstly, we explore some perfect fluid solutions of the Einstein field equations in the f(G) theory of gravity using algebraic technique. Secondly, we investigate proper conformal vector fields of the solutions obtained *via* the direct integration approach. Studying each case in detail we show that a very special class of static spherically symmetric space-times admit proper conformal vector fields.

Key words: f(G) theory; Conformal vector fields; Algebraic and direct integration techniques.

1. INTRODUCTION

Spherical symmetric (SS) space-time is of great interest due to it having a number of important physical and theoretical literature. This space-time is considered as one of the basic solutions of the *Einstein field equations* (EFEs). For instance, the initial and physically important vacuum solution to the EFEs is the Schwarzschild solution. The Schwarzschild solution was derived from spherical symmetry in 1916 just after the introduction of *general relativity* (GR). From the physical point of view, the Schwarzschild solution is used to explain the gravitational field exterior to static spherical stars without discussing angular momentum. In the context of GR, gravity is described as the curvature of space-time. A number of experiments have been performed to obtain the success of GR. One of the experiments demonstrate that the deviation of light rays are due to the curvature within the space-time. On the other hand, verification of the event horizon of a black hole is one of the remarkable experiments recently performed to test the validity of GR [1–4]. These experiments have been theoretically performed by means of the Schwarzschild's metric which is basically a SS solution of the EFEs. The Schwarzschild's metric plays a pivotal role in looking at mysteries like the presence of negative mass matter and dynamics of black holes in the universe [5]. The Schwarzschild's metric aims to modify the curvature of space-time due to this property it is essence in the discussion of planetary motion and the bending of light around the sun. SS space-times retain their importance in the theory of black holes as well as its utility to classify them according to their physical properties. Another vacuum solution to the EFEs is the Kiselev's black hole space-time [6]. This space-time has its own identity as it is a generalized version of the Schwarzschild, Reissner–Nordstrom and Schwarzschild– (anti) de Sitter solutions. These space-times are the SS solutions of the EFEs and are important from both the mathematical and physical points of view. SS space-times are also used to discuss the solar system tests and can be considered as key ingredients to compute associated physical quantities like pressure, density and gravitational fields [7]. On the other hand, exact SS solutions yield a deeper approach to the Steller models and related red-shifts [8]. No doubt, vacuum solutions of the EFEs have their own physical importance. However, the inclusion of matter as a perfect fluid opens a new way to understand several physical phenomena in metric theories of gravity [9]. In view of the important applications discussed above, SS solutions of the EFEs are also applied to explore the effects of the non-zero cosmological constant [10].

In the recent advancements of modified theories of gravity, SS space-times have been widely discussed [11–20]. These theories have their birth for different reasons. For instance, discovery of the present accelerated expansion of the universe and the data coming from the recent observations have led to the amendments in GR. In spite of this, most of our universe is still unknown which is usually taken as the dark sector of the universe and assume to be composed of some baryonic matter called dark matter (DM). In the continuation of advances searching for the reasons of such type of mysteries and their possible solutions, some modified theories of gravity have been introduced. Some of these modified theories of gravity have been discussed in [21–24]. Among these theories, f(G) theory of gravity, where G being the Gauss-Bonnet invariant has proved to be an important one due to it having the conundrum of the expanding universe [25]. In the f(G) theory of gravity, the problem of cosmological constant is well tackled. However, the presence of the Gauss-Bonnet invariant in the modified EFEs has created difficulty in finding solutions. Some successful attempts have been made in this regard and a variety of solutions considering different space-times have been found so far [26–36]. One of the possible suggestions to overcome the difficulty arising in finding the solution of EFEs formed in f(G) gravity is to use symmetries. Moreover, the quantities that remain invariant under some transformation act as a source to convert nonlinear behavior of the EFEs into linear one. Symmetries have proved to be an effective and well-known method to find the metric components of a space-time which may be the solution of EFEs. Symmetries of SS space-times have been discussed in a series of works [37–43]. Space-time symmetries are also important in the sense that these are used to construct conservation laws [44]. In addition, space-time symmetries possess a wide range of applications and the details can be seen in [45]. In particular, conformal symmetries are important which yield *conformal vector fields* (CVFs). The CVFs act as an extra mathematical tool for simplifying gravitational potentials. Due to the physical importance of conformal symmetry, some work have appeared in f(R) gravity [46–52]. In this paper, the key point is to find proper CVFs of static SS space-times in the f(G) theory of gravity. To perform the required task, first we look for the solutions of the EFEs in the f(G) gravity and then we use these solutions to find proper CVFs. A vector field X is said to be conformal if [53]

$$L_X g_{ab} \equiv g_{ab,c}X^c + g_{bc}X^c_{,a} + g_{ac}X^c_{,b} = 2\psi g_{ab}, \tag{1}$$

where ψ, L, g_{ab} and comma represent the conformal function, the Lie derivative, metric tensor and partial derivative, respectively. From equation (1), we see that CVFs can further be categorized as *homothetic vector fields* (HVFs) or *Killing vector fields* (KVFs). If ψ is constant in equation (1), then X is called a HVF (proper HVF if $\psi \neq 0$) and in the case when ψ vanishes, then X is called a KVF. A proper CVF is one which is not a HVF.

2. MAIN RESULTS

Consider static SS space-times in the usual coordinates (t,r,θ,ϕ) (given by (x^0,x^1,x^2,x^3), respectively) with the line element [54]

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right], \tag{2}$$

where $A = A(r)$ and $B = B(r)$ are nowhere zero functions of r only. The above space-times (2) admit four KVFs which are [54]

$$\frac{\partial}{\partial t},\ \frac{\partial}{\partial \phi},\ \cot\theta\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta},\ \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}. \tag{3}$$

The Gauss-Bonnet invariant G for the above space-times (2) is [19]

$$G = \frac{e^{-2B}}{r^2}\left[2\left(1-e^{B}\right)\left(A'^2 + 2A'' - A'B'\right) - B'\left(4A' + B'\right) + A'^2 e^{2B(A-B)}\right], \tag{4}$$

where prime represents the derivative with respect to r. It is important to mention here that the authors in [19] found some static isotropic and anisotropic SS solutions and then discussed related physical parameters using power law models of the f(G) theory of gravity. In this paper, we are interested in finding proper CVFs for the space-times (2). In order to find proper CVFs, first we find space-times in the f(G) theory of gravity. The EFEs in the f(G) theory of gravity are [19]

$$P_{ac} = kT_{ac} - 8\begin{bmatrix} R_{abcd} + R_{cb}g_{da} - R_{bd}g_{ac} - R_{ac}g_{bd} \\ +R_{ad}g_{bc} + \dfrac{R}{2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \end{bmatrix}\nabla^b\nabla^d F - \left(Gf_G - f\right)g_{ac}, \tag{5}$$

where P_{ac} denotes the Einstein tensor, R_{ac} is the Ricci tensor, R_{abcd} is the Riemann tensor, f(G) is the function of the Gauss-Bonnet invariant G, $F(G) \equiv \frac{d}{dG}f(G)$, k is the coupling constant, T_{ac} is the energy momentum tensor and ∇ denotes the covariant derivative. In the above equation (5), if $T_{ac} = 0$, then one can find vacuum solutions. Here, we are using the case when $T_{ac} \neq 0$. In particular, the cosmological perfect fluids are important source of physical interest. Recently, these were qualitatively discussed in the frame-work of f(R) gravity as well as in Gauss-Bonnet gravity [55, 56]. Moreover, the use of non-vanishing energy momentum tensor T_{ac} along with the space-times (2) describe interior of a static SS star. Here, we are taking matter distribution as a perfect fluid whose energy momentum tensor is of the form

$$T_{ac} = \left(\rho + \omega\right)u_a u_c + \omega g_{ac}, \tag{6}$$

where ρ, ω and u_a are the matter density, pressure of the universe and four-velocity vector, respectively. By using equations (2) and (6) in equation (5), we obtain

$$\begin{gathered} \frac{1}{r^2} - \frac{e^{-B}}{r^2} + \frac{B'e^{-B}}{r} + \left[\frac{12}{r^2}B'e^{-2B} - \frac{4}{r^2}B'e^{-B} - \frac{8}{r^2}e^{-2B} + \frac{8}{r^2}e^{-B}\right]F \\ +GF - f = k\rho, \end{gathered} \tag{7}$$

$$\begin{gathered} \frac{e^{-B}}{r^2} - \frac{1}{r^2} + \frac{A'e^{-B}}{r} + \frac{8F}{r^2}A'e^{-2B} + \left[\frac{4}{r^2}B'e^{-2B} + \frac{8}{r}B'e^{-2B} - \frac{4}{r^2}A'e^{-B}\right]F \\ -\left(2A'^2e^{-2B}\right)F + \left(2A'B'e^{-2B}\right)F - \left(4A''e^{-2B}\right)F - GF + f = k\omega, \end{gathered} \tag{8}$$

$$\begin{gathered} \left[\frac{4}{r}A'e^{-2B} - \frac{2}{r}A'^2e^{-2B} + \frac{4}{r}A'B'e^{-2B} - \frac{4}{r}A''e^{-2B}\right]F + \\ \left[\frac{2}{r}A'B'e^{A-3B} - \frac{4}{r}A'e^{A-3B}\right]F + \left[\frac{4}{r}B'e^{-2B} - \frac{2}{r}B'^2e^{-2B}\right]F + \\ \frac{1}{4r}e^{-2A'B'+2BB'-rBA'^2+rBA'B'} + \frac{A''}{2}e^{-B} - GF + f = k\omega. \end{gathered} \tag{9}$$

Comparing equations (8) and (9), we obtain

$$\begin{aligned}&\left[\frac{2}{r}A'B'-\frac{4}{r}A'\right]e^{A-3B}F+\left[\frac{4}{r}B'-\frac{2}{r}B'^2\right]e^{-2B}F+\\&\frac{1}{4r}e^{-2A'B'+2BB'-rBA'^2+rBA'B'}+\frac{A''}{2}e^{-B}\left[\frac{4}{r}A'-\frac{2}{r}A'^2+\frac{4}{r}A'B'-\frac{4}{r}A''\right]e^{-2B}F-\\&\left[\frac{4}{r^2}B'e^{-2B}+\frac{8}{r}B'e^{-2B}-\frac{4}{r^2}A'e^{-B}\right]F+\frac{1}{r^2}-\frac{e^{-B}}{r^2}-\\&\frac{A'e^{-B}}{r}-\frac{8F}{r^2}A'e^{-2B}-\left[2A'^2e^{-2B}+2A'B'e^{-2B}-4A''e^{-2B}\right]F=0.\end{aligned} \tag{10}$$

We see that the above equation (10) is non-linear having three unknowns namely, *A*, *B* and *F*. Therefore, we need an extra condition to find a solution. Further, the calculations become easier if we impose the condition $B = -A$. This condition helps to find the following equation relating the function *f*(*G*) and the metric potential *A*:

$$f(G)=\left(\frac{\begin{matrix}16r^2A'^2e^{2A}+16r^2A''e^{2A}-16rA''e^{2A}-32rA'^2e^{2A}-\\16A'e^{2A}+32rA'e^{2A}+16A'e^{A}-16rA'e^{4A}-8rA'^2e^{4A}\end{matrix}}{4e^{A}-4+4rA'e^{A}-2r^2A''e^{A}-re^{2A'^2+2AA'+2rAA'^2}}\right)^{-1}G+k_1, \tag{11}$$

where $k_1 \in \Re$. From the above relation (11), it is clear that one can find the values of the function $f(G)$ by imposing some restrictions on the metric potential A. In order to determine a particular form of the function $f(G)$, we adopt the following relations:

(a1) $16r^2A'^2e^{2A}+16r^2A''e^{2A}=0.$
(a2) $16r^2A'^2e^{2A}-16rA''e^{2A}=0.$
(a3) $16r^2A'^2e^{2A}-16A'e^{2A}=0.$
(a4) $16r^2A'^2e^{2A}+32rA'e^{2A}=0.$
(a5) $16r^2A'^2e^{2A}+16A'e^{A}=0.$
(a6) $16r^2A'^2e^{2A}-8rA'^2e^{4A}=0.$
(a7) $16r^2A''e^{2A}-16rA''e^{2A}=0.$
(a8) $16r^2A''e^{2A}+32rA'e^{2A}=0.$
(a9) $16r^2A''e^{2A}+16A'e^{A}=0.$
(a10) $16rA''e^{2A}+32rA'^2e^{2A}=0.$
(a11) $16rA''e^{2A}+16A'e^{2A}=0.$
(a12) $16rA''e^{2A}+8rA'^2e^{4A}=0.$
(a13) $32rA'^2e^{2A}+16A'e^{2A}=0.$
(a14) $32rA'^2e^{2A}-32rA'e^{2A}=0.$
(a15) $32rA'^2e^{2A}+16rA'e^{4A}=0.$
(a16) $32rA'e^{2A}-8rA'^2e^{4A}=0.$
(a17) $16A'e^{A}-16rA'e^{4A}=0.$
(a18) $16rA'e^{4A}+8rA'^2e^{4A}=0.$
(a19) $2r^2A''e^{A}-4rA'e^{A}=0.$
(a20) $4e^{A}+4rA'e^{A}=0.$

Using the above approach given in (a1) to (a20), we have the following cases:

(i) $A=\ln\left(c_1 r+c_2\right)$, $f(G)=\left(\dfrac{-J_1-J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$,

$B=\ln\left(c_1 r+c_2\right)^{-1}$, where $J_1=16rA''e^{2A}$, $J_2=32rA'^2e^{2A}$, $J_3=16A'e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the Gauss-Bonnet invariant G is given by $G=\dfrac{c_1}{r^2}\left[7c_1-4J_{12}-\dfrac{4c_1}{c_1 r+c_2}+c_1 e^{-4\left(\ln\left(c_1 r+c_2\right)\right)^2}\right]$, $J_{12}=\left(\dfrac{c_1 r+c_2-1}{c_1 r+c_2}\right)$ and $c_1,c_2\in\Re\left(c_1\neq 0\right)$.

(ii) $A=\ln\left(\dfrac{1+c_1 r}{1-c_1 r}\right)^{c_1c_2}$, $f(G)=\left(\dfrac{J_1-J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$,

$B=\ln\left(\dfrac{1+c_1 r}{1-c_1 r}\right)^{-c_1c_2}$, where $J_1=16r^2A''e^{2A}$, $J_2=32rA'^2e^{2A}$, $J_3=16A'e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the value of G is $G=\dfrac{4c_1^4c_2}{r^2}\dfrac{\left(1+c_1 r\right)^{2\left(c_1c_2-1\right)}}{\left(1-c_1 r\right)^{2\left(c_1c_2+1\right)}}\left[7c_2+4rJ_{12}-4c_2e^{B}+c_2e^{-4A^2}\right]$ with $J_{12}=\left(1-e^{B}\right)$, $c_1,c_2\in\Re\left(c_1\neq 0\right)$.

(iii) $A=\left(\dfrac{c_1 r-1}{r}\right)$, $f(G)=\left(\dfrac{J_1-J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$,

$B=\left(\dfrac{c_1 r-1}{r}\right)^{-1}$, where $J_1=16r^2A''e^{2A}$, $J_2=16rA''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and G has the value $G=\dfrac{e^{2A}}{r^6}\left[7-4e^{-A}+e^{-4A^2}-8r\left(1-e^{-A}\right)\right]$ with $c_1\in\Re\setminus\{0\}$.

(iv) $A=\ln\left(c_1 r^{-2}\right)$, $f(G)=\left(\dfrac{J_1-J_2-J_3-J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$,

$B=\ln\left(c_1 r^{-2}\right)^{-1}$, where $J_1=16r^2A''e^{2A}$, $J_2=16rA''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=16A'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$,

$J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the value of G is $G=c_1^2\left[36-24c_1^{-1}r^2+4e^{-4\left(\ln c_1 r^{-2}\right)^2}\right]$ with $c_1\in\Re\setminus\{0\}$.

(v) $A=\ln\left(\frac{16-16c_1r}{r}\right)$, $f(G)=\left(\frac{J_1-J_2-J_3-J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln\left(\frac{r}{16-16c_1r}\right)$, where $J_1=16r^2A''e^{2A}$, $J_2=16rA''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=16A'e^{2A}$, $J_5=32rA'e^{2A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the value of G is $G=\frac{128}{r^6}\left[22-16c_1r-\left(16-16c_1r\right)e^B+2e^{-4A^2}\right]$ with $c_1\in\Re\setminus\{0\}$.

(vi) $A=\ln\left(2r\right)^{1/2}$, $f(G)=\left(\frac{J_1-J_2-J_3-J_4+J_5+J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln\left(2r\right)^{-1/2}$, where $J_1=16r^2A''e^{2A}$, $J_2=16rA''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=16A'e^{2A}$, $J_5=32rA'e^{2A}$, $J_6=16A'e^{A}$, $J_7=16rA'e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the G in this case is $G=\frac{1}{2r^3}\left[-1+4e^{-A}+e^{-4A^2}\right]$.

(vii) $A=\left(c_1r+c_2\right)$, $f(G)=\left(\frac{J_1-J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\left(c_1r+c_2\right)^{-1}$, where $J_1=16r^2A''e^{2A}$, $J_2=32rA'^2e^{2A}$, $J_3=16A'e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and G has the form $G=\frac{c_1^2e^{2A}}{r^2}\left[7+e^{-4A^2}-4e^{-A}\right]$ with $c_1,c_2\in\Re\left(c_1\neq 0\right)$.

(viii) $A=\left(\frac{c_2r-c_1}{r}\right)$, $f(G)=\left(\frac{J_1-J_2-J_3-J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\left(\frac{c_2r-c_1}{r}\right)^{-1}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16rA''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=16A'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the G over here is $G=\frac{c_1e^{2A}}{r^6}\left[7c_1+c_1e^{-4A^2}-4c_1e^{-A}-\ 8r\left(1-e^{-A}\right)\right]$ with $c_1,c_2\in\Re\left(c_1\neq 0\right)$.

(ix) $A=\ln\left(r^{-1}\right)$, $f(G)=\left(\dfrac{J_1-J_2-J_3-J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln\left(r^{-1}\right)^{-1}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16rA''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=16A'e^{2A}$, $J_5=32rA'e^{2A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the G in this case is $G=\dfrac{1}{r^4}\left[7-4r+e^{-4A^2}\right]$.

(x) $A=\ln\left(2c_1r+c_2\right)^{1/2}$, $f(G)=\left(\dfrac{J_1+J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln\left(2c_1r+c_2\right)^{-1/2}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16A'e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^A$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and G here has the form $G=\dfrac{c_1^2}{r^2\left(2c_1r+c_2\right)}\left[-1+4e^B+e^{-4A^2}\right]$ with $c_1,c_2\in\Re\left(c_1\neq 0\right)$.

(xi) $A=\ln\left(c_2r^{c_1}\right)$, $f(G)=\left(\dfrac{J_1+J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln\left(c_2r^{c_1}\right)^{-1}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^A$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the G over here is $G=\dfrac{c_1e^{2A}}{r^4}\left[7c_1-4c_1e^B+c_1e^{-4A^2}-4\left(1-e^B\right)\right]$ with $c_1,c_2\in\Re\left(c_1\neq 0\right)$.

(xii) $A=\ln\left(\dfrac{3c_1r+4c_2}{4}\right)^{4/3}$, $f(G)=\left(\dfrac{J_1+J_2-J_3-J_4+J_5+J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln\left(\dfrac{3c_1r+4c_2}{4}\right)^{-4/3}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=32rA'^2e^{2A}$, $J_4=16A'e^{2A}$, $J_5=32rA'e^{2A}$, $J_6=16A'e^A$, $J_7=16rA'e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and Gauss-Bonnet term is $G=\dfrac{c_1^2}{r^2}\left(\dfrac{3c_1r+4c_2}{4}\right)^{2/3}\left[4-e^B+e^{-4A^2}\right]$ with $c_1,c_2\in\Re\left(c_1\neq 0\right)$.

(xiii) $A=\ln\left(\frac{c_1}{\sqrt{r}}\right)$, $f(G)=\left(\frac{J_1+J_2+J_3-J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln\left(\frac{c_1}{\sqrt{r}}\right)^{-1}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the value of G turn to be $G=\frac{(c_1r)^{-1}}{4r^4}\left[15-12(c_1r)^{1/2}+e^{-4\left(\ln(c_1r)^{-1/2}\right)^2}\right]$ with $c_1\in\Re\setminus\{0\}$.

(xiv) $A=(c_1+r)$, $f(G)=\left(\frac{J_1+J_2-J_3-J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=(c_1+r)^{-1}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=16A'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and G in this case has the value $G=\frac{e^{2(c_1+r)}}{r^2}\left[7+e^{-4(c_1+r)^2}-4e^{-(c_1+r)}\right]$ and $c_1\in\Re\setminus\{0\}$.

(xv) $A=\ln(c_1+r)^{-1/2}$, $f(G)=\left(\frac{J_1+J_2-J_3-J_4+J_5+J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln(c_1+r)^{1/2}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=16A'e^{2A}$, $J_5=32rA'e^{2A}$, $J_6=16A'e^{A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the G over here is $G=\frac{(c_1+r)^{-3}}{4r^2}\left[15-12(c_1+r)^{1/2}+e^{-4\left(\ln(c_1+r)^{-1/2}\right)^2}\right]$ and $c_1\in\Re\setminus\{0\}$.

(xvi) $A=\ln(8r+c_1)^{1/2}$, $f(G)=\left(\frac{J_1+J_2-J_3-J_4-J_5+J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln(8r+c_1)^{-1/2}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=32rA'^2e^{2A}$, $J_5=16A'e^{2A}$, $J_6=16A'e^{A}$, $J_7=16rA'e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the value of G is $G=\frac{16}{r^2(8r+c_1)}\left[-1+4e^{A}+e^{-4A^2}\right]$ with $c_1\in\Re\setminus\{0\}$.

(xvii) $A=\ln(r)^{-1/3}$, $f(G)=\left(\frac{J_1+J_2-J_3-J_4-J_5+J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, $B=\ln(r)^{1/3}$,

where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=32rA'^2e^{2A}$, $J_5=16A'e^{2A}$, $J_6=32rA'e^{2A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and G in this case is $G=\frac{1}{9r^{14/3}}\left[19-16r^{1/3}+e^{-4\left(\ln r^{-1/3}\right)^2}\right]$.

(xviii) $A=(-2r+c_1)$, $f(G)=\left(\frac{J_1+J_2-J_3-J_4-J_5+J_6+J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$,

$B=(-2r+c_1)^{-1}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=32rA'^2e^{2A}$, $J_5=16A'e^{2A}$, $J_6=32rA'e^{2A}$, $J_7=16A'e^A$, $J_8=4e^A-4$, $J_9=4rA'e^A$, $J_{10}=2r^2A''e^A$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and Gauss-Bonnet invariant is $G=\frac{e^{-4r+2c_1}}{r^2}$ $\left[28-16e^{2r-c_1}+4e^{-4(-2r+c_1)^2}\right]$ with $c_1\in\Re\setminus\{0\}$.

(xix) $A=\left(\frac{c_1r^3+3c_2}{3}\right)$, $f(G)=\left(\frac{J_1+J_2-J_3-J_4-J_5+J_6+J_7-J_8-J_9}{J_{10}-J_{11}}\right)^{-1}G+k_1$,

$B=\left(\frac{3}{c_1r^3+3c_2}\right)$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=32rA'^2e^{2A}$, $J_5=16A'e^{2A}$, $J_6=32rA'e^{2A}$, $J_7=16A'e^A$, $J_8=16rA'e^{4A}$, $J_9=8rA'^2e^{4A}$, $J_{10}=4e^A-4$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and G is of the form $G=c_1e^{2A}\left[7c_1r^2+\frac{8}{r}J_{12}+c_1r^2J_{13}\right]$ with $J_{12}=\left(1-e^{-A}\right)$, $J_{13}=\left(e^{-4A^2}-4e^{-A}\right)$ and $c_1,c_2\in\Re(c_1\neq0)$.

(xx) $A=\ln\left(c_1r^{-1}\right)$, $f(G)=\left(\frac{J_1+J_2-J_3-J_4-J_5+J_6+J_7-J_8-J_9}{J_{10}-J_{11}}\right)^{-1}G+k_1$,

$B=\ln\left(c_1r^{-1}\right)^{-1}$, where $J_1=16r^2A'^2e^{2A}$, $J_2=16r^2A''e^{2A}$, $J_3=16rA''e^{2A}$, $J_4=32rA'^2e^{2A}$, $J_5=16A'e^{2A}$, $J_6=32rA'e^{2A}$, $J_7=16A'e^A$, $J_8=16rA'e^{4A}$,

$J_9 = 8rA'^2e^{4A}$, $J_{10} = -4-2r^2A''e^A$, $J_{11} = re^{2A'^2+2AA'+2rAA'^2}$ and G here is $G = \frac{\left(c_1 r^{-1}\right)^2}{r^4}\left[11-8e^B+e^{-4A^2}\right]$ with $c_1 \in \Re\setminus\{0\}$.

Now, we find proper CVFs for each of the above cases. Writing equation (1) explicitly and using equation (2), we obtain

$$A'X^1 + 2X^0_{,0} = 2\psi, \tag{12}$$
$$e^A X^0_{,1} - e^B X^1_{,0} = 0, \tag{13}$$
$$e^A X^0_{,2} - r^2 X^2_{,0} = 0, \tag{14}$$
$$e^A X^0_{,3} - r^2\sin^2\theta X^3_{,0} = 0, \tag{15}$$
$$B'X^1 + 2X^1_{,1} = 2\psi, \tag{16}$$
$$e^B X^1_{,2} + r^2 X^2_{,1} = 0, \tag{17}$$
$$e^B X^1_{,3} + r^2\sin^2\theta X^3_{,1} = 0, \tag{18}$$
$$X^1 + rX^2_{,2} = r\psi, \tag{19}$$
$$X^2_{,3} + \sin^2\theta X^3_{,2} = 0, \tag{20}$$
$$X^1 + r\cot\theta X^2 + rX^3_{,3} = r\psi. \tag{21}$$

From equations (15), (17), (18) and (20), we have

$$\begin{aligned}
X^0 &= \sin\theta r^2 e^{-A}\left[\iint H^1_t(t,r,\phi)drd\phi + \sin\theta\int H^2_t(t,\theta,\phi)d\phi\right] + H^5(t,r,\theta),\\
X^1 &= -r^2e^{-B}\left[\sin\theta\int H^1(t,r,\phi)d\phi + \int H^3_r(t,r,\theta)d\theta\right] + H^4(t,r,\phi),\\
X^2 &= \cos\theta\iint H^1(t,r,\phi)drd\phi - \sin^2\theta\int H^2_\theta(t,\theta,\phi)d\phi + H^3(t,r,\theta),\\
X^3 &= \cos ec\theta\int H^1(t,r,\phi)dr + H^2(t,\theta,\phi),
\end{aligned} \tag{22}$$

where $H^1(t,r,\phi)$, $H^2(t,\theta,\phi)$, $H^3(t,r,\theta)$, $H^4(t,r,\phi)$ and $H^5(t,r,\theta)$ are functions of integration. The values of these functions will give final form of the CVFs. As we have already mentioned that we find the CVFs by using the values of metric components A and B given in cases (i) to (xx). Here, it is necessary to state that we skip the lengthy procedure of integration and only present the results.

Case (i)

In this case, we have that $A=\ln\left(c_1r+c_2\right)$, $B=\ln\left(c_1r+c_2\right)^{-1}$, $f(G)=\left(\frac{-J_1-J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, where $J_1=16rA''e^{2A}$, $J_2=32rA'^2e^{2A}$, $J_3=16A'e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$ and the Gauss-Bonnet invariant G is given by $G=\frac{c_1}{r^2}\left[7c_1-4J_{12}-\frac{4c_1}{c_1r+c_2}+c_1e^{-4\left(\ln\left(c_1r+c_2\right)\right)^2}\right]$, $J_{12}=\left(\frac{c_1r+c_2-1}{c_1r+c_2}\right)$ and $c_1,c_2\in\Re\left(c_1\neq 0\right)$. The space-times (2) take the form

$$ds^2=-\left(c_1r+c_2\right)dt^2+\left(c_1r+c_2\right)^{-1}dr^2+r^2\left[d\theta^2+\sin^2\theta d\phi^2\right]. \tag{23}$$

Solving equations (12) to (21) with the help of space-time (23), we find that

$$\begin{gathered}X^0=\left(\frac{c_1r+2c_2}{c_1\sqrt{c_1r+c_2}}\right)\left[c_3e^{\frac{c_1t}{2}}-c_4e^{\frac{-c_1t}{2}}\right]+c_7,\quad X^1=r\sqrt{c_1r+c_2}\left[c_3e^{\frac{c_1t}{2}}+c_4e^{\frac{-c_1t}{2}}\right],\\ X^2=-c_5\cos\phi+c_6\sin\phi,\quad X^3=\cot\theta\left[c_5\sin\phi+c_6\cos\phi\right]+c_8,\end{gathered} \tag{24}$$

where $c_3,c_4,c_5,c_6,c_7,c_8\in\Re\setminus\{0\}$. The conformal factor in this case is $\psi=\sqrt{c_1r+c_2}\left[c_3e^{\frac{c_1t}{2}}+c_4e^{\frac{-c_1t}{2}}\right]$. Clearly, the space-time (23) admits six CVFs in which four are KVFs which are given in equation (3) and two are proper CVFs. The proper CVF after eliminating KVFs from (24) is

$$\begin{gathered}X^0=\left(\frac{c_1r+2c_2}{c_1\sqrt{c_1r+c_2}}\right)\left[c_3e^{\frac{c_1t}{2}}-c_4e^{\frac{-c_1t}{2}}\right],\\ X^1=r\sqrt{c_1r+c_2}\left[c_3e^{\frac{c_1t}{2}}+c_4e^{\frac{-c_1t}{2}}\right],\quad X^2=0,\quad X^3=0.\end{gathered} \tag{25}$$

Case (ii)

Here, we have the values $A=\ln\left(\frac{1+c_1 r}{1-c_1 r}\right)^{c_1 c_2}$, $B=\ln\left(\frac{1+c_1 r}{1-c_1 r}\right)^{-c_1 c_2}$, $f(G)=\left(\frac{J_1-J_2-J_3+J_4+J_5-J_6-J_7}{J_8+J_9-J_{10}-J_{11}}\right)^{-1}G+k_1$, where $J_1=16r^2A''e^{2A}$, $J_2=32rA'^2e^{2A}$, $J_3=16A'e^{2A}$, $J_4=32rA'e^{2A}$, $J_5=16A'e^{A}$, $J_6=16rA'e^{4A}$, $J_7=8rA'^2e^{4A}$, $J_8=4e^{A}-4$, $J_9=4rA'e^{A}$, $J_{10}=2r^2A''e^{A}$, $J_{11}=re^{2A'^2+2AA'+2rAA'^2}$, $G=\frac{4c_1^4c_2}{r^2}\frac{(1+c_1 r)^{2(c_1c_2-1)}}{(1-c_1 r)^{2(c_1c_2+1)}}\left[7c_2+4rJ_{12}-4c_2e^{B}+c_2e^{-4A^2}\right]$ and $J_{12}=\left(1-\left(\frac{1+c_1 r}{1-c_1 r}\right)^{-c_1c_2}\right)$, $c_1,c_2\in\Re(c_1\neq 0)$. The space-times (2) in this case become

$$ds^2=-\left(\frac{1+c_1 r}{1-c_1 r}\right)^{c_1c_2}dt^2+\left(\frac{1+c_1 r}{1-c_1 r}\right)^{-c_1c_2}dr^2+r^2\left[d\theta^2+\sin^2\theta d\phi^2\right]. \quad (26)$$

Substituting the values of A and B in equations (12) to (21) and overlooking the details, we find that $\psi=0$ which means that no proper CVFs exist in this case. The CVFs are the KVFs which are given in equation (3). Other cases (iii) to (xx) provide the same result as in the present case.

3. SUMMARY

Spherical symmetric space-times are of great interest in the study of several mathematical and physical problems. These space-times retain their position in the modern study of black holes, gravstars and many physical phenomena. Due to the rich amount of valuable literature, the static SS space-time is considered in order to study proper CVFs in the f(G) theory of gravity. The approach which we have adopted for studying proper CVFs was based on the direct integration. From this study, we found that in the f(G) theory of gravity, a very special class of the static SS space-times admit proper CVFs. The space-time which admit proper CVFs is provided in equation (23) and the CVFs for this space-time are given in equation (24). In the remaining cases, the CVFs become KVFs. The KVFs provide laws of conservation, for example, ∂_t corresponds to energy conservation, ∂_ϕ represents spatial translation yielding the well-defined conservation of linear momentum whereas $\cot\theta\cos\phi\partial_\phi+\sin\phi\partial_\theta$ and $\cos\phi\partial_\theta-\cot\theta\sin\phi\partial_\phi$ are the rotations implying conservation of angular momentum [57, 58]. On the other hand, CVFs are used in the characterization of different spaces, to study the geometry of manifold and potential functions of wave and Klein-Gordon equations. More applications of the CVFs are stated in [45, 52, 59, 60]. It is important to mention here that the space-times extracted from this paper may further be used in equations (7) to (9) of the present work which yield pressure and density for each of the space-time. This is beyond the scope of our present study.

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